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ON THE $*$-HOLONOMY OF THE INVERSE IMAGE OF A STEFAN FOLIATION

Let $\mathcal{F}'$ be a Stefan foliation ([3]) of a manifold M' and let f: M → M' be a smooth mapping transverse to $\mathcal{F}'$ ([4]). We show that there exists a natural homomorphism of the $*$-holonomy groupoid ([2]) of $f^{-1}(\mathcal{F}')$ into the $*$-holonomy groupoid of $\mathcal{F}$.

1. INTRODUCTION

The notion of a Stefan foliation was introduced in ([3]). In 1986 V e r E e c k e [4] showed that if f: M → M' is a smooth mapping transverse to a Stefan foliation $\mathcal{F}'$ on M', then the decomposition $f^{-1}(\mathcal{F}')$ of M is a Stefan foliation. In section 2 of the present paper we prove this fact in terms of distinguished charts.

By the $*$-holonomy we mean the same object which was defined in [2] as holonomy. This new designation is introduced in order to distinguish it from the Ehresmann holonomy ([1], [4]). In section 2 we recall the definition of a $*$-holonomy.

The main theorem of our paper, given in section 4, is the following:

There exists a natural homomorphism of the $*$-holonomy groupoid of $f^{-1}(\mathcal{F}')$ into the $*$-holonomy groupoid of $\mathcal{F}'$.

The analogous result for the Ehresmann holonomy was proved in [4].

2. STEFAN FOLIATIONS AND A $*$-HOLONOMY

Let M be a paracompact and connected manifold of class C^∞. Let m = dim M. In [3], Stefan gave the following definition:

(2.1) A decomposition $\mathcal{F}$ of M into connected immersed submanifolds is called a foliation if, for each $x \in M$, there exists a chart φ of M which satisfies the conditions:

(i) $\varphi: D_\varphi \to U_\varphi \times W_\varphi$ is a diffeomorphism onto $U_\varphi \times W_\varphi$ where U_φ and W_φ are connected neighbourhoods of 0 in $\mathbb{R}^k$ and $\mathbb{R}^{m-k}$, respectively (k being the dimension of that element of $\mathcal{F}$, denoted by L_x, which contains x);

(ii) $\varphi(x) = (0,0)$;

(iii) for each $L \in \mathcal{F}$, the equality $\varphi(D_\varphi \cap L) = U_\varphi \times \ell$ with $\ell = \{w \in W_\varphi;\ \varphi^{-1}(0, w) \in L\}$ holds.

The chart φ satisfying the above conditions is called a distinguished chart of $\mathcal{F}$ around x. The elements of $\mathcal{F}$ are said to be leaves of the foliation $\mathcal{F}$. If $L \in \mathcal{F}$ then each connected component of $L \cap D_\varphi$ is called a plaque of φ in L. In particular, $P_\varphi := \varphi^{-1}(U_\varphi \times \{0\})$ is a plaque which is said to be central.

(2.2) Let φ be a distinguished chart around x and let G be an open neighbourhood of 0 in W_φ. It is easy to check that the mapping

$$\varphi_G := \varphi \mid \varphi^{-1}(U_\varphi \times G)$$

is a distinguished chart of $\mathcal{F}$ around x, too.

Let G be an arbitrary neighbourhood of 0 in W_φ. Define in G the equivalence relation $\sim\varphi_G$ in the following way: $w \sim \varphi_G w'$ if and only if $\varphi^{-1}(0, w)$ and $\varphi^{-1}(0, w')$ are contained in the same plaque of φ_G. In particular, we write $\sim_\varphi$ if $G = W_\varphi$.

(2.3) Let x and y be points of the same leaf $L \in \mathcal{F}$ and let φ and ψ be distinguished charts of $\mathcal{F}$ around x and y, respectively. As in [2], denote by $\mathcal{A}_{\varphi,\psi}$ the set of all diffeomorphisms f of an open neighourhood G of 0 in W_φ into W_ψ, such that $f(0) = 0$ and f, f^{-1} are compatible with the relations $\sim\varphi_G$ and $\sim\psi_{f(G)}$.

Define in $\mathcal{A}_{\varphi,\psi}$ the relation $\equiv$ in the following way: if $f_i: G_i \to H_i$ (i = 0, 1) are elements of $\mathcal{A}_{\varphi,\psi}$ then $f_0 \equiv f_1$ if and only if there exists a family $\{\hat{f}_t : t \in \langle 0, 1\rangle\}$ of mappings satisfying the conditions:

(i) there exists a neighbourhood $\hat{G}$ of 0 in $G_0 \cap G_1$ such that, for each $t \in \langle 0, 1\rangle$, $\hat{f}_t$ is an immersion of $\hat{G}$ into $H_0 \cap H_1$;

(ii) the mapping $\langle 0, 1\rangle \times \hat{G} \ni (t, w) \mapsto \hat{f}_t(w) \in H_0 \cap H_1$ is continuous;

(iii) $\hat{f}_0 = f_0|\hat{G}$, $\hat{f}_1 = f_1|\hat{G}$;

(iv) for each $t \in \langle 0, 1\rangle$, the mapping $\hat{f}_t$ is compatible with the relations $\sim_{\varphi_{\hat{G}}}$ and $\sim_{\psi}$;

(v) for each $w \in \hat{G}$, the curve $\langle 0, 1\rangle \ni t \mapsto \hat{f}_t(x) \in H_0 \cap H_1$ takes its values in an equivalence class of $\sim_{\psi}$.

It was shown in [2] that $\equiv$ is an equivalence relation.

(2.4) In [2], the following fact was proved:

PROPOSITION. If $f_i \in \mathcal{A}_{\varphi,\psi}$, $g_i \in \mathcal{A}_{\psi,\chi}$ $(i = 0, 1)$ and $f_0 \equiv f_1$, $g_0 \equiv g_1$, then $g_0 \circ f_0 \equiv g_1 \circ f_1$ in $\mathcal{A}_{\varphi,\chi}$.

(2.5) Let φ be a distinguished chart of $\mathcal{F}$ around x and let $\gamma : \langle 0, 1\rangle \to L_x$ be a continuous curve. For $t \in \langle 0, 1\rangle$, a pair (φ, t) is called a link on γ if $\gamma(t) \in P_\varphi$.

If (φ, t), (ψ, v) are two links on γ, then they are said to overlap if

$$\gamma^{-1}(D_\varphi)_t \cap \gamma^{-1}(D_\psi)_v \neq \emptyset$$

where, for an open set $V \subset M$, $\gamma^{-1}(V)_t$ denotes the connected component of $\gamma^{-1}(V)$ containing t.

A finite sequence $\mathcal{C} = (\varphi_0, t_0; \varphi_1, t_1; \ldots; \varphi_r, t_r; \psi, t_{r+1})$ $(t_0 = 0, t_{r+1} = 1)$ of links on γ is said to be a chain of charts along γ if, for each $i \in \{0, 1, \ldots, r\}$, (φ_i, t_i), (φ_{i+1}, t_{i+1}) are overlapping links $(\varphi_{r+1} = \psi)$.

(2.6) Let (φ, t), (ψ, v) be a pair of overlapping links on γ. Choose a point x belonging to that connected component of $P_\varphi \cap P_\psi$ which contains a connected set $\gamma(\gamma^{-1}(D_\varphi)_t \cap \gamma^{-1}(D_\psi)_v)$. Then there exists ([2]) an open neighbourhood G of 0 in W_φ such that the mapping

$$f_{\varphi,\psi;x}(w) = pr_2\,\psi\,\varphi^{-1}(pr_1\,\varphi(x), w)$$

is defined in G and is an element of $\mathcal{A}_{\varphi,\psi}$.

Let $L \in \mathcal{F}$ and let $\gamma : \langle 0, 1\rangle \to L$ be a continuous curve. Take arbitrary distinguished charts φ and ψ around $\gamma(0)$ and $\gamma(1)$, respectively. Let $\mathcal{C} = (\varphi, t_0; \varphi_1, t_1; \ldots; \varphi_r, t_r; \psi, t_{r+1})$ ($t_0 = 0$, $t_{r+1} = 1$) be an arbitrary chain of charts along γ. Choose a point x_i ($i = 0, 1, \ldots, r$) belonging to the connected component of $P_{\varphi_i} \cap P_{\varphi_{i+1}}$ ($\varphi_{r+1} = \psi$) containing

$$\gamma(\gamma^{-1}(D_{\varphi_i})_{t_i} \cap \gamma^{-1}(D_{\varphi_{i+1}})_{t_{i+1}}).$$

Define a mapping

$$f_{\mathcal{C}} := f_{\varphi_r, \varphi_{r+1}; x_r} \circ \ldots \circ f_{\varphi_1, \varphi_2; x_1} \circ f_{\varphi_0, \varphi_1; x_0}.$$

It was shown in [2] that $f_{\mathcal{C}} \in \mathcal{A}_{\varphi,\psi}$ and its equivalence class $[f_{\mathcal{C}}]$ relative to the relation $\equiv$ depends only on the homotopy class of the curve γ. The equivalence class $[f_{\mathcal{C}}]$ denoted also by $[f_{\gamma,\varphi,\psi}]$ is called a $*$-holonomy of L along γ.

(2.7) Let Λ be the family of all triplets (x, γ, y) where x, y are points of the same leaf L and $\gamma : \langle 0, 1\rangle \to L$ is a curve joining x to y. The elements (x, γ, y) and (x', γ', y') of Λ are identified (this relation is denoted by $\sim$) if and only if $x = x'$, $y = y'$ and $[f_{\gamma,\varphi,\psi}] = [f_{\gamma';\varphi,\psi}]$ for arbitrary distinguished charts φ and ψ around x and y, respectively. A class of (x, γ, y) of this equivalence relation is denoted by $[(x, \gamma, y)]$. Define the mappings $\alpha: \Lambda/\!\sim\, \ni [(x, \gamma, y)] \mapsto x \in M$, $\beta: \Lambda/\!\sim\, \ni [(x, \gamma, y)] \mapsto y \in M$. If $\beta([(x, \gamma, y)]) = \alpha([(x', \gamma', y')])$, then define the multiplication

$$[(x', \gamma', y')] \cdot [(x, \gamma, y)] = [(x, \gamma \cdot \gamma', y')].$$

The definition is correct by (2.4).

It is easy to see that the set $\Lambda/\!\sim$ with α, β and the multiplication is a groupoid over M which is called a $*$-holonomy groupoid of $\mathcal{F}$ and denoted by $*$-Hol($\mathcal{F}$).

(2.8) Let $M_{\mathcal{F}}$ be a topological space on M whose base consists of all plaques. Let $\pi_1(M_{\mathcal{F}})$ be the fundamental groupoid of this space. It is obvious that there exists a natural groupoid homomorphism

$$H_{\mathcal{F}}: \pi_1(M_{\mathcal{F}}) \ni [\gamma] \mapsto [(\gamma(0), \gamma, \gamma(1))] \in *\text{-Hol}(\mathcal{F}).$$

3. THE INVERSE IMAGE OF A STEFAN FOLIATION

Let $\mathcal{F}'$ be a Stefan foliation of an m'-dimensional manifold M', let M be an m-dimensional manifold and $g: M \to M'$ a smooth mapping. We denote leaves of $\mathcal{F}'$ by L', $L'_{g(x)}$, L'_x, etc.

(3.1) We say that g is transverse to $\mathcal{F}'$ if, for each $x \in M$, the equality

$$g_* T_x M + T_{g(x)} L'_{g(x)} = T_{g(x)} M'$$

holds. This is denoted by $g \pitchfork \mathcal{F}'$.

(3.2) It is well known that connected components of $g^{-1}(L')$ for $L' \in \mathcal{F}'$ give a decomposition of M into connected immersed submanifolds. Moreover, the codimension of $g^{-1}(L')$ equals the codimension of L'. Denote this decomposition by $g^{-1}(\mathcal{F}')$ or simply by $\mathcal{F}$.

PROPOSITION. $\mathcal{F}$ is a Stefan foliation.

(3.3) Ver Eecke proved this proposition in [4] but we prove it in a quite different way here.

Proof. The only fact which has to be proved is the existence of distinguished charts of $\mathcal{F}$.

Let $x \in M$ be an arbitrary point. Let $\varphi': D_{\varphi'} \to U_{\varphi'} \times W_{\varphi'}$, be a distinguished chart of $\mathcal{F}$ around $x = g(x)$. Denote by L the element of $\mathcal{F}$ containing x (dim L = k) and by L' the leaf of $\mathcal{F}'$ for which $g(L) \subset L'$ (dim L' = k'). Take a connected and relatively compact neighbourhood $\tilde{P}$ of x in $g^{-1}(P_{\varphi'})$ such that there exists a chart ψ of the submanifold $g^{-1}(L')$, defined in $\tilde{P}$ and satisfying $\psi(x) = 0$. It is easy to see that the mapping $h = pr_2 \circ \varphi' \circ g$ is a submersion in x, thus in an open neighbourhood $\tilde{W}$ of x contained in $g^{-1}(D_{\varphi'})$. Let $\mathcal{F}_1$ be a regular foliation of $\tilde{W}$ induced by h. Define P to be the connected component of

$\tilde{W} \cap \tilde{P}$ containing x. Let $\xi = (W, p)$ be a tubular neighbourhood of P such that $W \subset \tilde{W}$, and $p^{-1}(\tilde{x})$ is connected and transverse to leaves of $\mathcal{F}_1$ for each $\tilde{x} \in P$. The mapping

$$\tilde{\varphi} : W \ni y \mapsto (\psi\, p(y),\ h(y)) \in \mathbb{R}^k \times \mathbb{R}^{m'-k'} = \mathbb{R}^{k'} \times \mathbb{R}^{m-k}$$

is a diffeomorphism on some neighbourhood D_φ of x. One can suppose that $\tilde{\varphi}(D_\varphi)$ is of the form $U_\varphi \times W_\varphi$ where U_φ, W_φ are connected neighbourhoods of 0 in $\mathbb{R}^k$ and $\mathbb{R}^{m-k}$, respectively. Set $\varphi := \tilde{\varphi} \mid D_\varphi$. Note that

(*) $pr_2\varphi = pr_2\varphi' g$

by the definition of φ.

We show that φ is a distinguished chart of $\mathcal{F}$ around x. Conditions (i) and (ii) of definition (2.1) are obviously satisfied. Let $\tilde{L} \in \mathcal{F}$. We prove that

$$\varphi(\tilde{L} \cap D_\varphi) = U_\varphi \times \tilde{\ell}$$

where $\tilde{\ell} = \{w \in W_\varphi;\ \varphi^{-1}(0, w) \in \tilde{L}\}$. Let $(u, w) \in \varphi(\tilde{L} \cap D_\varphi)$. Then there exists $y \in \tilde{L} \cap D_\varphi$ such that $\varphi(y) = (u, w)$. Denote by $\tilde{L}' \in \mathcal{F}'$ the leaf for which $g(\tilde{L}) \subset \tilde{L}'$. Since $pr_2\varphi' g \varphi^{-1}(U_\varphi \times \{w\}) = \{w\}$ by (*) and $pr_2\varphi' g(y) = w$, we have $g\varphi^{-1}(U_\varphi \times \{w\}) \subset \varphi'^{-1}(U_{\varphi'} \times \{w\}) \subset \tilde{L}'$. Thus $\varphi^{-1}(U_\varphi \times \{w\}) \subset g^{-1}g\varphi^{-1}(U_\varphi \times \{w\}) \subset g^{-1}(\tilde{L}')$. The set $\varphi^{-1}(U_\varphi \times \{w\})$ is contained in $\tilde{L}$ since it is connected and contains y. In particular, $\varphi^{-1}(0, w) \in \tilde{L}$, so $w \in \tilde{\ell}$. We have $(u, w) \in U_\varphi \times \tilde{\ell}$.

Conversely, let $(u, w) \in U_\varphi \times \tilde{\ell}$. It is obvious that $\varphi^{-1}(u, w) \in D_\varphi$. We show that $\varphi^{-1}(u, w) \in \tilde{L}$. Since $w \in \tilde{\ell}$, therefore $\varphi^{-1}(0, w) \in \tilde{L}$. Analogously as above we prove that $g\varphi^{-1}(U_\varphi \times \{w\}) \subset \tilde{L}'$. Then the connected set $\varphi^{-1}(U_\varphi \times \{w\})$ containing $\varphi^{-1}(u, w)$ has to be contained in $\tilde{L}$. In particular, $\varphi^{-1}(u, w) \in \tilde{L}$, so $U_\varphi \times \tilde{\ell} \subset \varphi(\tilde{L} \cap D_\varphi)$. □

(3.4) Let $\varphi : D_\varphi \to U_\varphi \times W_\varphi$ be a distinguished chart of $\mathcal{F}$ constructed as above by using the distinguished chart $\varphi' : D_{\varphi'} \to U_{\varphi'} \times W_{\varphi'}$ of $\mathcal{F}'$. We have

PROPOSITION. $W_\varphi \subset W_{\varphi'}$. Equivalence classes of $\sim_\varphi$ are equal to connected components of intersections of W_φ with equivalence classes of $\sim_{\varphi'}$ in $W_{\varphi'}$.

(3.5) P r o o f. The first part of the proposition follows directly from the definition of φ. We show that the second part holds. Let $\tilde{\ell}^{(0)}$ be an equivalence class of $\sim_\varphi$. Then $\varphi^{-1}(U_\varphi \times \tilde{\ell}^{(0)}) \subset \tilde{L}$, so $g\varphi^{-1}(U_\varphi \times \tilde{\ell}^{(0)}) \subset L' \cap D_{\varphi'}$. Since the set $g\varphi^{-1}(U_\varphi \times \tilde{\ell}^{(0)})$ is connected, it is contained in a plaque of φ. Consequently, $\tilde{\ell}^{(0)} = pr_2\, \varphi' g\varphi^{-1}(U_\varphi \times \tilde{\ell}^{(0)}) \subset \tilde{\ell}'^{(0)}$ by (*), where $\tilde{\ell}'^{(0)}$ is a connected component of $\tilde{\ell}' = \{w \in W_{\varphi'};\ \varphi'^{-1}(0, w) \in \tilde{L}'\}$. Thus $\tilde{\ell}^{(0)}$, being connected, is contained in a connected component of $W_\varphi \cap \tilde{\ell}'^{(0)}$.

Conversely, let ℓ' be a connected component of the set $W_\varphi \cap \ell'^{(0)}$ where $\ell'^{(0)}$ is an equivalence class of $\sim_\varphi$. Consider the set $A := \varphi^{-1}(U_\varphi \times \ell')$. It is connected. Therefore $g(A)$ is connected and $pr_2\, \varphi' g(A) = pr_2 \varphi' g \varphi^{-1}(U_\varphi \times \ell') = \ell' \subset \tilde{\ell}'^{(0)} \subset \tilde{\ell}'$ by (*). Thus $g(A) \subset \tilde{L}'$, so $A \subset g^{-1}(\tilde{L}')$. Consequently, A is contained in a leaf of $\mathcal{F}$ since A is connected. Obviously, $A \subset D_\varphi$, so it is contained in a plaque of φ. Then we have $\ell' \subset \tilde{\ell}^{(0)}$. □

4. THE MAIN THEOREM

Let $g: M \to M'$ be a smooth mapping transverse to a Stefan foliation $\mathcal{F}'$ of M'. Then it is well known that there exists a natural groupoid homomorphism

$$G: \pi_1(M_{\mathcal{F}}) \ni [\gamma] \mapsto [g \circ \gamma] \in \pi_1(M'_{\mathcal{F}'})$$

where $\mathcal{F} = g^{-1}(\mathcal{F}')$.

(4.1) THEOREM. There exists a natural groupoid homomorphism $\tilde{G}$: *-Hol($\mathcal{F}$) → *-Hol($\mathcal{F}'$) such that the diagram

$$(**)\qquad \begin{array}{ccc} \pi_1(M_{\mathcal{F}}) & \xrightarrow{\ G\ } & \pi_1(M'_{\mathcal{F}'}) \\ \downarrow H_{\mathcal{F}} & & \downarrow H_{\mathcal{F}'} \\ \text{*-Hol}(\mathcal{F}) & \xrightarrow{\ \tilde{G}\ } & \text{*-Hol}(\mathcal{F}') \end{array}$$

commutes.

(4.2) P r o o f. It is easily seen that if we want diagram (**) to commute, the mapping $\tilde{G}$ has to be defined by the formula:

$$\tilde{G}([(x, \gamma, y)]) = [(g(x), g \circ \gamma, g(y))].$$

We show that this definition is correct. Let (x, γ_0, y), (x, γ_1, y) be triplets from Λ such that $f_{\gamma_0;\varphi,\psi} \equiv f_{\gamma_1;\varphi,\psi}$ in $\mathcal{A}_{\varphi,\psi}$, where φ, Ψ are arbitrarily chosen distinguished charts around x and y, respectively. Note that, for an arbitrary curve γ in L_x, it is possible to choose a chain $\mathcal{C}' = (\varphi', 0; \varphi'_1, t_1; \ldots; \varphi'_r, t_r; \psi', 1)$ along $g \circ \gamma$ such that the charts $\varphi, \varphi_1, \ldots, \varphi_r, \psi$ (defined as in (3.3) from the charts of $\mathcal{C}'$) form a chain $\mathcal{C} = (\varphi, 0; \varphi_1, t_1; \ldots; \varphi_r, t_r; \psi, 1)$ along γ. This can be obtained in the following way: for each $s \in \langle 0, 1\rangle$, there exists a distinguished chart $\varphi'_{(s)}$ of $\mathcal{F}'$ around $g\gamma(s)$. For every chart $\varphi'_{(s)}$, define the distinguished chart $\varphi_{(s)}$ of $\mathcal{F}$ around $\gamma(s)$ as in (3.3). There exists a finite subfamily $\{\varphi, \varphi_1, \ldots, \varphi_r, \psi\}$ of $\{\varphi_{(s)}; s \in \langle 0, 1\rangle\}$ with $\varphi = \varphi_{(0)}$, $\varphi_i = \varphi_{(s_i)}$, $\psi = \varphi_{(1)}$, such that $\mathcal{C} = (\varphi, 0; \varphi_1, t_1; \ldots; \varphi_r, t_r; \psi, 1)$ is a chain along γ, where $t_1, \ldots, t_r$ are suitably chosen parameters from $\langle 0, 1\rangle$. It is obvious that $(\varphi'_{(0)}, 0; \varphi'_{(s_1)}, t_1; \ldots; \varphi'_{(s_r)}, t_r; \varphi'_{(1)}, 1) = (\varphi', 0; \varphi'_1, t_1; \ldots; \varphi'_r, t_r; \psi', 1)$ is a chain along $g \circ \gamma$.

It is clear that the ranges of charts of this chain can be assumed to be convex.

Let $\mathcal{C}_i$, $\mathcal{C}'_i$ $(i = 0, 1)$ be the chains along γ_i and $g \circ \gamma_i$, respectively, constructed as above. By the assumption, $f_{\mathcal{C}_0} \equiv f_{\mathcal{C}_1}$ in $\mathcal{A}_{\varphi',\psi}$. We have to prove that $f_{\mathcal{C}'_0} \equiv f_{\mathcal{C}'_1}$ in $\mathcal{A}_{\varphi',\psi'}$. Note that, by proposition (3.4), the diffeomorphism $f_{\mathcal{C}_i}$ can be considered as an element of $\mathcal{A}_{\varphi',\psi'}$. By the transitivity of $\equiv$, it suffices to show that $f_{\mathcal{C}_i} \equiv f_{\mathcal{C}'_i}$ in $\mathcal{A}_{\varphi',\psi'}$. In view of proposition (2.4), it will be sufficient to prove this last equivalence in the case when the chains $\mathcal{C}_i$ and $\mathcal{C}'_i$ consist of two links. In other words we have to show that

(***) $f_{\varphi,\psi;x} \equiv f_{\varphi',\psi';g(x)}$ in $\mathcal{A}_{\varphi',\psi'}$.

Denote $f_{\varphi,\psi;x}$ by f_0 and $f_{\varphi',\psi';g(x)}$ by f_1 and recall that

$$f_0(w) = pr_2\, \psi\, \varphi^{-1}(pr_1 \varphi(x), w)$$

for w in some open neighbourhood of 0 in W_φ, and

$$f_1(w) = pr_2 \psi' \varphi'^{-1}(pr_1 \varphi' g(x), w)$$

for w in some open neighbourhood of 0 in $W_{\varphi'}$.

Define

$$\hat{f}_t(w) = pr_2 \psi' \varphi'^{-1}((1 - t)\, \varphi' g\, \varphi^{-1}(pr_1 \varphi(x), w) + t(pr_1\, \varphi' g(x), w)).$$

We show that there exists an open neighbourhood of 0 in W_φ on which all mappings $\hat{f}_t$ are defined. Note that the mapping

$$\alpha\colon \langle 0, 1\rangle \times W_\varphi \ni (t, w) \mapsto (1 - t)\, \varphi' g \varphi^{-1}(pr_1\, \varphi(x), w) + t(pr_1\, \varphi' g(x), w) \in U_{\varphi'} \times W_{\varphi'}$$

is continuous. For each $t \in \langle 0, 1\rangle$, we have $\alpha(t, 0) = \varphi' g(x) \in \varphi'(D_{\psi'})$ since $g(x) \in D_{\psi'}$. The set $\varphi'(D_{\psi'})$ is an open subset of $U_{\varphi'} \times W_{\varphi'}$. By the continuity of α, for each $t \in \langle 0, 1\rangle$, there exist a neighbourhood V_t of t in $\langle 0, 1\rangle$ and a neighbourhood G_t of 0 in W_φ, such that $\alpha(V_t \times G_t) \subset \varphi'(D_{\psi'})$. Let $\{V_{t_1}, \ldots, V_{t_s}\}$ form a covering of $\langle 0, 1\rangle$. Set $G := \bigcap_{j=1}^{s} G_{t_j}$. Then $\alpha(\langle 0, 1\rangle \times G) \subset \varphi'(D_{\psi'})$, which means that all mappings $\hat{f}_t$ are defined in G.

We now prove that $\hat{f}_t$ is the homotopy realizing equivalence (***). Every $\hat{f}_t$ is an immersion at 0. Indeed, let $v \in T_0 W_\varphi$ and assume that $\hat{f}_{t*} v = 0$. Then $\alpha(t, \cdot)_* v \in T_{\varphi' g(x)}(U_{\varphi'} \times \{0\})$. Therefore $pr_{2*}\, \alpha(t, \cdot)_* v = 0$ but, on the other hand, $pr_2\, \alpha(t, \cdot) = id_{W_\varphi}$ by (*).

Consequently, $pr_{2*}\, \alpha(t, \cdot)_* v = v$. Hence $v = 0$. Now, similarly as above, using the continuity of the differential of α, we can assert that there exists a neighbourhood $\hat{G}$ of 0 in G such that, for each $t \in \langle 0, 1\rangle$, the mapping $\hat{f}_t$ is an immersion in $\hat{G}$. Consequently, condition (i) of definition (2.3) holds.

Condition (ii) of (2,3) is quite obvious.

Note that

$$\hat{f}_0(w) = pr_2\, \psi'\varphi'^{-1}(\varphi' g \varphi^{-1}(pr_1 \varphi(x), w))$$
$$= pr_2\, \psi\varphi^{-1}(pr_1\varphi(x), w) = f_0(w)$$

by (*), and

$$\hat{f}_1(w) = pr_2\, \psi'\varphi'^{-1}(pr_1\, \varphi' g(x), w) = f_1(w).$$

Thus condition (iii) of (2.3) holds.

We now prove that, for each $t \in \langle 0, 1\rangle$, the mapping $\hat{f}_t$ is compatible with $\sim_{\varphi\hat{G}}$ and $\sim_{\psi'}$. Indeed, let ℓ_0 be an equivalence class of $\sim_{\varphi\hat{G}}$. Note that $\alpha(\{t\} \times \ell_0) \subset U_{\varphi'} \times \ell_0$ because $pr_2\alpha(t, w) = w$ for each $w \in \hat{G}$. Therefore $\varphi'^{-1}\alpha(\{t\} \times \ell_0)$ is contained in some leaf $L' \in \mathcal{F}'$ by (3.4). Hence $\hat{f}_t(\ell_0) \subset \ell'$ where $\ell' = \{w' \in W_{\psi'};\ \psi'^{-1}(0, w') \in L'\}$. Since $\hat{f}_t(\ell_0)$ is connected, it is contained in a connected component of ℓ', thus in an equivalence class of $\sim_{\psi'}$. So, condition (iv) of (2.3) holds.

To prove condition (v) of (2.3), note that $\varphi'^{-1}\alpha(\langle 0, 1\rangle \times \{w\})$ is contained in some leaf of $\mathcal{F}'$ because of the equality $pr_2\alpha(t, w) = w$. Consequently, the image of the curve $\langle 0, 1\rangle \ni t \mapsto \hat{f}_t(w) \in W_{\psi'}$ is contained in some set ℓ' and, since it is connected, in an equivalence class of $\sim_{\psi'}$.

This completes the proof of the correctness of the definition of $\tilde{G}$.

It is easy to check that $\tilde{G}$ is a groupoid homomorphism. □

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O *-HOLONOMII PRZECIWOBRAZU FOLIACJI STEFANA

Niech $\mathcal{F}'$ będzie foliacją Stefana ([3]) na rozmaitości M' i niech f: M → M' będzie gładkim odwzorowaniem transwersalnym do $\mathcal{F}'$ ([4]). W pracy tej pokazujemy, że istnieje naturalny homomorfizm grupoidu *-holonomii([2]) foliacji $f^{-1}(\mathcal{F}')$ w grupoid *-holonomii foliacji $\mathcal{F}'$.