A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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A NOTE ON THE EXTENSION OF FUNCTIONALS

In [4] there is defined an extension of a mapping $J_o: B \to \langle 0, \infty\rangle$ to $J^*: H \to \langle 0, \infty\rangle$, where B is a sublattice of a lattice H. In this paper we introduce the notion of an n-cover of an element of H with respect to a system $\{\mathcal{N}_t\}_{t\in T}$ of subsets of B. Then a relation between the elements of H having n-covers with respect to $\{\mathcal{N}_t\}_{t\in T}$ and the elements with small J^*-values is given.

Let B be a sublattice of a given lattice H, a mapping $J^*: H \to \langle 0, \infty\rangle$ be an extension of $J_o: B \to \langle 0, \infty\rangle$. For every nonnegative integer n, let us denote

$$\mathcal{P}_n = \{x \in H: J^*(x) < \tfrac{1}{n}\}, \qquad \mathcal{Q}_n = \{x \in H: J^*(x) \le \tfrac{1}{n}\}.$$

Further define a system $\{\mathcal{N}_t\}_{t\in T}$ of subests of B by $\mathcal{N}_t = \{x \in B: J_o(x) < t\}$, where $T \subset (0, \infty)$. We are going to introduce the notion of an n-cover of an element of H with respect to $\{\mathcal{N}_t\}_{t\in T}$. Now let $\mathcal{N}_n^*$ be the set of all elements of H having an n-cover with respect to $\{\mathcal{N}_t\}_{t\in T}$.

In this paper we give some simple conditions for T and for J_o, B, which guarantee the validity of

$$\mathcal{P}_n \subset \mathcal{N}_n^* \subset \mathcal{Q}_n$$

for every nonnegative integer n. The relation between sequences $\{\mathcal{N}_n^*\}$, $\{\mathcal{P}_n\}$ and $\{\mathcal{Q}_n\}$ constructed by analogous way for rings and σ-rings was investigated by J. L l o y d in [1] and P. C a p e k in [2]. The inclusion $\mathcal{N}_n^* \subset \mathcal{P}_n$ in [2] (see Lemma 3) is incorrect.

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Let H be a distributive, relatively σ-complete lattice with the least element 0. Suppose that there is given a binary operation $\setminus$ on H, satisfying the following conditions:

1) If $x, y, z \in H$, $x \leqq y$, then $z \setminus x \geqq z \setminus y$, $y \setminus z \geqq x \setminus z$

2) $x = (x \vee y) \setminus y$ whenever $x, y \in H$, $x \wedge y = 0$.

Let B be a sublattice of H, closed under the operation $\setminus$. As regards B, we assume in the following that for every $x \in H$ there is a $b \in B$ such that $x \leqq b$. Finally, we assume that there is given a mapping $J_o: B \to \langle 0, \infty\rangle$ satisfying the following conditions:

(i) $J_o(0) = 0$;

(ii) For $x \leqq y$, $x, y \in B$ is $J_o(x) \leqq J_o(y)$;

(iii) $J_o(x \vee y) \leqq J_o(x) + J_o(y)$ for all $x, y \in B$;

(iv) if $x_n \nearrow x$, $x_n \in B$, $n = 1, 2, \ldots$, $x \in H$, then $x \in B$ and $J_o(x) = \lim_{n\to\infty} J_o(x_n)$;

(v) $J_o(x) = J_o(x \wedge y) + J_o(x \setminus y)$ for all $x, y \in B$.

Let N denote the set of all positive integers and let us repeat that

$$\mathcal{N}_t = \{x \in B: J_o(x) < t\}, \quad t \in (0, \infty).$$

Let $\{\mathcal{N}_t\}_{t\in T}$ be a system of subsets of the lattice B, where T is a nonempty subset of the set of all positive real numbers such that 0 is a limit point of T.

DEFINITION 1. Let $n \in N$. By an n-cover of an element $x \in H$ with respect to the system $\{\mathcal{N}_t\}_{t\in T}$ we mean a system $\{x_i\}_{i\in I}$ of elements $x_i \in \mathcal{N}_{k_i}$ with $\{k_i\}_{i\in I}$ being a sequence in T, satisfying the following conditions:

1) $\bigvee_{i\in I} x_i$ exists in H,

2) $x \leqq \bigvee_{i\in I} x_i$,

3) $\sum_{i\in I} k_i \leqq \frac{1}{n}$.

Let $\mathcal{N}^*_n$ denote the set of all elements of H having n-covers with respect to $\{\mathcal{N}_t\}_{t\in T}$.

We extend $J_o: B \to \langle 0, \infty\rangle$ to $J^*: H \to \langle 0, \infty\rangle$. For every $x \in H$ we put

$$J^*(x) = \inf \{J_o(f): f \in B, \ x \leqq f\} \quad \text{(see [4], Definition 1.1.).}$$

The mapping $J^*: H \to \langle 0, \infty\rangle$ satisfies the following conditions:

1) J^* is an extension of J_o,

2) J^* is non decreasing,

3) $J^*(x \vee y) \leqq J^*(x) + J^*(y)$ for every $x, y \in H$.

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We repeat that

$$\mathcal{P}_n = \{x \in H: J^*(x) < \tfrac{1}{n}\}, \quad \mathcal{Q}_n = \{x \in H: J^*(x) \leqq \tfrac{1}{n}, \quad n \in N.$$

LEMMA 1. $\mathcal{N}^*_n \subset Q_n$ for every $n \in N$.

P r o o f. Let $x \in \mathcal{N}^*_n$, i.e. there exist a system $\{k_i\}_{i\in I}$ of numbers of T, and $x_i \in \mathcal{N}_{k_i}$, $i \in I \subset N$ such that $x \leqq \bigvee_{i\in I} x_i \in H$ and $\sum_{i\in I} k_i \leqq \frac{1}{n}$.

If the set I is finite, then $\bigvee_{i\in I} x_i = \bigvee_{i=1}^{\alpha} x_i \in B$ and since

$$J^*(x) \leqq J^*(\bigvee_{i=1}^{\alpha} x_i) = J_o(\bigvee_{i=1}^{\alpha} x_i) \leqq \sum_{i=1}^{\alpha} J_o(x_i) < \sum_{i=1}^{\alpha} k_i \leqq \frac{1}{n},$$

we have $x \in Q_n$.

If the set I is infinite, then by (iv) $\bigvee_{i\in I} x_i = \bigvee_{i=1}^{\infty} x_i \in B$ and

$$J^*(\bigvee_{i=1}^{\infty} x_i) = J_o(\bigvee_{i=1}^{\infty} x_i) = \lim_{n\to\infty} J_o(\bigvee_{i=1}^{n} x_i) \leqq \lim_{n\to\infty} \sum_{i=1}^{n} J_o(x_i) =$$

$$= \sum_{i=1}^{\infty} J_o(x_i) \leqq \sum_{i=1}^{\infty} k_i \leqq \frac{1}{n},$$

so

$$J^*(x) \leqq J^*(\bigvee_{i=1}^{\infty} x_i) \leqq \frac{1}{n},$$ which completes the proof.

LEMMA 2. If $T = \{\frac{1}{n}: n \in N\}$, then $\mathcal{P}_n \subset \mathcal{N}^*_n$ for all $n \in N$.
Proof is very simple, so we can omit it.

From Lemmas 1 and 2 we obtain the following theorem.

THEOREM 1. If $T = \{\frac{1}{n}: n \in N\}$, then $\mathcal{P}_n \subset \mathcal{N}^*_n \subset \mathcal{Q}_n$ for every $n \in N$.

Denote by (D) the following condition: for every $f, g \in B$, $f < g$, $J_o(g) < \infty$, the set $\{J_o(r): r \in B, f < r < g\}$ is dense in $\langle J_o(f), J_o(g)\rangle$.

The lattice Z is called complementary, if for every $x, y \in Z$, $x \leqq y$ there exists $z \in Z$ such that $x \vee z = y$ and $x \wedge z = 0$. This property of the lattice Z we denote by (C).

LEMMA 3. Suppose that the mapping $J_o: B \to \langle 0, \infty\rangle$ satisfies the condition (D) and the lattice $\{x \in B: x \leqq y\}$ has the property (C) for every $y \in B$. Then $\mathcal{P}_n \subset \mathcal{N}^*_n$ for all $n \in N$.

P r o o f. Let $x \in \mathcal{P}_n$, i.e. $x \in H$, $J^*(x) < \frac{1}{n}$. The definition of J^* gives the existence of $f \in B$ such that $x \leqq f$ and $J_o(f) < \frac{1}{n}$. Let $\{f\}$ do not be an n-cover of x. Put $\varepsilon = \frac{1}{n} - J_o(f)$ and choose $t_o \in T$, $0 < t_o < \varepsilon$ and $p \in N$, $p > 2$ such that

$$(p-1)t_o \leqq J_o(f) < pt_o \leqq J_o(f) + \varepsilon.$$

Now we define the sequence $\{f_j\}_{j=1}^{p-1}$, $f_j \in B$, $j = 1, 2, \ldots, p-1$ as follows. By the condition (D) there exists $f_1 \in B$, $f_1 < f$ with

$$\frac{J_o(f)}{p} \leqq J_o(f_1) < t_o.$$

Suppose that $f_j \in B$, $j = 1, 2, \ldots, i$ for $1 \leqq i \leqq p-2$ are defined having the following properties:

$$f_j < f, \quad j = 1, 2, \ldots, i$$

$$\frac{J_o(f)}{p} \leqq J_o(f_j) < t_o, \quad j = 1, 2, \ldots, i,$$

$$f_k \wedge f_l = 0 \quad \text{for} \quad k \neq l, \quad k, l = 1, 2, \ldots, i.$$

In virtue of (D) there exists $\tilde{f} \in B$ such that $\bigvee_{j=1}^{i} f_j < \tilde{f} < f$ and $J_o(\tilde{f})$ belongs to the interval

$$(J_o(\bigvee_{j=1}^{i} f_j) + \frac{J_o(f)}{p} + \frac{t_o - \frac{J_o(f)}{p}}{2} - \frac{t_o - \frac{J_o(f)}{p}}{p},$$

$$J_o(\bigvee_{j=1}^{i} f_j) + \frac{J_o(f)}{p} + \frac{t_o - \frac{J_o(f)}{p}}{2}).$$

Since the lattice $\{x \in B: x \leqq \tilde{f}\}$ has the property (C), there exists $f_{i+1} \in B$ such that $\bigvee_{j=1}^{i+1} f_j = \tilde{f}$ and $f_{i+1} \wedge (\bigvee_{j=1}^{i} f_j) = 0$.

Evidently $f_{j+1} < f$ and $f_{i+1} \wedge f_j = 0$ for $j = 1, 2, \ldots, i$. Further, by (v) and (2)

$$J_o(\bigvee_{j=1}^{i} f_j) = \sum_{j=1}^{i} J_o(f_j), \qquad J_o(\tilde{f}) = J_o(f_{i+1}) + J_o(\bigvee_{j=1}^{i} f_j),$$

$$\sum_{j=1}^{i} J_o(f_j) + \frac{J_o(f)}{p} < J_o(f_{i+1}) + \sum_{j=1}^{i} J_o(f_j) < \sum_{j=1}^{i} J_o(f_j) + \frac{t_o}{2} + \frac{J_o(f)}{2p},$$

hence

$$\frac{J_o(f)}{p} < J_o(f_{i+1}) < t_o.$$

Finaly, by (C) there exists $f_p \in B$ such that $\bigvee_{j=1}^{p} f_j = f$ and $f_p \wedge f_j = 0$ for $j = 1, 2, \ldots, p-1$. We have

$$J_o(f_p) = J_o(f) - J_o(\bigvee_{j=1}^{p-1} f_j) = J_o(f) - \sum_{j=1}^{p-1} J_o(f_j) \leqq$$

$$\leqq J_o(f) - (p-1)\frac{J_o(f)}{p} = \frac{J_o(f)}{p} < t_o.$$

Hence $f_j \in \mathcal{N}_{t_o}$ for every $j \in \{1, 2, \ldots, p\}$ and taking into account that $pt_o \leqq \frac{1}{n}$, we have that $\{f_j\}_{j=1}^{p}$ is an n-cover of the element x. The proof is complete.

From Lemmas 1 and 3 we obtain the next theorem.

THEOREM 2. Under the hypotheses of Lemma 3 $\mathcal{P}_n \subset \mathcal{N}^*_n \subset \mathcal{Q}_n$ for every $n \in N$.

LEMMA 4. If $\overline{T} \supset \langle 0, 1\rangle$, then $\mathcal{P}_n \subset \mathcal{N}^*_n$ for all $n \in N$.

Proof is evident. Lemmas 1 and 4 imply the following theorem.

THEOREM 3. If $\overline{T} \supset \langle 0, 1\rangle$, then $\mathcal{P}_n \subset \mathcal{N}^*_n \subset \mathcal{Q}_n$ for every positive integer n.

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O ROZSZERZENIACH FUNKCJONAŁÓW

W prezentowanym artykule rozważa się problem rozszerzenia odwzorowania J_o: $B \to \langle 0, \infty\rangle$ do $J^*: H \to \langle 0, \infty\rangle$, gdzie B jest podkartą karty H. Wprowadza się pojęcie n-pokrycia pewnych elementów z H względem ustalonego systemu $\{\mathcal{N}_t\}_{t\in T}$, podzbiorów zbioru B. Podany jest również związek pomiędzy zbiorem elementów z H mających n-pokrycie względem systemu $\{\mathcal{N}_t\}_{t\in T}$, a tymi elementami z H, dla których wartość odwzorowania J* jest mała.