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A NOTE ON INTERSECTIONS OF CERTAIN TOPOLOGIES ON R

Intersections of topologies related to the density topology and their category analogue are characterized in this paper.

In this note R will be denote the real line, T - the natural topology on R, int A and $\bar{A}$ - interior and closure of A with respect to the topology T. Let $\mathcal{S}$ be the collection of Lebesgue measurable sets, $|A|$ - a Lebesgue measure of a set $A \in \mathcal{S}$ and I - the σ-ideal of sets of the first category.

Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of X. We denote $\mathcal{U} \cap \mathcal{V} = \{W \subset X: W \in \mathcal{U}$ and $W \in \mathcal{V}\}$. If $\mathcal{U}$ and $\mathcal{V}$ are topologies, then $\mathcal{U} \cap \mathcal{V}$ is the largest topology contained in $\mathcal{U}$ and $\mathcal{V}$.

In [1] and [8] the density topology $\mathcal{T}_d = \{\Phi(A) - N: A \in \mathcal{S}, |N| = 0\}$ was presented, where $\Phi(A)$ is the set of all $x \in R$ at which the metric density of A is equal to 1. R. O'Malley in [7] introduced the a.e. - topology $\mathcal{T}_{a.e.} = \{U \in \mathcal{T}_d: |U| = |\text{int } U|\}$. N. F. G. Martin in [4] and H. Hashimoto in [2] proposed a topology - constructing method based on a topology on X and an ideal of subsets of X. The particular case of such a topology is $\mathcal{T}_H = \{G - N: G \in T, |N| = 0\}$.

THEOREM 1. $\mathcal{T}_H \cap \mathcal{T}_{a.e.} = \{U - M: U \in T, |U \cap \bar{M}| = 0\}$.

P r o o f. Suppose that $A \in \mathcal{T}_H$ and $A \in \mathcal{T}_{a.e.}$. Then $A = V - N$ where $V \in T$, $|N| = 0$.

$$\text{int } A = \text{int } (V - N) = \text{int } (V \cap N') = \text{int } V \cap \text{int } N' =$$
$$= \text{Int } V \cap (R \setminus \bar{N}) = V \cap (R \setminus \bar{N}) = V \setminus \bar{N}.$$

Besides, $A \in \mathcal{T}_{a.e.}$, $|A| = |\text{int } A| = |V - \bar{N}| = |V - (V \cap \bar{N})|$.

From $|A| = |V|$ we have $|V \cap \overline{N}| = 0$.

On the other hand, if $W = U - M$, $U \in T$, $|U \cap \overline{M}| = 0$, then, obviously, $W \in \mathcal{T}_H$, $W \in \mathcal{T}_d$ and

$$|\text{int } W| = |\text{int } (U - M)| = |U - \overline{M}| = |U - (U \cap \overline{M})| = |U| = |W|,$$

which ends the proof.

PROPOSITION 1. The collection $\{U \setminus M: U \in T, |U \cap \overline{M}| = 0\}$ is not identical with the collection $\{U - M: U \in T, |\overline{M}| = 0\}$.

P r o o f. Let X be a Cantor set of positive Lebesgue measure, $U = R - X = \bigcup_{n=1}^{\infty} U_n$ where U_n are components of U. Let now u_n be the central point of an interval U_n and let $(u_n^{(k)})_{k\in N}$ be a sequence of points of U_n such that

$$\lim_{k\to\infty} u_n^{(k)} = u_n \quad \text{and} \quad u_n^{(k)} \neq u_n, \text{ for } k \in N.$$

We put $A = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{u_n^{(k)}\}$. As $|U \cap \overline{A}| = |A \cup \{u_1, u_2, u_3, \ldots\}| = 0$, therefore $U \setminus A$ belongs to the first of the above - mentioned collections.

Suppose that $U - A = V - M$ with $V \in T$, $|\overline{M}| = 0$. For any $n \in N$ $u_n \in U - A$, so $u_n \in V$. As V is an open set it contains a neighbourhood V_n of u_n, and $\{u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, \ldots\} \cap V_n \subset M$, so the set M contains a sequence converging to u_n.

By the arbitrainess of n, $\{u_1, u_2, \ldots\} \subset \overline{M}$ and $X \subset \overline{\{u_1, u_2, u_3, \ldots\}} \subset \overline{M}$. This gives a contradiction as $|X| > 0$.

(Notice that the collection $\{U - M: |\overline{M}| = 0\}$ is not countably additive.)

Now, we shall consider the category analogue of the topologies $\mathcal{T}_d$, $\mathcal{T}_H$ and $\mathcal{T}_{a.e.}$.

In [6] W. P o r e d a, E. W a g n e r-B o j a k o w s k a and W. W i l c z y ń s k i considered the notion of an I-density point of a set having the Baire property. Using this notion, they defined the category analogue of the density topology - the so-called I-density topology (denoted by $\mathcal{T}_I$). We can now introduce the topologies

$\mathcal{T}^* = \{G - P: G \in T,\ P \in I\}$ (compare [3])

and

$\mathcal{T}^1_I = \{U \in \mathcal{T}_I: U - \text{int}\, U \in I\}$ (compare [5]).

THEOREM 2. $\mathcal{T}^* \cap \mathcal{T}^1_I = \{G - P: G \in T,\ (G \cap \overline{P}) \in I\}$.

P r o o f. Suppose that $A \in \mathcal{T}^*$ and $A \in \mathcal{T}^1_I$. Then $A = V - P$ where $V \in T$, $P \in I$. Since $(A - \text{int}\, A) \in I$, therefore

$$V \cap \overline{P} = (V \cap (\overline{P} - P)) \cup (V \cap P) = ((V - P) - (V - \overline{P})) \cup$$
$$\cup (V \cap P) = (A - \text{int}\, A) \cup (V \cap P) \in I.$$

Conversely, if $U = G - P$, $G \in T$, $(G \cap \overline{P}) \in I$, then, obviously, $U \in \mathcal{T}^*$, $U \in \mathcal{T}_I$ and

$$U - \text{int}\, U = (G - P) - (G - \overline{P}) = G \cap (\overline{P} - P) \subset G \cap \overline{P} \in I.$$

PROPOSITION 2. $\mathcal{T}^* \cap \mathcal{T}^1_I = \{G - P: G \in T,\ \overline{P} \in I\} = \{G - P: G \in T,$ P is a nowhere dense set$\}$.

P r o o f. Let $A \in \mathcal{T}^* \cap \mathcal{T}^1_I$, so $A = G - P$, $G \in T$, $(G \cap \overline{P}) \in I$. We can assume that $P \subset G$, we have $\overline{P} = (\overline{P} \cap G) \cup (\overline{P} - G) = (\overline{P} \cap G) \cup (\overline{G \cap P} - G) \subset (\overline{P} \cap G) \cup (\overline{G} - G) \in I$ because the set $\overline{G} - G$ is nowhere dense, so $\overline{P}$ and P are nowhere dense sets.

Since the collection of all continuous functions $f: R \to R$ is identical with the collections of all $\mathcal{T}_H$-continuous functions and $\mathcal{T}^*$-continuous functions ([4], theorem (4), therefore the collections of all $\mathcal{T}_H \cap \mathcal{T}_{a.e.}$-continuous and $\mathcal{T}^* \cap \mathcal{T}^1_I$ - continuous functions are also identical with the collection of all continuous functions.

Thus $\mathcal{T}_H \cap \mathcal{T}_{a.e.}$ and $\mathcal{T}^* \cap \mathcal{T}^1_I$ are not completely regular. They are, obviously, Hausdorff topologies.

PROPOSITION 3. The topologies $\mathcal{T}_H \cap \mathcal{T}_{a.e.}$ and $\mathcal{T}^* \cap \mathcal{T}^1_I$ are not regular.

P r o o f. Suppose that $\mathcal{T}_H \cap \mathcal{T}_{a.e.}$ is regular. Let

$$F = \langle 1,2 \rangle \cup \{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\},\quad x = 0.$$

The set F is closed with respect to the topology $\mathcal{T}_H \cap \mathcal{T}_{a.e.}$, so there are two open sets V and W such that $x \in V$, $F \subset W$, $V \cap W = \emptyset$. The set $V = U - N$ where $U \in T$, $|U \cap \overline{N}| = 0$, and there exist an interval J such that $0 \in J$, $J - N \subset V$ and a positive integer n such that $\frac{1}{n} \in J$. The point $\frac{1}{n} \in W$, V and W are disjoint sets, so $(J \cap W) \subset N$ and we have a contradiction.

The same example shows that $\mathcal{T}^* \cap \mathcal{T}_I^1$ is not a regular topology.

REFERENCES

[1] C. Goffman, C. J. Neugebauer, T. Nishiura, Density topology and approximate continuity, Duke Math. J., 28(1961), 497-505.

[2] H. Hashimoto, On the *topology and it's applications, Fund. Math., 91 (1976), 5-10.

[3] E. Łazarow, R. A. Johnson, W. Wilczyński, Topologies related to sets having the Baire property, Demonstr. Math. (in print).

[4] N. F. G. Martin, Generalized condensation points, Duke Math. J., 28 (1961).

[5] W. Poreda, E. Wagner-Bojakowska, The topology of I-approximately continuous functions, Rad. Mat., 2 (1986), 263-277.

[6] W. Poreda, E. Wagner-Bojakowska, W. Wilczyński, A category analogue of the density topology, Fund. Math., 125 (1985), 167-173.

[7] R. O'Malley, Approximately differentiable functions: the r-topology, Pacific J. Math., 72 (1977), 207-222.

[8] J. C. Oxtoby, Measure and Category, New York-Heidelberg-Berlin 1971.

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NOTATKA O PRZEKROJACH PEWNYCH TOPOLOGII NA PROSTEJ

Artykuł zawiera charakterystykę przekroju topologii (wprowadzonych przez R. O'Malleya i N. F. G. Martina) związanych z topologią gęstości na prostej oraz ich odpowiedników dla I-gęstości.