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STABILITY OF THE VOLTERRA INTEGRODIFFERENTIAL EQUATION

M. JANFADA AND GH. SADEGHI

Abstract. In this paper, the Hyers-Ulam stability of the Volterra integrodifferential equation

$$x'(t) = g(t, x(t)) + \int_0^t K(t, s, x(s))ds,$$

and the Volterra equation

$$x(t) = g(t, x(t)) + \int_0^t K(t, s, x(s))ds,$$

on the finite interval $[0, T]$, $T > 0$, are studied, where the state $x(t)$ take values in a Banach space X.

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1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?" If there exists an affirmative answer we say that the equation $\mathcal{E}$ is stable [8]. During the last decades several stability problems for various functional equations have been investigated by numerous mathematicians. We refer the reader to the survey articles [8, 9, 23] and monographs [6, 10, 13, 24] and references therein.

Consider the *Volterra integrodifferential equation*

$$x'(t) = g(t, x(t)) + \int_0^t K(t, s, x(s))ds. \tag{1}$$

If for given differentiable function $x(t)$, satisfying

$$\|x'(t) - g(t, x(t)) - \int_0^t K(t, s, x(s))ds\| \leq \phi(t),$$

$\phi(t) > 0, t \in [0, T]$, there exists a solution $y(t)$ of (1) such that for some $C > 0$,

$$\|x(t) - y(t)\| \leq C\phi(t),$$

then we say that (1) has the Hyers-Ulam stability. A similar definition can be considered for the *Voltera equation*

$$x(t) = g(t, x(t)) + \int_0^t K(t, s, x(s))ds. \tag{2}$$

These equations and their special and general versions with different view points, have been studied by many authors. See [1, 4, 5, 14, 15, 17, 18, 19, 20] and the references given therein.

For a nonempty set X, a function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

(M1) $d(x, y) = 0$ if and only if $x = y$.
(M2) $d(x, y) = d(y, x)$, for all $x, y \in X$.
(M3) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Trivially the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include infinity.
We now introduce one of fundamental results of fixed point theory. For the proof, we refer to [16]. This theorem will play an important role in proving our main results.

Theorem 1. *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < 1$, for some $x \in X$, then the following are true:*

(a) The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;
(b) x^* is the unique fixed point of Λ in

$$X^* = \{y \in X : d(\Lambda^k x, y) < \infty\};$$

(c) If $y \in X^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$.

In this paper, using this theorem, we shall study the Hyers-Ulam stability of (1) and (2). Next some applicable examples of these equations and their Hyers-Ulam stability will be considered.

2. Hyers–Ulam stability

Cădariu and Radu [2] studied the stability of the Cauchy additive functional equation using the fixed point method. Applying such a clever idea, they could present another proof for the Hyers-Ulam stability of that equation [3, 11, 22]. Also Soon-Mo Jung [12] used this idea for studying the stability of the following *Voltrra integral equation*

$$y(x) = \int_0^x f(\tau, y(\tau))d\tau.$$

As a recent work on the stability of integral equations, one can see [25].

In this section, by using the idea of Cădariu, Radu and Jung, we will study the Hyers-Ulam stability of the *integrodifferential Volterra integral equation* and the *Volterra integral equation*.

Theorem 2. *Suppose $\mathcal{X}$ is a Banach space and L, L_1, L_2 and T are positive constant for which $0 < L_1 + (L_1 + L_2)L + L_2TL < 1$. Let $g : [0,T] \times \mathcal{X} \to \mathcal{X}$, $K : [0,T] \times [0,T] \times \mathcal{X} \to \mathcal{X}$ and $\phi : [0,T] \to (0,\infty)$ be continuous and satisfy*

$$\begin{aligned} \|g(t,x) - g(t,y)\| &\le L_1\|x-y\|, \\ \|K(t,s,x) - K(t,s,y)\| &\le L_2\|x-y\|, \\ \text{and} \int_0^t \phi(s)ds &\le L\phi(t), \end{aligned} \tag{3}$$

for all $s,t \in [0,T]$ and $x,y \in \mathcal{X}$. If $f : [0,T] \to \mathcal{X}$ is a differentiable function satisfies

$$\|f'(t) - g(t,f(t)) - \int_0^t K(t,s,f(s))ds\| \le \phi(t), \ t \in [0,T], \tag{4}$$

then there exists a unique differentiable function $f_0 : [0,T] \to \mathcal{X}$ such that for each $t \in [0,T]$

$$f_0'(t) = g(t,f_0(t)) + \int_0^t K(t,s,f_0(s))ds, \tag{5}$$

and

$$\|f'(t) - f_0'(t)\| + \|f(t) - f_0(t)\| \le \frac{1+L}{1 - L_1 + (L_1+L_2)L + L_2TL}\phi(t). \tag{6}$$

Proof. Put

$$M := \{x : [0,T] \to \mathcal{X} : x \text{ is differentiable}\}$$

and define a mapping $d : M \times M \to [0,\infty]$ by

$$d(x,y) = \inf\{C \in [0,\infty] : \|x'(t) - y'(t)\| + \|x(t) - y(t)\| \le C\phi(t), t \in [0,T]\}.$$

We show that (M,d) is a complete generalized metric space. We just prove the triangle inequality and the completeness of this space. Assume that $d(x,y) > d(x,z) + d(z,y)$, for some $x,y,z \in M$. Then there exists $t_0 \in [0,T]$ with

$$\|x'(t_0) - y'(t_0)\| + \|x(t_0) - y(t_0)\| > \Big(d(x,z) + d(z,y)\Big)\phi(t_0). \tag{7}$$

Thus, by definition of d,

$$\begin{aligned} &\|x'(t_0) - y'(t_0)\| + \|x(t_0) - y(t_0)\| > \\ &\|x'(t_0) - z'(t_0)\| + \|x(t_0) - z(t_0)\| + \|z'(t_0) - y'(t_0)\| + \|z(t_0) - y(t_0)\|, \end{aligned}$$

which is a contradiction. Now we show that (M, d) is complete. Let $\{x_n\}$ be a Cauchy sequence in (M, d). This, by definition of d, implies that

$$(8)\quad \forall_{\varepsilon>0}\ \exists_{N_\varepsilon\in\mathbb{N}}\ \forall_{m,n\geq N_\varepsilon}\forall_{t\in[0,T]}\ \|x'_n(t)-x'_m(t)\|+\|x_n(t)-x_m(t)\|<\varepsilon\phi(t).$$

By continuity of ϕ on compact interval $[0, T]$, (8) implies that $\{x_n\}$ and $\{x'_n\}$ are uniformly convergent on $[0, T]$. So there exists a differentiable function x such that $\{x_n\}$ and $\{x'_n\}$ are uniformly convergent to x and x', respectively. Hence $x \in M$ and from (8), letting $m \to \infty$, we have

$$\forall_{\varepsilon>0}\ \exists_{N_\varepsilon\in\mathbb{N}}\forall_{n\geq N_\varepsilon}\forall_{t\in[0,T]}\ \ \|x'_n(t)-x'(t)\|+\|x_n(t)-x(t)\|\leq\varepsilon\phi(t).$$

Consequently

$$\forall_{\varepsilon>0}\ \exists_{N_\varepsilon\in\mathbb{N}}\forall_{n\geq N_\varepsilon}\ \ d(x_n,x)\leq\varepsilon$$

and so (M, d) is complete.

Now define $\Lambda : M \to M$ by

$$(9)\qquad \Lambda(x(t))=\int_0^t g(\tau,x(\tau))d\tau+\int_0^t\int_0^t K(\tau,s,x(s))dsd\tau.$$

First we show that Λ is strictly contractive. Suppose $x, y \in M$, $C_{xy} \in [0,\infty]$ and $d(x, y) \leq C_{xy}$. Thus for all $t \in [0, T]$,

$$\|x'(t)-y'(t)\|+\|x(t)-y(t)\|\leq C_{xy}\phi(t).$$

Hence by (3)

$$\begin{aligned}
&\left\|\frac{d}{dt}\Big(\Lambda x(t)-\Lambda y(t)\Big)\right\|+\|\Lambda x(t)-\Lambda y(t)\|=\\
&=\left\|g(t,x(t))-g(t,y(t))+\int_0^t\Big(K(t,s,x(s))-K(t,s,y(s))\Big)ds\right\|+\\
&+\left\|\int_0^t g(\tau,x(\tau))-g(\tau,y(\tau))d\tau+\int_0^t\int_0^t K(\tau,s,x(s))-K(\tau,s,y(s))dsd\tau\right\|\\
&\leq L_1\|x(t)-y(t)\|+L_2\int_0^t\|x(\tau)-y(\tau)\|\,d\tau+\\
&+L_1\int_0^t\|x(\tau)-y(\tau)\|\,d\tau+L_2T\int_0^t\|x(s)-y(s)\|\,ds\\
&\leq\Big(L_1+(L_1+L_2)L+L_2TL\Big)C_{xy}\phi(t).
\end{aligned}$$

This implies that

$$(10)\qquad d(\Lambda x,\Lambda y)\leq(L_1+(L_1+L_2)L+L_2TL)d(x,y).$$

So Λ is strictly contractive, since $0 < L_1 + (L_1 + L_2)L + L_2TL < 1$. On the other hand, trivially $f \in M$ and by (4)

$$\left\|\frac{d}{dt}\Big(\Lambda f(t) - f(t)\Big)\right\| + \|\Lambda f(t) - f(t)\| \leq \phi(t) + \int_0^t \phi(s)ds$$
$$= (1 + L)\phi(t).$$

Consequently

$$d(\Lambda f, f) \leq 1 + L < \infty. \tag{11}$$

It follows from Theorem 1 (a) that there exists a unique element $f_0 \in M^* = \{y \in M : d(\Lambda f, y) < \infty\}$ such that $\Lambda f_0 = f_0$, or equivalently

$$f_0(t) = \int_0^t g(\tau, f_0(\tau))d\tau + \int_0^t \int_0^t K(\tau, s, f_0(s))dsd\tau.$$

Now the facts that f_0 is differentiable and g, K are continuous, imply that

$$f_0'(t) = g(t, f_0(t)) + \int_0^t K(t, s, f_0(s))ds.$$

Also from Theorem 1 (c) and (11), we have

$$d(f, f_0) \leq \frac{1}{1 - (L_1 + (L_1 + L_2)L + L_2TL)} d(\Lambda f, f) \leq$$
$$\leq \frac{(1 + L)}{1 - (L_1 + (L_1 + L_2)L + L_2TL)}.$$

In view of definition of d we can conclude that the inequality (6) holds, for all $t \in [0, T]$. Put $\theta = \frac{(1+L)}{1-(L_1+(L_1+L_2)L+L_2TL)}$.

Let h be another differentiable function satisfying (5), (6). Then $f \in M$, $d(f, h) < \theta$ and

$$h'(t) = g(t, h(t)) + \int_0^t K(t, s, h(s))ds. \tag{12}$$

For proving the uniqueness of f_0, it is enough to show that h is a fixed point of Λ and $h \in M^*$. Using (12), one can see that $\Lambda h = h$. We show that

$d(\Lambda f, h) < \infty$. From (12) and the fact that $d(f, h) < \theta$, we obtain

$$\left\|\frac{d}{dt}(\Lambda f(t) - h(t))\right\| + \|\Lambda f(t) - h(t)\| =$$
$$= \|g(t, f(t)) - g(t, h(t)) - \int_0^t K(t, s, f(s))ds + \int_0^t K(t, s, h(s))ds\| +$$
$$+ \left\|\int_0^t \Big(g(t, f(\tau)) - g(\tau, h(\tau))\Big)d\tau - \int_0^t\int_0^t \Big(K(\tau, s, f(s))ds + K(\tau, s, h(s))ds\Big)d\tau\right\|$$
$$\leq L_1\|f(t) - h(t)\| + L_2\int_0^t \|f(s) - h(s)ds +$$
$$+ T\Big(L_1\|f(t) - h(t)\| + L_2\int_0^t \|f(s) - h(s)ds\Big)\|$$
$$\leq (L_1 + L_2)(1 + T)\theta\phi(t),$$

which implies that $d(\Lambda f, h) \leq (L_1 + L_2T)\theta < \infty$. This completes the proof. □

In the next theorem the Hyers-Ulam stability of the *Volterra integral equation* is studied. This theorem is an extension of Theorem 2.1 of [12].

Theorem 3. *Suppose $\mathcal{X}$ is a Banach space and L, L_1, L_2 and T are positive constants for which $0 < (L_1 + L_2)L < 1$. Let $g : [0, T] \times \mathcal{X} \to \mathcal{X}$, $K : [0, T] \times [0, T] \times \mathcal{X} \to \mathcal{X}$ and $\phi : [0, T] \to (0, \infty)$ be continuous functions satisfying*

$$\begin{aligned} \|g(t, x) - g(t, y)\| &\leq L_1\|x - y\| \\ \|K(t, s, x) - K(t, s, y)\| &\leq L_2\|x - y\| \\ \int_0^t \phi(s)ds &\leq L\phi(t), \end{aligned} \tag{13}$$

for all $s, t \in [0, T]$ and $x, y \in \mathcal{X}$. If $f : [0, T] \to \mathcal{X}$ is a continuous function satisfies

$$\|f(t) - g(t, f(t)) - \int_0^t K(t, s, f(s))ds\| \leq \phi(t), \ t \in [0, T]. \tag{14}$$

Then there exists a unique continuous function $f_0 : [0, T] \to \mathcal{X}$ such that

$$f_0(t) = g(t, f_0(t)) + \int_0^t K(t, s, f_0(s))ds \tag{15}$$

and

$$\|f(t) - f_0(t)\| \quad \leq \frac{1}{1 - (L_1 + L_2)L}\phi(t). \tag{16}$$

Proof. With

$$M := \{x : [0,T] \to \mathcal{X} : x \text{ is continuous } \},$$

define $d : M \times M \to [0,\infty]$ by

$$d(x,y) = \inf\{C \in [0,\infty] : \|x(t) - y(t)\| \leq C\phi(t), t \in [0,T]\}.$$

With a similar argument to the proof of Theorem 2, one can see that (M,d) is a complete generalized metric space. Now define Λ on M as in the proof of Theorem 2. One can verify that for any $x,y \in M$,

$$d(\Lambda x, \Lambda y) \leq (L_1 + L_2)Ld(x,y).$$

The fact that $0 < (L_1 + L_2)L < 1$ implies that Λ is strictly contractive. Also from (14), we obtain $d(\Lambda f, f) \leq 1 < \infty$ and so by Theorem 1, Λ has a unique fixed point f_0 in the set $M^* := \{y \in M : d(\Lambda f, y) < \infty\}$.

Let h be another continuous function satisfying (15), (16). Thus $f \in M$, $d(f,h) < \frac{1}{1-(L_1+L_2)L}$ and

$$h(t) = g(t,h(t)) + \int_0^t K(t,s,h(s))ds. \tag{17}$$

For proving the uniqueness of h, it is enough to show that h is a fixed point of Λ and $h \in M^*$. Using (15), we have $\Lambda h = h$. Also form (15) and the fact that $d(f,h) < \frac{1}{1-(L_1+L_2)L}$, we obtain

$$\begin{aligned} \|\Lambda f(t) - h(t)\| = \|g(t,f(t)) - g(t,h(t)) - \int_0^t K(t,s,f(s))ds+ \\ + \int_0^t K(t,s,h(s))ds\| \leq \\ \leq \frac{(L_1+L_2)}{1-(L_1+L_2)L}\phi(t), \end{aligned}$$

which implies that $d(\Lambda f, h) < \infty$. This completes the proof. □

In the sequel some examples and applications of our discussion are presented.

We recall that for a Banach space $\mathcal{X}$, a one-parameter family $\{\mathfrak{T}(t)\}_{t\geq 0}$ in $\mathcal{B}(\mathcal{X})$, the space of all bounded linear operators, is called a C_0-semigroup of operators if for all $s,t \geq 0$,

$$\mathfrak{T}(s+t) = \mathfrak{T}(s)\mathfrak{T}(t) \quad \text{and} \quad \mathfrak{T}(0) = \mathcal{I}(\text{ the identity operator}),$$

and for any $x \in X$, $\lim_{t\to 0^+} \mathfrak{T}(t)x = x$. In this case the operator $\mathfrak{A} : \mathfrak{D}(\mathfrak{A}) \subset \mathcal{X} \to \mathcal{X}$, where

$$\mathfrak{D}(\mathfrak{A}) = \left\{ x \in \mathcal{X} : \lim_{t\to 0^+} \frac{\mathfrak{T}(t)x - x}{t} \text{ exists} \right\},$$

defined by $\mathfrak{A}(x) = \lim_{t\to 0^+} \frac{\mathfrak{T}(t)x-x}{t}$ is called the infinitesimal generator $\{\mathfrak{T}(t)\}_{t\geq 0}$. One can see [21] or [7] for a comprehensive reference of semigroup of operators theory.

Example 1. *Let $\mathcal{X}$ be a Banach space, $\mathfrak{A} \in \mathcal{B}(\mathcal{X})$ with $\|\mathfrak{A}\| \leq 1$, $T \in (0,\infty)$ and $a(.) \in \mathcal{W}^{1,1}(\mathbb{R}_+, \mathbb{C})$, where $\mathcal{W}^{1,1}$ is the* Sobolev space. *Put $\mathfrak{T}(t) = e^{t\mathfrak{A}}$, $t \geq 0$. For any $f \in \mathcal{W}^{1,1}(\mathbb{R}_+, \mathcal{X})$, from Corollary 7.27* [7], *we know that there exists a unique solution $u \in \mathcal{C}^1(\mathbb{R}, \mathcal{X}) \cap \mathcal{C}(\mathbb{R}, \mathcal{X})$ satisfying the Volterra integrodifferential equation*

$$u'(t) = \mathfrak{A}u(t) + f(t) + \int_0^t a(t-s)\mathfrak{A}u(s)ds. \tag{18}$$

Now define $g : [0,T] \times \mathcal{X} \to \mathcal{X}$ and $K : [0,T] \times [0,T] \times \mathcal{X} \to \mathcal{X}$ by

$$g(t,x) = \mathfrak{A}x + f(t),\ K(t,s,x) = a(t-s)\mathfrak{A}x.$$

Trivially

$$\|g(t,x) - g(t,y)\| \leq \|\mathfrak{A}\|\|x-y\|,$$

and continuity of a implies that

$$\|K(t,s,x) - K(t,s,y)\| = |a(t-s)|\|\mathfrak{A}\|\|x-y\| \leq M\|\mathfrak{A}\|\|x-y\|,$$

for some $M > 0$. Suppose

$$0 < L < \frac{1-\|\mathfrak{A}\|}{\|\mathfrak{A}\|(1+M+MT)},$$

$\alpha \geq \frac{1}{L}$ and $\phi(t) = \rho e^{\alpha t}$, $\rho > 0$. If

$$\|u'(t) - \mathfrak{A}u(t) - f(t) - \int_0^t a(t-s)\mathfrak{A}u(s)ds\| \leq \phi(t),$$

for appropriate f and $a(.)$, then with $L_1 = \|\mathfrak{A}\|$, and $L_2 = M\|\mathfrak{A}\|$, by Theorem 2, there exists a unique solution $u_0(t)$ of (18) such that

$$\|u(t) - u_0(t)\| + \|u'(t) - u_0'(t)\| \leq \frac{1+L}{1-1-(L_1+(L_1+L_2)L+L_2TL)}\phi(t)$$

Thus we obtain the Hyers–Ulam stability of the equation 18.

Example 2. *Suppose $\mathcal{X}$ is a Banach space, $T \in (0,\infty)$ and $\{\mathfrak{T}(t)\}_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup of bounded linear operators with the infinitesimal generator $(\mathfrak{A}, D(\mathfrak{A}))$. Let $B \in C([0,T], \mathcal{B}(\mathcal{X}))$, the space of all continuous function from $[0,T]$ into $\mathcal{B}(\mathcal{X})$. For $x_0 \in D(\mathfrak{A})$, consider the integral equation*

$$u(t) = \mathfrak{T}(t)x_0 + \int_0^t \mathfrak{T}(t-s)B(s)u(s)ds,\ t \in [0,T]. \tag{19}$$

This equation has a solution (see 9.21 of [7]*). Define* $g : [0,T] \times \mathcal{X} \to \mathcal{X}$ *and* $K : [0,T] \times [0,T] \times \mathcal{X} \to \mathcal{X}$*, by*

$$g(t,x) = \mathfrak{T}(t)x_0, \ K(t,s,x) = \mathfrak{T}(t-s)B(s)x.$$

Trivially for any $x, y \in \mathcal{X}$*,* $\|g(t,x) - g(t,y)\| = 0$*. Also from Theorem I.2.2*[21]*, we know that there exist* $M, \omega > 0$*, such that for all* $t \geq 0$*,* $\|\mathfrak{T}(t)\| \leq Me^{tw}$*. On the other hand continuity of* $B : [0,T] \to \mathcal{B}(\mathcal{X})$ *implies that* $\|B(s)\| \leq M_0$*, for some* $M_0 > 0$ *and all* $s \in [0,T]$*. Thus*

$$\|K(t,s,x) - K(t,s,y)\| = \|\mathfrak{T}(t-s)B(s)(x-y)\| \leq MM_0e^{T\omega}\|x-y\|.$$

Now suppose $0 < L \leq \frac{1}{MM_0e^{T\omega}}$*. For a fixed* $\alpha \geq \frac{1}{L}$ *and* $\rho > 0$*, if* $\phi(t) = \rho e^{\alpha t}$*, then the conditions (13) hold. Thus if*

$$\|u(t) - \mathfrak{T}(t)x_0 - \int_0^t \mathfrak{T}(t-s)B(s)u(s)ds\| \leq \phi(t), t \in [0,T],$$

by Theorem 3, there exists a unique solution $u_0(t)$ *of (19) such that*

$$\|u(t) - u_0(t)\| \leq \frac{1}{1 - MM_0e^{T\omega}L}\phi(t).$$

Hence it conclude the Hyers-Ulam Stability of (19).

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Mohammad Janfada
Department of Pure Mathematics, Ferdowsi University of Mashhad
Mashhad, P.O. Box 1159-91775, Iran
E-mail: mjanfada@gmail.com

Gh. Sadeghi
Department of Mathematics, Hakim Sabzevary University
Sabzevar, P.O. Box 397, Iran
E-mail: ghadir54@yahoo.com