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TWO REMARKS ABOUT SURFACES

It is shown that among continuous functions defined on the unit square and non-decreasing with respect to each variable separately there is neither a function with the greatest nor a function with the smallest surface area.

We shall introduce the following denotations. Let

$\mathcal{F}_1 = \{f : [0,1] \to [0,1] : f$ is a continuous, non-decreasing function, $f(0) = 0$ and $f(1) = 1\}$

$\mathcal{F}_2 = \{z : [0,1]^2 \to [0,1] : z$ is a continuous function, $z(0,0) = 0$ and $z(1,1) = 1$, $z(x,y)$ is non-decreasing as a function of one variable$y$ for each $x \in [0,1]$, $z(x,y)$ is non-decreasing as a function of one variable x for each $y \in [0,1]\}$.

We shall denote by $L_1(f)$ the length of a curve $f \in \mathcal{F}_1$ and by $L_2(z)$ the area of the surface $z \in \mathcal{F}_2$. Respectively $|A|_1$ will denote the measure of a linear set $A \subset [0,1]$, $|B|_2$ the measure of a planar set $B \subset [0,1] \times [0,1]$. It can be easily shown that $\mathcal{F}_1$ with the metric

$$\rho(f_1, f_2) = \sup_{x \in [0,1]} |f_1(x) - f_2(x)|$$

is a complete space. Also we can prove that $\mathcal{F}_2$ with the metric

$$\rho(z_1, z_2) = \sup_{(x,y) \in [0,1]^2} |z_1(x,y) - z_2(x,y)|$$

is a complete space. It is known that

$$\sup_{f \in \mathcal{F}_1} L_1(f) = 2, \ \inf_{f \in \mathcal{F}_1} L_1(f) = \sqrt{2}$$

where least upper bound is reached by the most of functions since $\{f \in \mathcal{F}_1 : L_1(f) = 2\}$ is a residual set in $\mathcal{F}_1$ and greatest lower bound is reached for one function $f(x) = x$.

We shall recall some definitions concerning the surface areas. We say that the continuous function $P : [0,1]^2 \to [0,1]$ defines a polyhedral if there exists a subdivision of $[0,1]^2$ into a finite number of non-overlapping triangles $T_1, T_2, ..., T_n$ such that

$$P(x,y) = a_i x + b_i y + c_i \text{ for } (x,y) \in T_i, i = 1, 2, ..., n$$

where a_i, b_i, c_i are constant coefficients for a fixed triangle T_i. The sum of the areas of the faces in the sense of elementary geometry i.e. the number

$$\sum_i |T_i|_2 (a_i^2 + b_i^2 + 1)^{\frac{1}{2}} = \iint_{[0,1]^2} \left(\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2 + 1 \right)^{\frac{1}{2}} dxdy$$

is called an elementary area. Let $F : [0,1]^2 \to [0,1]$ be any continuous function defining a surface. By the surface area $L_2(F)$ we shall mean the lower limit of elementary areas of polyhedrals uniformly convergent to F, i.e. the lower bound of all numbers s such that for any $\epsilon > 0$ there exists a polyhedral $P : [0,1]^2 \to [0,1]$ such that for any $(x,y) \in [0,1]^2$ $|P(x,y) - F(x,y)| < \epsilon$ and $L_2(p) \leq s$.

The variation of functions of two real variables in the sense of Tonelli is defined in the following way :

Let $F : [0,1]^2 \to [0,1]$ be any continuous function. For any $x \in [0,1]$ let $w_1(F,x,[0,1])$ be the total variation of $F(x,y), 0 \leq y \leq 1$ as a function of y only. For any $y \in [0,1]$ let $w_2(F,y,[0,1])$ be the total variation of $F(x,y), 0 \leq x \leq 1$ as a function of x only. Because of the continuity of $F(x,y)$ non-negative functions $w_1(F,x,[0,1])$, $w_2(F,y,[0,1])$ are lower semicontinuous functions of variables x and y respectively. When integrals

$$\int_0^1 w_1(F,x,[0,1])dx \text{ and } \int_0^1 w_2(F,y,[0,1])dy$$

are finite, function F is said to be of bounded variation in $[0,1]^2$ in the sense of Tonelli (B.V.T.). Hence we have immediately that any function of bounded variation of two variables (x,y) is a function of bounded variation as a function of x for almost all y, and it is a function of bounded variation as a function of y for almost all x.

Obviously for $z \in \mathcal{F}_2$ we have

$$w_1(F,x,[0,1]) \leq 1 \text{ for any } x \in [0,1]$$

and

$$w_2(F,y,[0,1]) \leq 1 \text{ for any } y \in [0,1]$$

thus

$$\int_0^1 w_1(F,x,[0,1])dx \leq 1$$

and

$$\int_0^1 w_2(F,y,[0,1])dy \leq 1.$$

By Tonelli theorem (1926) [see Cesari, p. 4] we have that if $z \in \mathcal{F}_2$ then

$$\|[0,1]^2|_2 \leq L_2(z) \leq \|[0,1]^2|_2 + \int_0^1 w_1(F,x,[0,1])dx + \int_0^1 w_2(F,y,[0,1])dy \leq 3.$$

Hence

$$\sup_{z\in\mathcal{F}_2} L_2(z) \le 3 \text{ and } \inf_{z\in\mathcal{F}_2} L_2(z) \ge 1.$$

Theorem 1.

$$\sup_{z\in\mathcal{F}_2} L_2(z) = 3$$

Proof. Let

$$z_n(x,y) = \begin{cases} 0 & \text{for } \begin{cases} 0 \le x \le 1-\frac{1}{n} \\ 0 \le y \le 1-\frac{1}{n} \end{cases} \\ n\,x + (1-n) & \text{for } \begin{cases} 1-\frac{1}{n} \le x \le 1 \\ 0 \le y \le x \end{cases} \\ n\,y + (1-n) & \text{for } \begin{cases} 1-\frac{1}{n} \le y \le 1 \\ 0 \le x \le y \end{cases} \end{cases}$$

for any $n \in \mathcal{N} - \{1\}$.

Then the surface area of z_n is equal to

$$L_2(z_n) = \left(1-\frac{1}{n}\right)^2 + 2\frac{1+1-\frac{1}{n}}{2}\sqrt{1+\frac{1}{n^2}}$$
$$= \left(1-\frac{1}{n}\right)^2 + \left(2-\frac{1}{n}\right)\sqrt{1+\frac{1}{n^2}},$$

so

$$\lim_{n\to\infty} L(z_n) = 3.$$

Hence we have immediately $\sup_{z\in\mathcal{F}_2} L_2(z) = 3$.

Theorem 2. *If $z \in \mathcal{F}_2$ then $L_2(z) < 3$.*

Proof. Let $z \in \mathcal{F}_2$. Then obviously

$$0 \le z(0,0) \le z(1,0) \le z(1,1) = 1.$$

At least one of the above inequalities must be proper. Suppose it is the first one. The proof in the other case is analogous. Thus we have $0 = z(0,0) < z(1,0)$. By the property of Darboux of the function $z(x,0)$ we have easily that there is a point $x_o \in [0,1]$ such that $0 < z(x_o,0) < 1$. Then we have for $x \in [x_o,1]$

$$z(x_o,0) \leq z(x,0).$$

Simultanously, as $w_1(z,x,[0,1]) = z(x,1) - z(x,0)$ so for $x \in [x_o,1]$ the inequality

$$w_1(z,x,[0,1]) \leq 1 - z(x_o,0) < 1$$

holds. Hence

$$\begin{aligned}\int_0^1 w_1(z,x,[0,1])dx &= \int_0^{x_o} w_1(z,x,[0,1])dx \\ &\quad + \int_{x_o}^1 w_1(z,x,[0,1])dx \\ &\leq x_o \cdot 1 + (1-x_o)(1-z(x_o,0)) < 1\end{aligned}$$

which immediately results in the inequality $L_2(z) < 3$.

Theorem 3.

$$\inf_{z\in\mathcal{F}_2} L_2(z) = 1.$$

Proof. Let

$$z_n(x,y) = \begin{cases} 0 & \text{for } \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \\ y \leq 2 - \frac{1}{n} - x \end{cases} \\ n\,x + n\,y + 1 - 2n & \text{for } \begin{cases} 1 - \frac{1}{n} \leq x \leq 1 \\ 2 - \frac{1}{n} - x \leq y \leq 1 \end{cases} \end{cases}$$

for any $n \in \mathcal{N} - \{1\}$.

$$L(z_n) = 1 - \frac{1}{2n^2} + \sqrt{\frac{2n^2+1}{n^2}}.$$

Hence $\lim_{n\to\infty} L(z_n) = 1$ so $\inf_{z\in\mathcal{F}_2} L_2(z) = 1$.

Theorem 4. *If $z \in \mathcal{F}_2$ then $L_2(z) > 1$.*

Proof. Suppose that $L_2(z) = 1$. Then (see Saks p. 181, Theorem 8.1, a,b,c) as $z \in \mathcal{F}_2$ so

$$1 = L_2(z) \geq \int_0^1 \int_0^1 \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 \right)^{\frac{1}{2}} dxdy \geq 1.$$

Thus the equalities hold. In particular from

$$\int_0^1 \int_0^1 \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 \right)^{\frac{1}{2}} dxdy = 1.$$

it follows that the subintegral function is almost everywhere equal to 1. Hence

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 0 \ a.e.$$

Thus $\frac{\partial z}{\partial x} = 0$ *a.e.* Since

$$L_2(z) = \int_0^1 \int_0^1 \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 \right)^{\frac{1}{2}} dxdy$$

so $z(x, y)$ is absolutely continuous in the sense of Tonelli (shortly A.C.T.) so for almost all $y_o \in [0, 1]$, $z(x, y_o)$ is absolutely continuous as a function of the variable y. Let

$$E_1 = \left\{ (x, y) \in [0, 1]^2 : \frac{\partial z}{\partial x}(x, y) = 0 \right\}$$
$$E_2 = \left\{ (x, y) \in [0, 1]^2 : \frac{\partial z}{\partial y}(x, y) = 0 \right\}.$$

We know that $|E_1|_2 = 1$ and $|E_2|_2 = 1$. Let

$$A_1 = \{ y \in [0, 1] : |(E_1)^y|_1 = 1 \}$$

where

$$(E_1)^y = \{x \in [0,1] : x, y) \in E_1\}$$

and

$$A_2 = \{x \in [0,1] : |(E_1)_x|_1 = 1\}$$

where

$$(E_2)_x = \{y \in [0,1] : (x,y) \in E_2\}.$$

Obviously $|A_1|_1 = 1$ and $|A_2|_1 = 1$. Let

$$B_1 = \{y \in [0,1] : z(x,y) \text{ is a.c. as a function of } x\}$$
$$B_2 = \{x \in [0,1] : z(x,y) \text{ is a.c. as a function of } y\}.$$

We know by A.C.T. that $|B_1|_1 = |B_2|_1 = 1$. Let $y \in A_1 \cap B_1$. Then $z(x, y_o)$ is a.c. since $y_o \in B_1$ and for almost all $x_o \in [0,1]$ $\frac{\partial z}{\partial x}(x, y_o) = 0$ since $y_o \in A_1$. Hence

$$(1) \qquad z(1, y_o) - z(0, y_o) = 0.$$

Let $x_o \in A_2 \cap B_2$. Then $z(x_o, y)$ is a.c. since $x_o \in B_2$ and for almost all $y \in [0,1]$ $\frac{\partial z}{\partial y}(x_o, y) = 0$ since $x_o \in A_2$. Hence

$$(2) \qquad z(x_o, 1) - z(x_o, 0) = 0$$

By (1) we have

$$(3) \qquad z(x, y_o) = z(0, y_o) \text{ for all } x \in [0,1]$$

and by (2) we have

$$(4) \qquad z(x_o, y) = z(1, y) \text{ for all } y \in [0,1].$$

Thus by (3) and (4) we have $z(0, y_o) = z(x_o, y_o) = z(x_o, 1)$. Thus $z(0, y_o) = z(x_o, 1)$.

Since $|A_1 \cap B_1|_1 = |A_2 \cap B_2|_1 = 1$ thus $A_1 \cap B_1$ is dense in $[0,1]$ and $A_1 \cap B_1$ is dense in $[0,1]$. We shall take the sequence $\{x_n\}_{n \in \mathcal{N}} \subset A_2 \cap B_2$ and such that x_n tends to 1 increasingly and the sequence $\{y_n\}_{n \in \mathcal{N}} \subset A_1 \cap B_1$ and such that y_n tends to 0 decreasingly. We have $z(0, y_n) = z(x_n, 1)$ for any $n \in \mathcal{N}$. By the continuity of the function $z(x,y)$ we have $z(0,0) = z(1,1)$, and this contradicts the fact that $z(0,0) = 0$ and $z(1,1) = 1$.

References

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[2] L. Cesari, *Surface area*, New York, 1965.

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DWIE UWAGI O POWIERZCHNIACH

Pokazujemy, że wśród funkcji ciągłych określonych na kwadracie jednostkowym i niemalejących ze względu na każdą zmienną nie istnieje ani funkcja o największym, ani o najmniejszym polu powierzchni.

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