

A CLASSICAL APPROACH TO DYNAMICS OF PARABOLIC COMPETITIVE SYSTEMS

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Abstract. We study the reaction-diffusion system, its stationary solutions, the behavior of the system near them and discuss similarities and differences for different boundary conditions.

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1. INTRODUCTION

We deal with a nonlinear parabolic system of the form

$$(1) \quad u_t = D\Delta u + f(u),$$

with Neumann homogeneous boundary condition

$$(2) \quad \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega} = 0$$

or, incidentally, Dirichlet homogeneous boundary condition

$$(3) \quad u \Big|_{\partial\Omega} = 0,$$

in a bounded domain $\Omega \subset \mathbb{R}^m$ with boundary $\partial\Omega$ being $m - 1$ -dimensional sufficiently smooth manifold. Here, we consider only classical solutions $u : \overline{\Omega} \rightarrow \mathbb{R}^n$ (we emphasize that we have vector-valued functions, since (1) is, in fact, a system of n parabolic equations. We denote by $D := \text{diag}(d_1, \dots, d_n)$ a diagonal matrix with positive diagonal entries d_i , $i = 1, \dots, n$, and by $f : \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ a continuous function defined on the cone of nonnegative vectors $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, where $x_i \geq 0$, $i = 1, \dots, n$. Such parabolic systems are largely investigated since they model kinetics of chemical reactions – each coordinate of u measures density one of interacting components, or they model development of several biological species living on the same area Ω and interacting in different ways (preys and predators, symbiosis, competition). We keep in mind the second application that causes Neumann condition is more natural (comp. [2]). System (1-2) defines a semiflow Φ_t , $t \geq 0$, on an appropriate space X_α , if one uses the theory of sectorial operators (comp. [6, 9]) or a semiflow on a cone of nonnegative

continuous functions, if one applies the theory of monotone dynamical systems (comp. [18]). We work in spaces of continuous functions, since our main example of $f = (f_1, \dots, f_n)$ is

$$(4) \quad f_i(u) = u_i \left(1 - \sum_{j=1}^n a_{ij} u_j \right),$$

where all coefficients a_{ij} are positive. If f is a C^1 -function such that

$$f_i(u) = 0 \quad \text{for} \quad u_i = 0, \quad i = 1, \dots, n,$$

and

$$(5) \quad \frac{\partial f_i}{\partial u_j} \leq 0, \quad \text{for} \quad i \neq j,$$

then (1) defines a semiflow Φ_t , $t \geq 0$, on the cone of nonnegative continuous functions $u : \bar{\Omega} \rightarrow [0, \infty)^n$ which is competitive in the sense of Hirsch (see [18, 19]). In particular, it means that all solutions of the system starting with nonnegative functions $\varphi = u(\cdot, 0) \geq 0$ are global in time and nonnegative for any $t > 0$. Moreover, if

$$u(\cdot, T) \leq \bar{u}(\cdot, T)$$

are two such solutions comparable at any time $T > 0$, then

$$u(\cdot, t) \leq \bar{u}(\cdot, t) \quad \text{for} \quad t < T.$$

2. STEADY-STATE SOLUTIONS, SINGLE SPECIES

First, we are interested in steady-state solutions, i.e. time independent solutions. They satisfy elliptic system:

$$(6) \quad \Delta u = -D^{-1} f(u), \quad \frac{\partial u}{\partial \nu} |_{\partial \Omega} = 0.$$

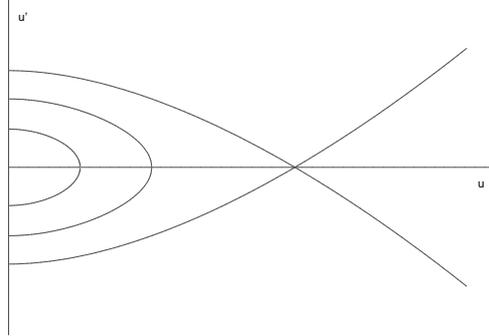
The obvious examples of such functions are zeros of the nonlinear term f . An important question is if there are not exist other steady-state solutions. They correspond nonuniform distributions of populations on the environment Ω which do not change in time. We can exactly investigate the case of one species ($n = 1$). If $\Omega \subset \mathbb{R}$, the analysis is standard and rather simple.

$$(7) \quad u'' = -D^{-1} u(1 - u) \quad \text{in} \quad (0, L), \quad u'(0) = 0 = u'(L), \quad u \geq 0.$$

The second order ODE is an example of a conservative system with one degree of freedom so it has a first integral (energy)

$$E(u, u') = \frac{u'^2}{2} + \frac{u^2}{2D} - \frac{u^3}{3D}.$$

It enables to find the phase portrait of the system:



One can see that there is no nonnegative solutions for any diffusion coefficient D and any length L of the environment. On the other hand, if one use the Dirichlet boundary condition $u(0) = 0 = u(L)$, then a priori such solutions can exist. They will correspond trajectories cutting axe u' twice. One can compute the time (we use standard notions from the theory of dynamical systems however, here, the independent variable is interpreted as a spatial one) between two passes of this axe. Denote (after [12]) by F the real function $F(u) = u^2/2 - u^3/3$ and let μ be the value of u where the trajectory cut axe u . Then this time equals

$$L(\mu) = \sqrt{2D} \int_0^\mu \frac{du}{\sqrt{F(\mu) - F(u)}} = \int_0^1 \frac{\mu dz}{\sqrt{F(\mu) - F(\mu z)}}.$$

It is obvious that μ changes between 0 and 1, that L is an increasing function of μ tending to ∞ as $\mu \rightarrow 1^-$. One can also compute the limit

$$\lim_{\mu \rightarrow 0^+} L(\mu) = 2\sqrt{D} \int_0^1 \frac{dz}{\sqrt{1 - z^2}} = \pi\sqrt{D}.$$

It can be interpreted that there is exactly one nonconstant steady-state solution iff the length of the environment L is greater than this limit $L_{KISS} = \pi\sqrt{D}$ called the KISS size. If $L < \pi\sqrt{D}$, then there are only constant steady-state solutions $u \equiv 0$ and $u \equiv 1$. From another point of view, if the length L is fixed, then nonconstant time-independent solutions exist when the diffusion coefficient D is sufficiently small.

Now, we study the case, when $\Omega \subset \mathbb{R}^m$ with $m > 1$. The most interesting dimension from the biological point of view is $m = 2$. $\Omega = B(0, R)$ (the disk centered at 0 with radius R) is the simplest set and we can easily look for radial solutions of (6). Our analysis can be repeated in larger dimension without troubles. If we denote by u' the derivative with respect to the radial

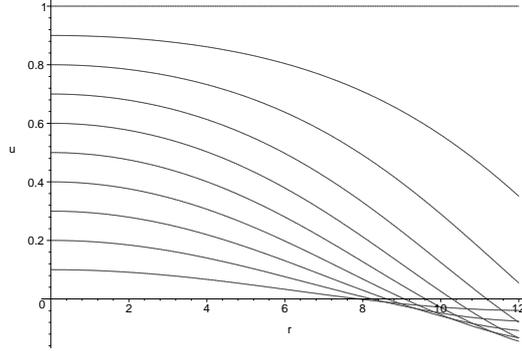
coordinate, then we get the following boundary value problem:

$$(8) \quad u'' + \frac{1}{r}u' = -\frac{1}{D}u(1-u) \quad \text{in } (0, R), \quad u'(0) = 0 = u'(R).$$

For Dirichlet's boundary condition $u|_{\partial\Omega} = 0$, we have for radial solutions: $u'(0) = 0 = u'(R)$. For both problems, we look at solutions of the second order ODE with initial values $u(0) = c$, $u'(0) = 0$, where $c > 0$. It is easy to see that for $c > 1$ the right hand side of ODE (8) is positive, thus the solution increase to infinity in finite time and cannot satisfy the Dirichlet nor Neumann conditions at R . If $c \in (0, 1)$, then the solution is concave near 0, u decreases but, by comparison with equation (7), slower then in the last system. It means that, for any c , $u(r_1) = 0$ when $r_1 > L_{KISS}/2$. Thus, for Dirichlet's boundary condition, we have the same phenomenon as in dimension 1, but with larger KISS size. Similarly, there is no nonconstant solutions for Neumann's problem. Below, we present numerical solutions for initial problems

$$u'' + \frac{1}{r}u' = -\frac{1}{D}u(1-u), \quad u(0) = c, \quad u'(0) = 0$$

with several $c > 0$ and $D = 0.1$.



3. STEADY-STATE SOLUTIONS, TWO SPECIES

For two species the situation is completely different. Consider the system

$$(9) \quad \begin{cases} u_t = d_1 \Delta u + u(1-u-bv), & \frac{\partial u}{\partial \nu} |_{\partial\Omega} = 0, & u(\cdot, 0) = \varphi \\ v_t = d_2 \Delta v + v(1-cu-v), & \frac{\partial v}{\partial \nu} |_{\partial\Omega} = 0, & v(\cdot, 0) = \psi, \end{cases}$$

where interspecies-competition coefficients b, c are positive constants, diffusion coefficients d_1, d_2 are positive and Ω is an open bounded set in \mathbb{R}^m with

smooth boundary. We study only nonnegative solutions. There are always three equilibria constant in space and time:

$$P_0 = (0, 0), \quad P_u = (1, 0), \quad P_v = (0, 1),$$

and if $b, c < 1$ or $b, c > 1$, the fourth equilibrium

$$P_1 = \left(\frac{1-b}{1-bc}, \frac{1-c}{1-bc} \right).$$

If $d_1 = d_2 = 0$, there is no diffusion and we have the standard Lotka-Volterra ODE with the above equilibria. The dynamical system given by this ODE is easily investigated: P_0 is its repeller, P_u is an attractor if $c > 1$, P_v is an attractor if $b > 1$, P_1 is global attractor (for nonnegative solutions) if $b, c < 1$. More exactly, if both b and c are greater than 1, then P_1 is a saddle point and its stable manifold $W^s(P_1)$ is the sum of a heteroclinic trajectory from P_0 to P_1 and a trajectory from infinity to P_1 . This manifold cuts the set $u, v \geq 0$ in two sets: all trajectories starting from the set containing P_u tend to P_u , trajectories starting from the second set tend to P_v . The method of monotone dynamical systems enables us to state a similar result for full parabolic system (9):

(i) if $b, c < 1$, then all solutions with nontrivial $\varphi, \psi \geq 0$ tend to P_1 as $t \rightarrow \infty$;

(ii) if $b < 1$ and $c > 1$, then all solutions with nontrivial $\varphi, \psi \geq 0$ tend to P_u as $t \rightarrow \infty$;

(iii) if $b > 1$ and $c < 1$, then all solutions with nontrivial $\varphi, \psi \geq 0$ tend to P_v as $t \rightarrow \infty$;

(iv) if $b, c > 1$, then the stable manifold $W^s(P_1)$ is again one-dimensional, P_u attracts solutions with $\varphi \geq u_0, \psi \leq v_0$ for some $(u_0, v_0) \in W^s(P_1)$, and P_v attracts solutions with $\varphi \leq u_0, \psi \geq v_0$ for some $(u_0, v_0) \in W^s(P_1)$.

The proof can be found in [19].

The last case is the most interesting since we do not know the dynamics for any (φ, ψ) not comparable in the above sense with any function from $W^s(P_1)$. For example, some nonconstant equilibria can exist and even they can be asymptotically stable. However, we can use the following result due to Conway, Hoff and Smöller [3] in many cases:

Let u be a solution to

$$u_t = D\Delta u + f(u), \quad \frac{\partial u}{\partial \nu} |_{\partial\Omega} = 0, \quad u(\cdot, 0) = u_0,$$

where $u = (u_1, \dots, u_m)$, D is a symmetric positively definite matrix, Ω is an open bounded subset of \mathbb{R}^n with smooth boundary, f is a C^2 function and $u_0 \in L^2(\Omega)$. Assume that there exists a bounded positively invariant set $\Sigma \subset \mathbb{R}^m$ such that u_0 takes values in Σ . Denote by λ the first eigenvalue

of $-\Delta$ with $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$ (here, one can take $m = 1$, and we mean the first positive eigenvalue – 0 is also eigenvalue), by d the lowest eigenvalue of matrix D , $M := \sup_{u \in \Sigma} \|f'(u)\|$, and at last

$$\sigma := \lambda d - M.$$

Then there are four positive constants c_i , $i = 1, 2, 3, 4$ such that

$$(10) \quad \|\nabla u(\cdot, t)\|_{L^2(\Omega)} \leq c_1 e^{-\sigma t},$$

$$(11) \quad \|u(\cdot, t) - \bar{u}(t)\|_{L^2(\Omega)} \leq c_2 e^{-\sigma t},$$

where \bar{u} is a spatial average of u and it satisfies $\bar{u}' = f(\bar{u}) + g(t)$, $\bar{u}(0) = \int_{\Omega} u_0 / \mu(\Omega)$, g is a function satisfying $|g(t)| \leq c_3 e^{-\sigma t}$,

$$(12) \quad \|u(\cdot, t) - \bar{u}(t)\|_{L^\infty(\Omega)} \leq c_4 e^{-\sigma t}$$

if D is a diagonal matrix.

In our investigations, $\Sigma = \{(u, v) : u, v \geq 0, u \leq 1 - bv \text{ or } v \leq 1 - cu\}$ if (φ, ψ) takes values in this set or Σ is the smallest triangle with vortices P_0 , $(a, 0)$, $(0, a)$ containing $(\sup \varphi, \sup \psi)$. Since the first set attracts all solutions, it contains all equilibria. If one has $\sigma > 0$, then $\lim_{t \rightarrow \infty} \bar{u}(t) = \text{const}$ and we the solution $u(x, t)$ tends to this constant as $t \rightarrow \infty$. It follows that there is no nonconstant in space steady-state solution. After easy though laborious computations one can find the constant M . We have

$$f'(u, v) = \begin{bmatrix} 1 - 2u - bv & -bu \\ -cv & 1 - 2v - cu \end{bmatrix},$$

$\|f'(u, v)\|^2 = (b^2 + c^2 + 4)(u^2 + v^2) + 4(b + c)uv - 4(u + v) - 2(cu + bv) + 2$ and the maximum of the last function on the set Σ equals 2 (it is reached at the origin). Thus, we have obtained

Theorem 1. *If both diffusion constants d_1, d_2 are sufficiently large, namely*

$$\min(d_1, d_2) > \frac{\sqrt{2}}{\lambda}$$

where λ is the first positive eigenvalue for $-\Delta$ with Neumann homogeneous condition, then there is no nonconstant steady-state solution. For the case $\Omega = (0, L) \subset \mathbb{R}$ as in the previous section, we have $\lambda = \frac{\pi^2}{L^2}$ and we have no nonconstant equilibrium if

$$L < 2^{-1/4} \pi \min(d_1, d_2).$$

Compare this number with the KISS size from the previous section and notice that, here, a priori we have nonconstant equilibria for the Neumann boundary condition. There is a numerically studied example of Matano and Mimura [16] where Ω is a set in the plane consisting two squares joined by

a thin strip and $b, c > 1$ such that there is a nonconstant positive equilibrium. Moreover, it is asymptotically stable. If domain Ω is convex all equilibria are stable (see [11]), hence it is not surprising that in this example the domain is such. On the other hand, one can understand ecological sense of the shape of the domain: in the first square the first species wins, in the other the second one; the thin strip makes migrations between squares more difficult hence both species can coexist.

Most results from the theory of monotone dynamical systems one used concern systems in ordered Banach spaces called SOP in the monograph by H. Smith [18]. Competitive parabolic systems are SOP if we use order:

$$(u, v) \prec (\check{u}, \check{v}) \Leftrightarrow u(x) \leq \check{u}(x), \quad v(x) \geq \check{v}(x) \quad \text{for any } x.$$

The semiflow generated by (9) preserves this order, i.e. if $(\varphi, \psi) \prec (\check{\varphi}, \check{\psi})$, then $(u(\cdot, t), v(\cdot, t)) \prec (\check{u}(\cdot, t), \check{v}(\cdot, t))$ for any $t > 0$. It enables us to get some information on ω -limit sets of our system. This choice of the order is typical for competition of two species and can be explain qualitatively by ecological arguments: if the first species dominates the second one and one increases the population of the first species and decreases of the second one then the domination conserves.

4. STEADY-STATE SOLUTIONS, MANY SPECIES

For more than two species the situation is much more complicated. If $n = 3$ then, roughly speaking the first species can dominate the second one, the second one can dominate the third one and this last species can dominate the first one. The simplest mathematical model of this case is given by May and Leonard [17] for ODE:

$$(13) \quad \begin{cases} \dot{x} = x(1 - x - \alpha y - \beta z) \\ \dot{y} = y(1 - \beta x - y - \alpha z) \\ \dot{z} = z(1 - \alpha x - \beta y - z) \end{cases}$$

with $\alpha, \beta > 0$, $\alpha + \beta = 2$. The typical ω -limit set is a limit cycle but there are three heteroclinic trajectories joining three equilibria $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. We have studied a slightly more general case in [10] however the behavior of the system is very similar.

Consider the parabolic system for three species:

$$(14) \quad \begin{aligned} U_t &= D\Delta U + f(U), \\ \frac{\partial U}{\partial \nu} |_{\partial\Omega} &= 0, \\ U(\cdot, 0) &= \varphi, \end{aligned}$$

where $U = (u, v, w)$, $D = \text{diag}(d_1, d_2, d_3)$,

$$(15) \quad f(U) = \begin{bmatrix} u(1 - a_1u - b_1v - c_1w) \\ v(1 - a_2u - b_2v - c_2w) \\ w(1 - a_3u - b_3v - c_3w) \end{bmatrix},$$

all constants in the above formulas d_i , a_i , b_i , c_i are positive, $x \in \Omega \subset \mathbb{R}^m$.

We study the system (14) and its spatially homogeneous ODE system

$$(16) \quad U' = f(U)$$

in the open set

$$\mathcal{D} := \{(u, v, w) : u, v, w > 0\}$$

which is obviously invariant and the same is true for its closure $\overline{\mathcal{D}}$. Fixed points of the system (16) can be easily found – four of them always lie in $\overline{\mathcal{D}}$:

$$P_0 = (0, 0, 0), \quad P_u = (a_1^{-1}, 0, 0), \quad P_v = (0, b_2^{-1}, 0), \quad P_w = (0, 0, c_3^{-1}),$$

next four are

$$P_{uv} = \left(\frac{b_2 - b_1}{a_1b_2 - a_2b_1}, \frac{a_1 - a_2}{a_1b_2 - a_2b_1}, 0 \right), \quad P_{uw} = \left(\frac{c_3 - c_1}{a_1c_3 - a_3c_1}, 0, \frac{a_1 - a_3}{a_1c_3 - a_3c_1} \right),$$

$$P_{vw} = \left(0, \frac{c_3 - c_2}{b_2c_3 - b_3c_2}, \frac{b_2 - b_3}{b_2c_3 - b_3c_2} \right)$$

and $P_1 = (\alpha, \beta, \gamma)$ being the solution of the equation

$$M \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{where} \quad M = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

We assume that $W := \det M \neq 0$ what means that none of two of planes

$$H_i : \quad a_i u + b_i v + c_i w = 1, \quad i = 1, 2, 3$$

are parallel.

It is easy to see that $f'(P_0) = I$ and its unique eigenvalue 1 is positive, thus P_0 is a source for (16). Similarly

$$f'(P_u) = \begin{bmatrix} -1 & -b_1/a_1 & -c_1/a_1 \\ 0 & 1 - a_2/a_1 & 0 \\ 0 & 0 & 1 - a_3/a_1 \end{bmatrix}$$

and if $\max(a_2, a_3) < a_1$, then one eigenvalue -1 is negative and two remaining ones are positive. The corresponding eigenspaces are: $\{(u, 0, 0) : u \in \mathbb{R}\}$ and $\{(0, v, w) : v, w \in \mathbb{R}\}$. The stable manifold of P_u is 1-dimensional:

$$\{(u, 0, 0) : u > 0\}$$

and by the Stable and Unstable Manifold Theorem all trajectories of (16) except starting in the stable manifold cannot tend to P_u as $t \rightarrow +\infty$. Similar arguments work for P_v and P_w under conditions:

$$\max(b_1, b_3) < b_2, \quad \max(c_1, c_2) < c_3.$$

In [10], we proved even more:

Lemma 1. *If*

$$(17) \quad \min(a_2, a_3) < a_1, \quad \min(b_1, b_3) < b_2, \quad \min(c_1, c_2) < c_3,$$

then P_u, P_v and P_w do not belong to the ω -limit set $\omega(P)$ of any point $P \in \mathcal{D}$.

Dividing the set $\overline{\mathcal{D}}$ into three pieces:

$$\mathcal{D}_+ := \{(u, v, w) \in \overline{\mathcal{D}} : \min_i(a_i u + b_i v + c_i w) > 1\},$$

$$A := \{(u, v, w) \in \overline{\mathcal{D}} : \min_i(a_i u + b_i v + c_i w) \leq 1 \leq \max_i(a_i u + b_i v + c_i w)\},$$

$$\mathcal{D}_- := \{(u, v, w) \in \overline{\mathcal{D}} : \max_i(a_i u + b_i v + c_i w) < 1\}$$

(\mathcal{D}_+ (resp. \mathcal{D}_-) is the set of points sitting under (resp. over) all three planes $H_i, i = 1, 2, 3, A = \overline{\mathcal{D}} \setminus (\mathcal{D}_+ \cup \mathcal{D}_-)$) we got [10] following result for (16):

Lemma 2. *The set A is positively invariant and all trajectories in \mathcal{D} eventually come into A . Moreover, A contains any compact invariant set that contains no fixed points.*

Thus, $\omega(P) \subset A$ for any $P \in \mathcal{D}$. Notice that $P_u, P_v, P_w \in A$ and similarly P_{uv}, P_{uw}, P_{vw} if they belong to $\overline{\mathcal{D}}$.

The last fixed point $P_1 = (\alpha, \beta, \gamma)$ can lie in \mathcal{D} (and then in A) or outside this cone. The following theorem excludes the existence of a periodic trajectory for (16) in \mathcal{D} if $P_1 \in \mathcal{D}$.

Lemma 3 ([18], p. 44, Prop. 4.3). *Let Γ be a nontrivial periodic trajectory of a competitive system in $\mathcal{D} \subset \mathbb{R}^3$ and*

$$\Gamma \subset [p, q] := \{\xi : p_i \leq \xi_i \leq q_i, i = 1, 2, 3\} \subset \mathcal{D}.$$

Then the set K of all points x which are not related to any point $y \in \Gamma$ (relation $x \leq y$ means $x_i \leq y_i$ for any i) has two components, one of them is bounded and contains a fixed point.

Hence, if we want to have a nontrivial periodic trajectory, then P_1 must belong to \mathcal{D} and we have two options:

(i) $W > 0$ and three other determinants

$$W_u := \det \begin{bmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{bmatrix},$$

$$W_v := \det \begin{bmatrix} a_1 & 1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & 1 & c_3 \end{bmatrix},$$

$$W_w := \det \begin{bmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{bmatrix}$$

are positive or

(ii) $W < 0$ and the above three determinants are negative.

The main result of [10] is the following

Theorem 2. *Assume (17). Let all four determinants W, W_u, W_v, W_w be positive and $p := a_1\alpha + b_2\beta + c_3\gamma - 1 < 0$. Then, for any point $P \in \mathcal{D}$ that does not belong to the half-line*

$$\begin{cases} u = \alpha s, \\ v = \beta s, \\ w = \gamma s \end{cases} \quad s > 0,$$

the ω -limit set $\omega(P)$ for (16) is a periodic trajectory. For P from this half-line, $\omega(P) = P_1$.

If we combine this theorem with the result [3] cited in the previous section and a classical result of L. Markus [15] that ω -limit sets of (14) are the same as corresponding (16), we obtain

Theorem 3. *Suppose that inequalities (17) hold, four determinants W, W_u, W_v, W_w are positive, $p < 0$ and all diffusion coefficients d_i are sufficiently large, then for any continuous, nonnegative function φ with all coordinates nontrivial, either*

$$\lim_{t \rightarrow \infty} U(x, t) = P_1$$

or there exists a periodic function $\tilde{U} : \mathbb{R} \rightarrow A \subset \mathcal{D}$ such that

$$\lim_{t \rightarrow \infty} |U(x, t) - \tilde{U}(t)| = 0$$

and in both cases limits are uniform with respect to x . The first case takes place only if $\int_{\Omega} \varphi$ belongs to the half-line starting from the origin.

The condition for the diffusion coefficients is given by:

$$\lambda_1 \min\{d_i : i = 1, 2, 3\} > \sup\{\|f'(U)\| : U \in A\}$$

where λ_1 is the first positive eigenvalue of $-\Delta$ with Neumann's boundary condition, but numerical experiments show that the assertion of the above theorem holds also for essentially larger these coefficients and even if only one of them is great and two others are very small. However, for all coefficient being small there exists spatially heterogeneous steady-state solutions which

attracts other solutions. Obviously, these functions take values in positively invariant subset A .

If the number p defined in Theorem 2 is positive, then all eigenvalues of $f'(P_1)$ have negative real parts – calculations in [10] show that the characteristic polynomial equals

$$Q(\lambda) = (\lambda + 1)(\lambda^2 + p\lambda + \alpha\beta\gamma W).$$

It follows that

$$\lim_{t \rightarrow \infty} U(x, t) = P_1$$

uniformly in $x \in \Omega$ for any starting point φ as in the last theorem. The case $p = 0$ is difficult for investigations since the behavior of the system near P_1 cannot be found by the linearization; P_1 is not hyperbolic. In the special case from the paper of May and Leonard [17] $a_1 = b_2 = c_3 = 1$, $b_1 = c_2 = a_3$, $c_1 = a_2 = b_3$ if $p = 0$ there are two first integrals of (16) and trajectories can be found apparently. The whole triangle spanned by P_u , P_v and P_w is fulfilled by limit cycles around P_1 . For sufficiently small diffusion coefficients these limit cycles describe the asymptotic behavior of the system (14).

If P_1 does not belong to \mathcal{D} , then all solutions tends to one of the others steady-state solutions at least for small diffusion coefficients, when there is no nonconstant steady-state solutions due to the above mentioned arguments. It means that at least one of the species extincts.

5. SOME NUMERICAL SIMULATIONS

The most interesting case considered in the previous section is presented in the theorem: almost all solutions of (14) tend to periodic functions given by the system of ODEs (16). We can investigate numerically equation (14) with $\Omega = (0, 1) \subset \mathbb{R}$ and f of the form from the previous section. Put matrix M of competition coefficients

$$M = \begin{bmatrix} 2 & 1.1 & 3.1 \\ 3.1 & 2 & 0.9 \\ 0.95 & 2.9 & 2 \end{bmatrix}.$$

This choice ensure the assumptions of Theorem 3 hold. The first positive eigenvalue of this degenerate Laplacian equals $\lambda_1 = \pi^2$ and one can compute the maximal value of the norm of the derivative $\|f'(U)\|$ on the set A :

$$\|f'(U)\| \leq \sqrt{3}.$$

Thus the critical value of diffusion coefficients is $\tilde{d} := \frac{\sqrt{3}}{\pi^2}$. For $\min\{d_i : i = 1, 2, 3\} > \tilde{d}$, almost all trajectories tend to periodic, spatially constant functions. Since the theorem of Conway, Hoff and Smöller gives only the

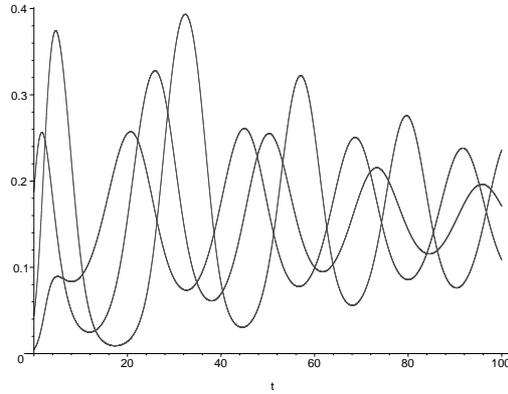
sufficient condition, it is not surprising that even for smaller d_i 's the assertion is true.

We have found numerically (by using Maple 10) solutions of (14) with the above matrix M and

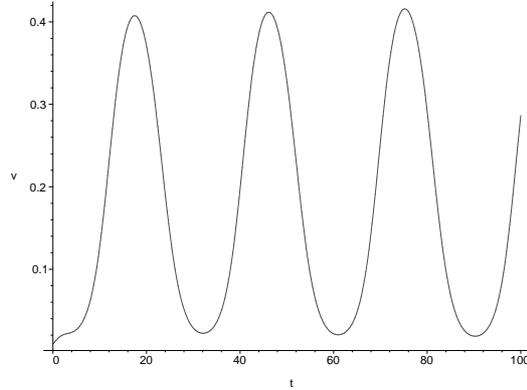
$$d_1 = 10^{-3}, \quad d_2 = 2 \cdot 10^{-3}, \quad d_3 = 0.5 \cdot 10^{-3}$$

$$\varphi(x) = [6x^2(1-x)^3, x^4(1-x)^2, 2x^3(1-x)^2].$$

The type of this initial function is natural if we are seeking classical solutions: the normal derivative of the initial function should vanish at both boundary points $x = 0, x = 1$. Below, we present the plot of the graph of the second coordinate of U for three values of x : 0.1, 0.5 and 0.9 as the function of time t for the range $[0, 100]$:



The plots for three values of x are completely different, hence the solutions of the parabolic system tend as $t \rightarrow \infty$ to functions which are not spatially constant. Nevertheless, they seem to be periodic as functions of time. Compare this plot with the plot of the second coordinate v of the solution to ODE (16) with initial point $(0.1, 0.0095238, 0.0333333)$ which is the spatial average of φ .



6. STABILITY OF STEADY-STATE SOLUTIONS

The stability analysis in the linear approximation for constant solutions of the parabolic system is standard (see, for example [20, 9]). Consider (14) not necessarily with U taking values in \mathbb{R}^3 but in \mathbb{R}^n . If P is a zero of f , then $U \equiv P$ is a solution both (14) and (16) and it is asymptotically stable for both systems again if all eigenvalues of $f'(P)$ have negative real parts. Below, we shall study the analogous problem for periodic solutions of (16).

Let $t \mapsto U(t)$ be such a solution. Denote by $A(t) = f'(U(t))$, by λ_k , $k = 0, 1, 2, \dots$, the sequence of all eigenvalues of $-\Delta$ with Neumann boundary conditions ($\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$) and by $x \mapsto e_k(x)$ corresponding eigenfunctions. Since the spectrum is discrete and the operator is self-adjoint, the eigenfunctions form a complete orthonormal system in $L^2(\Omega)$. One can solve the linearized problem

$$(18) \quad U_t = D\Delta U + A(t)U, \quad \frac{\partial U}{\partial \nu} \Big|_{\partial\Omega} = 0, \quad U(\cdot, 0) = \varphi$$

by using the Fourier method. If we put

$$U(x, t) = \sum_{k=0}^{\infty} e_k(x)g_k(t)$$

in (18), we have Neumann's condition satisfied and vector-valued function g_k should be a solution of the initial problem:

$$(19) \quad g'_k(t) = (-\lambda_k D + A(t))g_k(t), \quad g_k(0) = c_k$$

for any k , where c_k is a coefficient of the Fourier expansion of φ with respect to the orthonormal system $\{e_k : k = 0, 1, \dots\}$, i.e.

$$c_k = \int_{\Omega} \varphi e_k.$$

Since matrix D is diagonal, it commutes with $A(t)$ and multipliers (Floquet theory) of $-\lambda_k D + A(\cdot)$ are of the form $\exp(-\lambda_k d)\varrho$, where d belongs to the interval $[\min\{d_i\}, \max\{d_i\}]$ and ϱ is a multiplier for matrix $A(\cdot)$. Hence g_k decays exponentially as t tends to $+\infty$, if $|\varrho| < \exp(\lambda_k \min\{d_i\})$ for any multiplier ϱ . It is well known that one of multipliers for A equals 1, thus this inequality cannot hold for $k = 0$ as $\lambda_0 = 0$. But, in spite of this, if the remaining multipliers of A sit in the open unit disc and 1 is simple, then solution U of (14) is orbitally asymptotically stable – see [20] Theorem 8.2.3, p.251, i.e. there exists a neighborhood of the periodic orbit $\Gamma := \{U(t) : t \in \mathbb{R}\}$ such that solutions U_1 with initial function φ taking values in this nhbd tend to U in the sense

$$\lim_{t \rightarrow +\infty} \|U_1(\cdot, t) - U(t + t_0)\| = 0,$$

where the above norm means usual one in H^1 and t_0 is a number depending on φ called asymptotic phase. For 1-dimensional domains Ω , one has the canonical embedding of H^1 and the above limit is uniform in x . For more natural environments ($\Omega \subset \mathbb{R}^2$), we cannot use this argument to get the uniform limit.

7. CONCLUDING REMARKS

Recently, a lot of important papers on spatial heterogeneity models of competing species [4, 5, 7, 8, 13, 14], for instance. They consider mutual interplay of diffusion and competition which gives many interesting phenomena for such systems. However, most of results are obtained for two species but spatial heterogeneity is included in the model – some coefficients depend on variable x . It seems that some new effects can be obtained if we study interaction of more than two species. The present author hope that some possible directions of such investigations have been indicated above.

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