A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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ADDENDA TO MY WORK: "ESTIMATION OF THE FUNCTIONAL $|a_3 - \alpha a_2^2|$ IN THE CLASS S OF HOLOMORPHIC AND UNIVALENT FUNCTIONS FOR α COMPLEX"

In this paper there has been investigated a set of values of a mapping connected with a maximal value of the functional $|a_3 - \alpha a_2^2|$ in the well-known class S of functions holomorphic and univalent in the unit disc were α is a complex parameter [5].

At the XIII Rolf Nevanlinna-Colloquium in Joensun, Finland, in August 1987, during his lecture "On the functional $a_3 - \alpha a_2^2$ in the class S" [2], Prof. A. Pflüger pointed out a gap in the proof of Lemma 4 of my paper [5]. This lemma has an essential meaning for the estimation of the functional $|a_3 - \alpha a_2^2|$ in the class S for a complex parameter α [5]. Since the Lemma was directly applied in the paper by H. S i e j k a [3] and indirectly by H. S i e j k a and O. T a m m i in [4], I felt obliged to explain this problem.

The present paper completes the gap as well as indicates a more general method of proceeding which can be applied not only in the task given in paper [5].

Let us now repeat some notations and definitions from papers [5], p. 162-165.

Let us set

$B = \{(\rho, \psi) : 0 < \rho < 1 \text{ and } -\frac{\pi}{2} < \psi < \frac{\pi}{2}\}.$

By B^+ and B^- we denote the parts of the set B lying in the plane ρ, ψ above and below the abscissa axis.

Let

$$G = \mathbb{C} - (E \cup \{(x, y): x \geqq 1 \text{ and } y = 0\},$$

where $\mathbb{C}$ is the complex plain and E a set consisting of the segment $\langle 0, 1\rangle$ of the axis $0x$ and closed segments parallel to the axis $0y$ such that their terminal points are defined by the conditions $y = \pm y(x)$, $0 \leqq x < 1$, where $y = y(x)$ is the function given by the equation

$$\text{(i)}\quad \left[(1-x)^2 + y^2\right]\sqrt{1 - e^{2\left[1 - \frac{1-x}{(1-x)^2+y^2}\right]}} +$$

$$- \left[(1-x)^2 + y^2\right] e^{1 - \frac{1-x}{(1-x)^2+y^2}} \operatorname{arc\,cos} e^{1 - \frac{1-x}{(1-x)^2+y^2}} +$$

$$+ y\, e^{1 - \frac{1-x}{(1-x)^2+y^2}} = 0.$$

The graph of the function contains the point (0, 0) and besides this point, it is contained in the half-plane $y < 0$ and in the disc $|\alpha - \frac{1}{2}| < \frac{1}{2}$ where $\alpha = x + iy$, with that $\lim\limits_{x\to 1^-} y(x) = 0$ ([5], p. 165).

Let us denote by G^+ and G^- the sets in the plane x, y which are parallel to B^+ and B^-.

Let now α be a mapping of B in the plane x, y defined as follows ([5], p. 163):

$$\text{(ii)}\quad \alpha = 1 + \frac{2 + (\rho + \frac{1}{\rho})e^{2i\psi}}{\left[2 + (\rho + \frac{1}{\rho})e^{2i\psi}\right]\operatorname{Log}\frac{2 + (\rho + \frac{1}{\rho})e^{2i\psi}}{(\frac{1}{\rho} - \rho)e^{2i\psi}} + (\rho + \frac{1}{\rho} + 2e^{2i\psi})\operatorname{Log}\frac{1-\rho}{1+\rho} - 4}$$

The following lemma and its proof were given in paper [5], p. 165.

LEMMA. The set of values of mapping (ii) for $(\rho, \psi) \in B$ is identical with the set G.

Professor A. Pflüger remarked in Joensun that the justification of the non-obvious inclusion $\delta G \subset \delta\alpha(B)$ is missing in the proof of the Lemma, while it contains the justification of the opposite inclusion.

The justification of both the inclusions and the conclusion that $\alpha(B) = G$ can be carried out with the help of the following observations.

1. The functional α from (ii) enlarges in a continuous manner onto three boundary segments: $0 < \rho \leqq 1$, $\psi = 1$, $\rho = 1$, $0 < \psi < \frac{\pi}{2}$; $0 < \rho < 1$, $\psi = \frac{\pi}{2}$; maps them onto: the graph of the function $y = 0$, $-\infty < x \leqq 0$, the graph of the function $y = y(x)$ $0 < x < 1$, defined by (i); the graph of the function $y = 0$, $1 \leqq x < \infty$, all of them lying in the plane of variables x, y.

We obtain this immediately for the points on the segments where the right-hand side of (ii) is defined; for the remained points $(1, 0)$ and $(1, \frac{\pi}{2})$ we easily evaluate that there exist limits (ii) equal to $(0, 0)$ and $(0, 1)$, respectively.

In an analogous manner we show the product $\rho\alpha$ enlarges in a a continuous manner onto the fourth boundary segment: $\rho = 0$, $0 \leqq \psi \leqq \frac{\pi}{2}$ and takes there the values $e^{2i\psi/4}$. Consequently, for the points (ρ, ψ) tending to the point $(0, \psi_o)$ on the considered segment, the corresponding α tends to ∞, and the ratio $\alpha/|\alpha|$ tends to $e^{-2i\psi_o}$.

2. The boundary δG^- is the image of the three boundary segments described above, by means of the mapping α. This follows immediately from the definition of the set G^+ and the definition of G^-.

3. The image $\alpha(B^+)$ is open. This follows from the fact that the Jacobian of the mapping α is different from zero ([5], p. 163).

4. For an arbitrary angle Δ in the plane of x, y with the vertex at 0, for the rest entirely lying above the real axis, the points of the set Δ sufficiently large do not belong to the image $\alpha(B^+)$. Consequently, the complement of the set $\alpha(B^+)$ contains the interior points.

Really, in opposite case there would exist a sequence of points (ρ_n, ψ_n), $n = 1, 2, \ldots$ of the set B^+ converging to a limit (ρ_o, ψ_o) and a sequence of corresponding values α_n from

(ii) lying in Δ, above the real axis and tending to ∞. Then, according to 1, there would be $\rho_o = 0$, $0 \leq \psi_o \leq \frac{\pi}{2}$, simultaneously, the sequence $\alpha_n/|\alpha_n|$ should tend to the limit $-e^{2i\psi_o}$ and this limit should lie in Δ, while it is lying below or on the real axis.

5. The boundary $\delta\alpha(B^+)$ disconnects (cuts) the plane x, y i.e. its complement is not connected. This follows immediately from the fact the complement of the boundary is the sum of disjoint open sets: the interior of the image $\alpha(B^+)$ and the interior of the complement of this image, both of them being nonempty according to 3 and 4.

6. The boundary of the image $\delta\alpha(B^+)$ is contained in the boundary δG^-. This easily follows from 1, 2, 3.

7. No proper subset of the boundary δG^- disconnects the plane, i.e. its complement is a connected set. This follows immediately from the structure of this set, being in the light of 2 and 1, the sum of three graphs [1].

8. The image $\alpha(B^+)$ and the domain G^- have some points in common. Really, according to 1, for every ρ, ψ sufficiently close to 0, ψ_o where $\psi_o \neq 0, \frac{\pi}{2}$, the corresponding image α is lying in the plane of x, y, arbitrary far and the ratio $\alpha||\alpha|$-arbitrary close to $e^{-2i\psi_o}$; consequently, α is lying beyond the set E and below the real axis. Thus α belongs to G^-. From the observations made above, it follows finally that the boundary $\delta\alpha(B^+)$ is identical with the boundary δG^-.

Indeed, in the opposite case, according to 6 the boundary $\delta\alpha(B^+)$ would be a proper subset of δG^- and, according to 7, would not disconnect the plane, contrary to 5. From this and 8, it follows at last that $\alpha(B^+) = G^-$ and analogously $\alpha(B^-) = G^+$. Putting together these relations and taking into account, according to 1, the behaviour of the mapping α on the segment $0 < \rho \leq 1$, $\psi = 0$, we arrive at last to the required relation $\alpha(B) = G$.

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REFERENCES

[1] K u r a t o w s k i C., Topologia, Vol. 2, Warszawa 1952, 358.

[2] P f l ü g e r A., On the Functional $a_3 - \alpha a_2^2$ in the Class S, Comp. Variab., 10 (1988), 83-95.

[3] S i e j k a H., On estimation of the functional $|a_3 - \alpha a_2^2|$, $\alpha \in C$, in the classes of bounded univalent functions, Demonst. Math., 16/1 (1983).

[4] S i e j k a H., T a m m i O., On maximizing a homogeneous functional in the class of bounded univalent functions, Ann. Acad. Sci. Fenn., Ser. Math., 6 (1981), 273-288.

[5] S z w a n k o w s k i A., Estimation of the functional $|a_3 - \alpha a_2^2|$ in the class S of holomorphic and univalent functions for α complex, Acta Univ. Lodz., Folia math., 1 (1984) 151-177; Abstracts Conference on Analytic Functions, Kozubnik 1979, 54.

[6] C h a r z y ń s k i Z., Sur les fonctions univalentes bornées, Colloq. Math. 2, (1948).

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PEWNE UWAGI DO PRACY: "OSZACOWANIE FUNKCJONAŁU $|a_3 - \alpha a_2^2|$ W KLASIE S FUNKCJI HOLOMORFICZNYCH I JEDNOKROTNYCH DLA ZESPOLONYCH LICZB α"

W niniejszym artykule badamy zbiór wartości pewnego odwzorowania związanego z maksimum wartości funkcjonału $|a_3 - \alpha a_2^2|$ w znanej klasie funkcji holomorficznych i jednokrotnych w kole jednostkowym, gdzie α jest parametrem zespolonym.