FINITELY GENERATED SUBRINGS OF $R[x]$  

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ABSTRACT. In this article all rings and algebras are commutative with identity, and we denote by $R[x]$ the ring of polynomials over a ring $R$ in one variable $x$. We describe rings $R$ such that all subalgebras of $R[x]$ are finitely generated over $R$.

INTRODUCTION

Let $K$ be a field and let $L$ be a subfield of $K(x_1,\ldots,x_n)$ containing $K$. In 1954, Zariski in [15], proved that if $n \leq 2$, then the ring $L \cap K[x_1,\ldots,x_n]$ is finitely generated over $K$. This is a result concerning the fourteenth problem of Hilbert. Today we know ([8], [9], [7]) that a similar statement for $n \geq 3$ is not true. Many results on this subject one can find, for example, in [4], [5], [10], [13], and also in the author articles ([11], [12]) published by University of Lodz in Materials of the Conferences of Complex Analytic and Algebraic Geometry.

We are interested in the case $n = 1$. It is well known that every $K$-subalgebra $A$ of $K[x_1]$ is finitely generated over $K$. In this case we do not assume that $A$ has a form $L \cap K[x_1]$. We recall it (with a proof) as Theorem 2.1. An elementary proof one can find, for example, in [6]. The assumption that $K$ is a field is here very important. What happens in the case when $K$ is a commutative ring and $K$ is not a field? In this article we will give a full answer to this question.

Throughout this article all rings and algebras are commutative with identity, and we denote by $R[x]$ the ring of polynomials over a ring $R$ in one variable $x$. We say that a ring $R$ is an $sfg$-ring, if every $R$-subalgebra of $R[x]$ is finitely generated over $R$. We already know that if $R$ is a field then $R$ is an $sfg$-ring. We will show
that the rings \( \mathbb{Z} \) and \( \mathbb{Z}_4 \) are not sfg-rings. But, for instance, the rings \( \mathbb{Z}_6 \) and \( \mathbb{Z}_{105} \) are sfg-rings.

The main result of this article states that \( R \) is an sfg-ring if and only if \( R \) is a finite product of fields. For a proof of this fact we prove, in Section 3, many various lemmas. A crucial role plays the Artin-Tate Lemma (Lemma 1.3). If \( R \) is an sfg-ring then we successively prove that \( R \) is Noetherian, reduced, that every prime ideal of \( R \) is maximal, and by this way we obtain that \( R \) is a finite product of fields. Moreover, in the last section, we present a proof that every finite product of fields is an sfg-ring.

1. Preliminary lemmas and notations

We start with the following well known lemma (see for example [2] Proposition 6.5).

**Lemma 1.1.** If \( R \) is a Noetherian ring and \( M \) is a finitely generated \( R \)-module, then \( M \) is a Noetherian module.

Let \( A \) be an algebra over a ring \( R \). If \( S \) is a subset of \( A \), then we denote by \( R[S] \) the smallest \( R \)-subalgebra of \( A \) containing \( R \) and \( S \). Several times we will use the following obvious lemma.

**Lemma 1.2.** Let \( A = R[S] \). If the algebra \( A \) is finitely generated over \( R \), then there exists a finite subset \( S_0 \) of \( S \) such that \( A = R[S_0] \).

The next lemma comes from [14] (Lemma 2.4.3). This is a particular case of the Artin and Tate result published in [1]. Since this lemma plays an important role in our article, we present also its simple proof.

**Lemma 1.3** (Artin, Tate, 1951). Let \( R \) be a Noetherian ring, \( B \) a finitely generated \( R \)-algebra, and \( A \) an \( R \)-subalgebra of \( B \). If \( B \) is integral over \( A \), then the algebra \( A \) is finitely generated over \( R \).

**Proof.** Let \( B = R[b_1, \ldots, b_s] \), where \( b_1, \ldots, b_s \) are some elements of \( B \). Since each \( b_i \) is integral over \( A \), we have equalities of the form

\[
b_i^{n_i} + a_{i1} b_i^{n_{i1} - 1} + \cdots + a_{in_i} = 0, \quad \text{for } i = 1, \ldots, s,
\]

where all coefficients \( a_{ij} \) belong to \( A \), and \( n_1, \ldots, n_s \) are positive integers. Let \( \{a_1, \ldots, a_m\} \) be the set of all the coefficients \( a_{ij} \), and put

\[
A' = R[a_1, \ldots, a_m].
\]

It is clear that \( A' \) is a Noetherian ring and \( B \) is an \( A' \)-module generated by all elements of the form \( b_1^{j_1} b_2^{j_2} \cdots b_s^{j_s} \), where \( 0 \leq j_1 < n_1, \ldots, 0 \leq j_s < n_s \). Thus, \( B \) is a finitely generated \( A' \)-module and so, by Lemma 1.1, \( B \) is a Noetherian \( A' \)-module. This means that every submodule of \( B \) is finitely generated. In particular,
A is a finitely generated $A'$-module. Assume that $a_{m+1}, a_{m+2}, \ldots, a_n \in A$ are its generators. Then

$$A = A'a_{m+1} + \cdots + A'a_n = R[a_1, \ldots, a_n],$$

and we see that the algebra $A$ is finitely generated over $R$. \hfill \square

Let us fix some notations. For a given subset $I$ of a ring $R$, we denote by $I[x]$ the set of all polynomials from $R[x]$ with the coefficients belonging to $I$. If $I$ is an ideal of $R$, then $I[x]$ is an ideal of $R[x]$, and then the rings $R[x]/I[x]$ and $(R/I)[x]$ are isomorphic.

Let $f : S \rightarrow T$ be a homomorphism of rings. We denote by $\overline{f}$ the mapping from $S[x]$ to $T[x]$ defined by the formula

$$\overline{f} \left( \sum_j s_j x^j \right) = \sum_j \varphi(s_j)x^j$$

for all $\sum_j s_j x^j \in S[x]$. This mapping is a homomorphism of rings and $\text{Ker} \overline{f} = (\text{Ker} f)[x]$. We will say that $\overline{f}$ is the homomorphism associated with $f$. If $f$ a surjection, then $\overline{f}$ is also a surjection. It is clear that if $S$ and $T$ are $R$-algebras, and $f : S \rightarrow T$ is a homomorphism of $R$-algebras, then $\overline{f} : S[x] \rightarrow T[x]$ is also a homomorphism of $R$-algebras.

In next sections we will use the following two lemmas.

**Lemma 1.4.** Let $I$ be an ideal of a ring $R$, and let $A = R[a_1x; \quad a \in I]$. If the ideal $I$ is not finitely generated, then the algebra $A$ is not finitely generated over $R$.

**Proof.** Assume that $I$ is not finitely generated and suppose that $A$ is finitely generated over $R$. Then, by Lemma 1.2, there exists a finite subset $\{a_1, \ldots, a_n\}$ of $I$ such that $A = R[a_1, \ldots, a_n]$. Then of course $(a_1, \ldots, a_n) \not\subseteq I$ so, there exists $b \in I \setminus (a_1, \ldots, a_n)$. Since $bx \in A = R[a_1, \ldots, a_n]$, we have $bx = F(a_1x, \ldots, a_nx)$, where $F$ is a polynomial belonging to $R[t_1, \ldots, t_n]$. Let

$$F = r_0 + r_1 t_1 + r_2 t_2 + \cdots + r_n t_n + G$$

where $r_0, r_1, \ldots, r_n \in R$ and $G \in R[t_1, \ldots, t_n]$ is a polynomial in which the degrees of all nonzero monomials are greater than 1. Then, in the ring $R[x]$ we have

$$bx = F(a_1x, \ldots, a_nx) = r_0 + r_1 a_1 x + \cdots + r_n a_n x + hx^2,$$

where $h$ is some element of $R[x]$. This implies that $b = r_1 a_1 + \cdots + r_n a_n \in (a_1, \ldots, a_n)$, but it is a contradiction, because $b \not\in (a_1, \ldots, a_n)$. \hfill \square

**Lemma 1.5.** Let $A = R[ bx, bx^2, \ldots , bx^n ]$, where $n \geq 1$, $0 \neq b \in R$ and $b^2 = 0$. Then every element $u$ of $A$ is of the form $u = r_0 + r_1 bx + r_2 bx^2 + \cdots + r_n bx^n$ for some $r_0, r_1, \ldots , r_n \in R$. 
Proof. Let \( u \in A \). Then \( u = F(bx, bx^2, \ldots, bx^n) \) for some \( n \), where \( F \) is a polynomial in \( n \) variables belonging to the polynomial ring \( R[t_1, \ldots, t_n] \). Let
\[
F(t_1, \ldots, t_n) = r_0 + r_1 t_1 + r_2 t_2 + \cdots + r_n t_n + G(t_1, \ldots, t_n),
\]
where \( r_0, \ldots, r_n \in R \) and \( G \in R[t_1, \ldots, t_n] \) is a polynomial such that the degrees of all nonzero monomials of \( F \) are greater than 1. Then \( G(bx, \ldots, bx^n) = b^2 H(x) \), gdzie \( H(x) \in R[x] \). But \( b^2 = 0 \), so \( u = r_0 + r_1 bx + r_2 bx^2 + \cdots + r_n bx^n \). \( \blacksquare \)

2. Subalgebras of \( K[x] \)

Let us start with the following consequence of Lemma 1.3.

Example 2.2. Let \( K[x, y] \) be the polynomial ring in two variables over a field \( K \), and
\[
A = K \left[ xy, xy^2, xy^3, \ldots \right].
\]
The algebra \( A \) is not finitely generated over \( K \).

Proof. For every positive integer \( n \), consider the ideal \( I_n \) of \( A \), generated by the monomials \( xy, xy^2, \ldots, xy^n \). Observe that \( xy^{n+1} \notin I_n \). Indeed, suppose \( xy^{n+1} = F_1 xy + F_2 xy^2 + \cdots + F_n xy^n \), where \( F_1, \ldots, F_n \in A \). Every element of \( A \) is of the form \( a + Gxy \) with \( a \in K \) and \( G \in K[x, y] \). In particular, \( F_j = a_j + G_j xy \), where \( a_j \in K, G_j \in K[x, y] \) for all \( j = 1, \ldots, n \). Thus, in \( K[x, y] \) we have
\[
y^{n+1} = a_1 y + a_2 y^2 + \cdots + a_n y^n + (G_1 y^2 + G_2 y^3 + \cdots + G_n y^n) x.
\]
Let \( \varphi : K[x, y] \to K[y] \) be the homomorphism of \( K \)-algebras defined by \( x \mapsto 0 \) and \( y \mapsto y \). Then in the ring \( K[y] \), we have the false equality \( y^{n+1} = \varphi (y^{n+1}) = a_1 y + a_2 y^2 + \cdots + a_n y^n \). Hence, the infinite sequence \( I_1 \subset I_2 \subset I_3 \subset \cdots \) is strictly increasing. The ring \( A \) is not Noetherian. In particular, the algebra \( A \) is not finitely generated over \( K \). \( \blacksquare \)
In Theorem 2.1 we assumed that $K$ is a field. This assumption is here very important. For instance, if $K$ is the ring of integers $\mathbb{Z}$, then a similar assertion is not true.

**Example 2.3.** Let $A = \mathbb{Z}[2x, 2x^2, 2x^3, \ldots]$. Then $A$ is a subalgebra of $\mathbb{Z}[x]$ and $A$ is not finitely generated over $\mathbb{Z}$.

**Proof.** For every positive integer $n$, consider the ideal $I_n$ of $A$, generated by the monomials $2x, 2x^2, \ldots, 2x^n$. Observe that $2x^{n+1} \notin I_n$. Indeed, suppose $2x^{n+1} = 2xF_1 + 2x^2F_2 + \cdots + 2x^nF_n$, where $F_1, \ldots, F_n \notin A$. Every element of $A$ is of the form $a + 2xG$ with $a \in \mathbb{Z}$ and $G \in \mathbb{Z}[x]$. In particular, $F_j = a_j + 2xG_j$, where $a_j \in \mathbb{Z}$, $G_j \in \mathbb{Z}[x]$ for all $j = 1, \ldots, n$. Thus, in $\mathbb{Z}[x]$ we have the equality

$$x^{n+1} = a_1x + a_2x^2 + \cdots + a_nx^n + 2(G_1x^2 + G_2x^3 + \cdots + G_nx^{n+1}).$$

For an integer $u$, denote by $\overline{u}$ the element $u$ modulo 2. Then, in the ring $\mathbb{Z}_2[x]$ we have the false equality $x^{n+1} = \overline{a_1}x + \overline{a_2}x^2 + \cdots + \overline{a_n}x^n$. Hence, the infinite sequence $I_1 \subset I_2 \subset I_3 \subset \cdots$ is strictly increasing. The ring $A$ is not Noetherian. In particular, the algebra $A$ is not finitely generated over $\mathbb{Z}$. □

### 3. Properties of sfg-rings

Let us recall that a ring $R$ is said to be an *sfg-ring*, if every $R$-subalgebra of $R[x]$ is finitely generated over $R$. We already know (by Theorem 2.1) that if $R$ is a field then $R$ is an sfg-ring. Moreover we know (by Example 2.3) that $\mathbb{Z}$ is not an sfg-ring. In this section we will prove that every sfg-ring is a finite product of fields. For a proof of this fact we need the following 9 successive lemmas. In all the lemmas we assume that $R$ is an sfg-ring.

**Lemma 3.1.** $R$ is Noetherian.

**Proof.** Suppose $R$ is not Noetherian. Then there exists an ideal $I$ of $R$ which is not finitely generated. Consider the $R$-algebra $A := R[ax; a \in I]$. It follows from Lemma 1.4 that this algebra is not finitely generated over $R$. But this contradicts our assumption that $R$ is an sfg-ring. □

Now we know, by this lemma, that if $R$ is an sfg-ring, then every $R$-subalgebra of $R[x]$ is a Noetherian ring.

**Lemma 3.2.** If $I$ is an ideal of $R$, then $R/I$ is also an sfg-ring.

**Proof.** Put $\overline{R} := R/I$. Let $\varphi : R \to \overline{R}$, $r \mapsto r+I$ be the natural ring homomorphism, and let $\overline{\varphi} : R[x] \to \overline{R}[x]$ be the homomorphism associated with $\varphi$. Let $B$ be an $\overline{R}$-subalgebra of $\overline{R}[x]$. We need to show that $B$ is finitely generated over $\overline{R}$. For this aim consider the $R$-algebra $A := \overline{\varphi}^{-1}(B)$. It is an $R$-subalgebra of $R[x]$. Since $R$ is an sfg-ring, the algebra $A$ is finitely generated over $R$. Let $W \subset A$ be a finite set of generators of $A$. Then it is easy to check that $\overline{\varphi}(W)$ is a finite set of generators of $B$ over $\overline{R}$. □
**Lemma 3.3.** Every non-invertible element of $R$ is a zero divisor.

*Proof.* Suppose there exists a non-invertible element $b \in R$ such that $b$ is not a zero divisor of $R$. Then $b \neq 0$ and $b$ is not a zero divisor of $R[x]$. Consider the $R$-subalgebra $A = R[bx, bx^2, bx^3, \ldots]$. For every positive integer $n$, let $I_n$ be the ideal of $A$, generated by the monomials $bx, bx^2, \ldots, bx^n$. Observe that $bx^{n+1} \notin I_n$. Indeed, suppose $bx^{n+1} = bx F_1 + bx^2 F_2 + \cdots + bx^n F_n$, where $F_1, \ldots, F_n \in A$. Every element of $A$ is of the form $a + bx G$ with $a \in R$ and $G \in R[x]$. In particular, $F_j = a_j + bx G_j$, where $a_j \in \mathbb{R}$, $G_j \in R[x]$ for all $j = 1, \ldots, n$. Since the element $b$ is not a zero divisor of $R[x]$, we have in $R[x]$ the following equality

$$x^{n+1} = a_1 x + a_2 x^2 + \cdots + a_n x^n + b \left( G_1 x^2 + G_2 x^3 + \cdots + G_n x^n \right).$$

Consider the factor ring $R/(b)$. Let $\varphi : R \rightarrow R/(b)$, $r \mapsto r + (b)$, be the natural homomorphism and $\overline{\varphi} : R[x] \rightarrow R/(b)[x]$ be the homomorphism associated with $\varphi$. Using $\overline{\varphi}$, from the above equality we obtain that $x^{n+1} = \varphi(a_1)x + \varphi(a_2)x^2 + \cdots + \varphi(a_n)x^n$. This is a false equality in the polynomial ring $\mathbb{R}/(b)[x]$. Therefore, $bx^{n+1} \notin I_n$. Hence, the infinite sequence $I_1 \subset I_2 \subset I_3 \subset \cdots$ is strictly increasing. This means that the ring $A$ is not Noetherian. In particular, by Lemma 3.1, the algebra $A$ is not finitely generated over $R$. But this contradicts our assumption that $R$ is an sfg-ring. \hfill \square

It follows from the above lemma that every ring without zero divisors, which is not a field, is not an sfg-ring. Thus, we see again, for instance, that $\mathbb{Z}$ is not an sfg-ring.

**Lemma 3.4.** $R$ is a reduced ring, that is, $R$ is without nonzero nilpotent elements.

*Proof.* Suppose that there exists $c \in R$ such that $c \neq 0$ and $c^m = 0$ for some $m \geq 2$. Assume that $m$ is minimal and put $b := c^{m-1}$. Then $0 \neq b \in R$ and $b^2 = 0$. Consider the $R$-algebra $A = R[bx, bx^2, bx^3, \ldots]$. It is an $R$-subalgebra of $R[x]$. Since $R$ is an sfg-ring, this algebra is finitely generated over $R$. Hence, by Lemma 1.2, $A = R[bx, bx^2, \ldots, bx^n]$ for some fixed $n$. But $bx^{n+1} \in A$ so, by Lemma 1.5,

$$bx^{n+1} = r_0 + r_1 bx + r_2 bx^2 + \cdots + r_n bx^n,$$

where $r_0, r_1, \ldots, r_n \in R$. It is an equality in the polynomial ring $R[x]$. This implies that $b = 0$ and we have a contradiction. Therefore, the algebra $A$ is not finitely generated over $R$, and this contradicts our assumption that $R$ is an sfg-ring. \hfill \square

**Lemma 3.5.** $(b) = (b^2)$ for all $b \in R$.

*Proof.* It is clear when $R$ is a field. Assume that $R$ is not a field. Let $b \in R$ and suppose $(b^2) \neq (b)$. Then $b \notin (b^2)$. Consider the ideal $I := (b^2)$ and the factor ring $\overline{R} := R/I$. Let $\overline{b} = b + I$. Then $0 \neq \overline{b} \in \overline{R}$ and $\overline{b}^2 = 0$, so the ring $\overline{R}$ has a nonzero nilpotent. Hence, by Lemma 3.4, $\overline{R}$ is not an sfg-ring. However, by Lemma 3.2, this is an sfg-ring. Thus, we have a contradiction. \hfill \square
Lemma 3.6. The Jacobson radical $J(R)$ is equal to zero.

Proof. Put $J := J(R)$. It follows from Lemma 3.1 that $J$ is a finitely generated $R$-module. If $b \in J$ then, by Lemma 3.5, $b = ub^2$ for some $u \in R$, and so, $b \in J^2$. Thus, we have the equality $J^2 = J$. Now, by Nakayama’s Lemma, $J = 0$. □

Lemma 3.7. If $R$ is local, then $R$ is a field.

Proof. Assume that $R$ is local and $M$ is the unique maximal ideal of $R$. Then $M$ is the Jacobson radical of $R$. It follows from Lemma 3.6 that $M = 0$. Thus $R$ is a field. □

Lemma 3.8. Every prime ideal of $R$ is maximal.

Proof. Let $P$ be a prime ideal of $R$ and suppose $P$ is not maximal. Then there exists a maximal ideal $M$ such that $P \subset M$ and $M \neq P$. Let $b \in M \setminus P$. It follows from Lemma 3.5 that $b = ub^2$ for some $u \in R$. Then
$$b(1 - ub) = 0 \in P.$$ But $b \notin P$, so $1 - ub \in P \subset M$. Hence, $b \in M$ and $1 - ub \in M$. This implies that $1 \in M$, that is, $M = R$. However $M \neq R$, so we have a contradiction. □

Lemma 3.9. $R$ is Artinian.

Proof. We already know by Lemma 3.1 that $R$ is Noetherian. Moreover we know, by Lemma 3.8 that the Krull dimension of $R$ is equal to 0. Using a basic fact of commutative algebra (see for example [2] or [3] 99) we deduce that $R$ is Artinian. □

Now we are ready to prove the mentioned proposition which is the main result of this section.

Proposition 3.10. Every sfg-ring is a finite product of fields.

Proof. Let $R$ be an sfg-ring. We already know (by Lemma 3.9) that $R$ is Artinian. It is known (see for example [2] or [3]) that every Artinian ring is a finite product of some local Artinian rings. Hence,
$$R = R_1 \times R_2 \times \cdots \times R_s,$$ where $R_1, \ldots, R_s$ are local Artinian rings. Since all projections $\pi_j : R \to R_j$ (for $j = 1, \ldots, s$) are surjections of rings, it follows from Lemma 3.2 that all the rings $R_1, \ldots, R_s$ are sfg-rings. Moreover, they are local so, by Lemma 3.7, they are fields. □

According to the above proposition we know that if $R$ is an sfg-ring, then $R$ is a finite product of fields. In the next sections we will prove that the opposite implication is also true.
4. Initial coefficients

Let us assume that $R$ is a ring which is not a field, and $A$ is an $R$-subalgebra of the $R$-algebra $R[x]$. Let us denote by $\mathcal{W}_A$ the set of all nonzero initial coefficients of polynomials of positive degree belonging to $A$. Note three lemmas concerning this set.

Lemma 4.1. Let $a \in \mathcal{W}_A$. Then the polynomial $ax$ is integral over $A$.

Proof. There exists a polynomial $f(x) = ax^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0 \in A$, with $n \geq 1$ and $r_0, \ldots, r_{n-1} \in R$. Let $g(x) = a^{n-1}f(x)$. Then

$$g(x) = (ax)^n + r_{n-1}(ax)^{n-1} + ar_{n-2}(ax)^{n-2} + \cdots + r_1a^{n-2}(ax) + r_0a^{n-1}$$

is also a polynomial belonging to $A$. Consider the polynomial

$$H(t) = t^n + r_{n-1}t^{n-1} + ar_{n-2}t^{n-2} + \cdots + r_1a^{n-2}t + r_0a^{n-1} - g(x).$$

It is a monic polynomial in the variable $t$ and all its coefficients are in $A$. Since $H(ax) = g(x) - g(x) = 0$, the element $ax$ is integral over $A$. \qed

Lemma 4.2. If $R$ is Noetherian and $\mathcal{W}_A$ contains an invertible element, then the algebra $A$ is finitely generated over $R$.

Proof. Let $a \in \mathcal{W}_A$ be invertible in $R$. Then, by Lemma 4.1, the variable $x$ is integral over $A$ and this means that the ring $R[x]$ is integral over $A$. Hence, by Lemma 1.3, the algebra $A$ is finitely generated over $R$. \qed

Lemma 4.3. Let $a, r \in R$. If $a \in \mathcal{W}_A$ and $ra \neq 0$, then $ra \in \mathcal{W}_A$.

Proof. Assume that $f = ax^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in A$ with $n \geq 1$. Then $rf$ is a polynomial belonging to $A$ and the initial coefficient equals $ra \neq 0$. Hence, $ra \in \mathcal{W}_A$. \qed

Consider for example the ring $\mathbb{Z}_6$. Using the above lemmas we will show that $\mathbb{Z}_6$ is an sfq-ring. Let $R = \mathbb{Z}_6$, and let $A \subset R[x]$ be an $R$-subalgebra. We need to show that $A$ is finitely generated over $R$. It is clear if $\mathcal{W}_A = \emptyset$, because in this case $A = R$. If $\mathcal{W}_A$ contains an invertible element of $R$ (in our case 1 or 5) then, by Lemma 4.2, it is also clear.

Let us assume that $\mathcal{W}_A \subset \{2,3,4\}$. Since $2 \cdot 2 = 4$ and $2 \cdot 4 = 2$ in $\mathbb{Z}_6$, we have $4 \in \mathcal{W}_A \iff 2 \in \mathcal{W}_A$. If $3 \in \mathcal{W}_A$ and $4 \in \mathcal{W}_A$ then, by Lemma 4.1, the polynomials $4x$ and $3x$ are integral over $A$, and then $R[x]$ is integral over $A$, because $x = 4x - 3x$, and in this case, by Lemma 1.3, the algebra $A$ is finitely generated over $R$.

Assume that $\mathcal{W}_A = \{2,4\}$, and let $f(x) = 4x^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0 \in A$ where $n \geq 1$ and $r_0, \ldots, r_{n-1} \in \mathbb{Z}_6$. Since $r_0 = r_0 \cdot 1 \in A$, we may assume that $r_0 = 0$. The polynomial $3f(x)$ also belongs to $A$. Hence, $3r_{n-1}x^{n-1} + \cdots + 3r_1x \in A$. 
Suppose that for some \( j \in \{1, \ldots, n - 1\} \) we have \( 3r_j \neq 0 \). Let us take the maximal \( j \). Then \( 3r_j \in W_A = \{2,4\} \), so \( r_j = 0, 2 \) or 4 and in every case we have a contradiction, because \( 3r_j \neq 0 \). Therefore, all the elements \( 3r_1, \ldots, 3r_{n-1} \) are zeros. This means that \( r_i = 4b_i \) with \( b_i \in \mathbb{Z}_6 \), for all \( i = 1, \ldots, n - 1 \). Observe that 4 is an idempotent in \( \mathbb{Z}_6 \). We have \( 4 = 4m \) for every positive integer \( m \). Hence,

\[
    f(x) = 4x^n + 4b_{n-1}x^{n-1} + 4b_{n-2}x^{n-2} + \cdots + 4b_1x \\
    = (4x)^n + b_{n-1}(4x)^{n-1} + \cdots + b_1(4x)^1
\]

and hence, \( A \) is a \( \mathbb{Z}_6 \)-subalgebra of the \( \mathbb{Z}_6 \)-algebra \( \mathbb{Z}_6[4x] \). In this case \( 4 \in W_A \) so, by Lemma 4.1, the monomial \( 4x \) is integral over \( A \) and so, the ring \( \mathbb{Z}_6[4x] \) is integral over \( A \). Therefore, by Lemma 1.3, the algebra \( A \) is finitely generated over \( R = \mathbb{Z}_6 \).

Now let us assume that \( W_A = \{3\} \). In this case we use a similar way, as in the previous case. We show that \( A \) is a subalgebra of \( \mathbb{Z}_6 \)-algebra \( \mathbb{Z}_6[3x] \) and, using again Lemma 1.3, we see that \( A \) is finitely generated over \( \mathbb{Z}_6 \). Therefore we proved that \( \mathbb{Z}_6 \) is an sfg-ring.

5. Finite products of fields

In this section we prove that every finite product of fields is an sfg-ring. Throughout this section

\[
R = K_1 \times K_2 \times \cdots \times K_n,
\]

where \( K_1, \ldots, K_n \) are fields. It is clear that the ring \( R \) is Noetherian, and even Artinian. Let \( A \) be an \( R \)-subalgebra of \( R[\times] \). We will show that \( A \) is finitely generated over \( R \). We know, by Theorem 2.1, that it is true for \( n = 1 \). Now we assume that \( n \geq 2 \).

Let us fix the following notations:

\[
N = \{1, 2, \ldots, n\}; \\
e_1 = (1, 0, \ldots, 0), \ e_2 = (0, 1, \ldots, 0), \ \ldots, \ e_n = (0, 0, \ldots, 1); \\
I = \{i \in N; e_i \in W_A\}; \\
J = N \setminus I; \\
\varepsilon = \sum_{i \in I} e_i.
\]

Observe that if \( I = \emptyset \), then \( A = R \) and nothing to prove. We know, by Lemma 4.1, that if \( i \in I \), then \( e_ix \) is an integral element over \( A \). If \( I = N \), then the variable \( x \) is integral over \( A \), because \( x = (1, 1, \ldots, 1)x = \sum_{i=1}^n e_ix \), and in this case, by Lemma 1.3, the algebra \( A \) is finitely generated over \( R \). Hence, we will assume that \( I \neq \emptyset \) and \( I \neq N \). Without loss of generality we may assume that

\[
I = \{1, 2, \ldots, s\}, \quad J = \{s+1, \ldots, n\}, \quad \text{where} \quad 1 \leq s < n,
\]

and \( \varepsilon = e_1 + \cdots + e_s \). Note two simple lemmas. The first one is obvious.
Lemma 5.1. Let \( u \) be an element of \( R \) such that \( ue_j = 0 \) for all \( j \in J \). Then \( u = \varepsilon u \).

Lemma 5.2. Let \( u \in R \). If \( u \in W_A \), then \( u = \varepsilon u \).

Proof. Let \( u = (u_1, \ldots, u_n) \) and assume that \( u \in W_A \). Suppose there exists \( j \in J \) such that \( ue_j \neq 0 \). Then \( u_j \) is a nonzero element of the field \( K_j \), and \( vu = e_j \), where \( v = (0, \ldots, 0, u_{j}^{-1}, 0, \ldots, 0) \). Hence, \( e_j = v \cdot ue_j \) and so, by Lemma 4.3, the element \( e_j \) belongs to \( W_A \). But it is a contradiction, because \( j \in J = N \setminus I \). Therefore, \( ue_j = 0 \) for all \( j \in J \) and so, by Lemma 5.1, we have \( u = \varepsilon u \). \( \square \)

Now consider the \( R \)-subalgebra \( B \) of \( R[x] \), defined by
\[
B = R[e_1x, e_2x, \ldots, e_sx].
\]
We will prove that \( A \subseteq B \), that is, that \( B \) is a subalgebra of \( A \).

Let \( f \) be an arbitrary element of \( A \). If \( \deg f = 0 \), then obviously \( f \in B \). Assume that \( \deg f \geq 1 \) and \( u \in R \) is the initial coefficient of \( f \). Since \( R \subseteq A \), we may assume that the constant term of \( f \) is equal to zero. Then we have
\[
f = ux^n + d_1x^{n_1} + d_2x^{n_2} + \cdots + d_px^{n_p},
\]
where \( d_1, \ldots, d_p \) are nonzero elements of \( R \), and \( n > n_1 > n_2 > \cdots > n_p \geq 1 \). It follows from Lemma 5.2 that \( u = \varepsilon u \).

Let \( j \in J \). Then \( ue_j = u(\varepsilon e_j) = u0 = 0 \) and then
\[
e_jf = e_jd_1x^{n_1} + e_jd_2x^{n_2} + \cdots + e_jd_px^{n_p} \in A.
\]
Suppose \( e_jd_q \neq 0 \) for some \( q \in \{1, \ldots, p\} \). Let us take the minimal \( q \). Then \( 0 \neq e_jd_q \in W_A \). Put \( d_q = (c_1, \ldots, c_n) \) with \( c_i \in K_i \) for all \( i = 1, \ldots, n \). Since \( e_jd_q \neq 0 \), we have \( e_j \neq 0 \) and so, \( vd_q = e_j \), where \( v = (0, \ldots, 0, e_j^{-1}, 0, \ldots, 0) \). This implies that \( e_j = v(e_jd_q) \in W_A \). But \( e_j \notin W_A \), because \( j \in J = N \setminus I \). Hence, we have a contradiction.

Therefore, all the elements \( e_jd_1, \ldots, e_jd_p \) are zeros, and such situation is for all \( j \in J \). This means, by Lemma 5.1, that \( d_1 = \varepsilon d_1 \), \( \ldots, d_p = \varepsilon d_p \). Observe that the element \( \varepsilon \) is an idempotent of \( R \), so \( \varepsilon = \varepsilon^m \) for \( m \geq 1 \). Hence,
\[
f = u\varepsilon^n + d_1\varepsilon x^{n_1} + d_2\varepsilon x^{n_2} + \cdots + d_p\varepsilon x^{n_p} = u\varepsilon^n + d_1\varepsilon x^{n_1} + d_2\varepsilon x^{n_2} + \cdots + d_p\varepsilon x^{n_p} = \varepsilon\varepsilon^n + d_1\varepsilon x^{n_1} + d_2\varepsilon x^{n_2} + \cdots + d_p\varepsilon x^{n_p} = u(\varepsilon x)^n + d_1(\varepsilon x)^{n_1} + d_2(\varepsilon x)^{n_2} + \cdots + d_p(\varepsilon x)^{n_p},
\]
and hence, the polynomial \( f \) belongs to the ring \( R[\varepsilon x] \). But
\[
R[\varepsilon x] \subseteq R[e_1x, e_2x, \ldots, e_sx] = B,
\]
so \( f \in B \). Thus, we proved that \( A \) is an \( R \)-subalgebra of \( B \). Let us recall that all the monomials \( e_1x, \ldots, e_sx \) are integral over \( A \). Hence, the ring \( B \) is integral over \( A \). It follows from Lemma 1.3 that \( A \) is finitely generated over \( R \). Therefore, we proved the following proposition.
**Proposition 5.3.** Every finite product of fields is an sfg-ring.

Immediately from this proposition and Proposition 3.10 we obtain the following main result of this article.

**Theorem 5.4.** A ring $R$ is an sfg-ring if and only if $R$ is a finite product of fields.

Now, by this theorem and the Chinese Remainder Theorem, we have

**Corollary 5.5.** The ring $\mathbb{Z}_m$ is an sfg-ring if and only if $m$ is square-free.

**References**


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