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ON COMPACT SETS OF COMPACT OPERATORS ON BANACH SPACES NOT CONTAINING A COPY OF l^1

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Abstract. F. Galaz-Fontes (Proc. AMS., 1998) has established a criterion for a subset of the space of compact linear operators from a reflexive and separable space X into a Banach space Y to be compact. F. Mayoral (Proc. AMS., 2000) has extended this criterion to the case of Banach spaces not containing a copy of l^1. The purpose of this note is to give a new proof of the result of F. Mayoral. In our proof, we use l^∞-spaces, a well known result of H. P. Rosenthal and L.E. Dor which characterizes the spaces without a copy of l^1 and a recent result obtained by G. Nagy in 2007 concerining compact sets in normed spaces. We point out that another proof of Mayoral's result was given by E. Serrano, C. Pineiro and J.M. Delgado (Proc. AMS., 2006) by using a different method.

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1. Introduction

Throughout this work, X, Y will be two (real or complex) Banach spaces. $\|.\|$ will denote both the norm on the Banach space X and on the Banach space Y. The Banach space of all linear bounded operators from X to Y will be denoted by $\mathcal{L}(X, Y)$ and will be equipped with its usual norm. We denote the space of all compact linear operators by $\mathcal{K}(X, Y)$. It is a closed subspace of $\mathcal{L}(X, Y)$.

As in [6] and [7], we are interested by the following question: How to characterize the compact subsets of $\mathcal{K}(X, Y)$?. This question has been discussed and studied by many authors in Functional analysis (see for example [12], [1] , [9]). In [12], K. Vala has discussed this question in a general context. In [6], assuming X is reflexive and separable, F. Galaz-Fontes has given another type of criterion for a set $\mathcal{M} \subset \mathcal{K}(X, Y)$ to be compact. In [7], F. Mayoral has proved that the criterion of F. Galaz-Fontes is also valid when the Banach space X does not contain a copy of l^1.

E. Serrano, C. Pineiro and J.M. Delgado ([11]) have given another proof of Mayoral's result by using a different method.

The main result of [7] reads as follows.

Theorem 1. *Let X, Y be Banach spaces, where X does not contain an isomorphic copy of l^1, and suppose that $\mathcal{M} \subset \mathcal{K}(X, Y)$ is a given subset. Then the following conditions are equivalent:*
(1) *$\mathcal{M}$ is relatively compact in the operator norm.*
(2) *$\mathcal{M}$ satisfies the following conditions:*
(*a*) *The sets $\{Tx : T \in \mathcal{M}\}$ are relativeley compact in Y for each $x \in X$, and*
(*b*) $\lim_{n\to\infty} (\sup_{T\in\mathcal{M}} \|Tx_n\|) = 0$, *whenever $(x_n)_n$ is a weak-null sequence in X (that is, the set $\mathcal{M}$ is sequentially uniformly w-continuous).*

The aim of this note is to give a new proof of Theorem 1.1. The methods used here are different from those used in [6], [7] and [11]. We recall that the proof in [7] is based on a version of Arzela-Ascoli theorem in uniform spaces, a result due to Palmer [9] and Anselone [1] and a result of H. P. Rosenthal and L. E. Dor (see [10] and [4]).

The tools used here in this paper are essentially: an observation (see Lemma 2.3) wich is easily derived from the result of H. P. Rosenthal and L. E. Dor (see [10] and [4]), a general result obtained by G. Nagy (see [8]) concerning compact subsets of normed spaces and standard arguments from classical analysis related to compact sets in complete metric spaces.

2. Recalls and Preliminaries

Throughout this paper, $\|.\|$ will denote both the norm on the Banach space X and on the Banach space Y and $B_X = \{x \in X : \|x\| \leq 1\}$ the closed unit ball of X. If a sequence $\{x_n\}$ in X is weakly convergent to $x \in X$, then we denote $x_n \xrightarrow{w} x$.

A linear operator $T : X \to Y$ is compact if, for each bounded sequence $\{x_n\}$ in X, there is a (norm) converging subsequence $\{Tx_{n_k}\}$ in Y.

As usual, if (M, d) is a metric space, then $A \subset M$ is said to be relatively compact if its closure is a compact set. When the metric space (M, d) is complete, we know that a subset $A \subset M$ is relatively compact if and only if A is precompact.

One helpful characterization of pre-compactness for sets in a complete metric space is the following well-known criterion, which we recall without proof.

Proposition 1. *Let (Y, d) be a complete metric space. For a subset $M \subset Y$, the following are equivalent:*
(i) M is relatively compact in Y, i.e. its closure $\overline{M}$ in Y is compact;
(ii) M contains no infinite sequence (t_n), satisfying

$$d(t_n, t_m) > \rho, \quad \forall n, m, \quad \textit{with} \quad n \neq m,$$

for some positive number ρ.

The following proposition is proved in [8]. It will be used in the proof of our result.

Proposition 2. *Let X be a (real or complex) normed vector space, let $S \subset X$ be a compact subset, and let B_{X^*} be the closed unit ball in X^* Ũ the (topological) dual of X Ũ equipped with the w^*Ũtopology. If we consider the Banach algebra $\mathcal{A} = \mathcal{C}(S)$, equipped with the uniform topology, then the restriction map $\Theta : (B_{X^*}, w^*) \ni \phi \longmapsto \phi|_S \in (\mathcal{A}, \|.\|)$ is continuous. In particular, the set $\Theta(B_{X^*})$ is compact in $\mathcal{A}$.*

Note that if X does not contain a copy of l^1, then by applying Rosenthal-Dor Theorem (see [10] and [4] or ([3] , Ch. IX)) every bounded sequence in X has a weakly-Cauchy subsequence and, therefore, a bounded linear operator $T : X \to Y$ is compact if and only if it is completely continuous, that is, if and only if it takes weakly convergent sequences in X to convergent ones in Y.

By using Proposition 2.1 and the Rosenthal-Dor Theorem, it is easy to show the following lemma.

Lemma 1. *Let X be a Banach space without a copy of l^1 and let Y be another Banach space. Let $\psi : X \to Y$ be a continuous linear mapping. Suppose that $\psi : B_X \to Y$ is sequentially weak-norm continuous. Then the closure $\overline{\psi(B_X)}$ of $\psi(B_X)$ is precompact in Y.*

3. A new proof of Theorem 1.1

$\mathbb{N}$ is the set on non-negative integers, and $\mathbb{K}$ will designate the field of real or complex numbers.

Let X, Y be Banach spaces (over $\mathbb{K}$), where X does not contain an isomorphic copy of l^1, and suppose that $\mathcal{M} \subset \mathcal{K}(X, Y)$ is a given subset.

3.1. Let us prove that $(1) \Rightarrow (2)$.

The property (a) is evident, since for every $x \in X$, the mapping $T \mapsto Tx$ is continuous from $\mathcal{K}(X, Y)$ to Y.

To establish the property (b), let $\epsilon > 0$ be given and let $\{x_n\}$ be a sequence in B_X converging weakly to zero. Then, by using the compactness of $\mathcal{M}$, we can find a finite set of operators $\{T_1, ..., T_n\}$ in $\mathcal{M}$ $(n \geq 1)$ such that for every $T \in \mathcal{M}$, there exists $k \in \{1, \ldots, n\}$ with $\|T - T_k\| \leq \frac{\epsilon}{2}$. Since the operators $T_1, \ldots, T_n$ are compact, there exists an integer N_ϵ such that

$$\|T_k(x_n)\| \leq \frac{\epsilon}{2}, \quad \text{for all} \quad n \geq N_\epsilon, \quad \text{and all} \quad k \in \{1, \ldots, n\}. \tag{3.1}$$

Now we are done, since if we start with some arbitrary $T \in \mathcal{M}$, and we choose $k \in \{1, \ldots, n\}$ such that $\|T - T_k\| \leq \frac{\epsilon}{2}$, then by using (3.1), we get

$$\|T(x_n)\| \leq \|T(x_n) - T_k(x_n)\| + \|T_k(x_n)\| \leq \|T - T_k\|\|x_n\| + \frac{\epsilon}{2} \leq \epsilon, \tag{3.2}$$

for all intger $n \geq N_\epsilon$. Thus (b) is satisfied.

3.2. Let us prove that (2) $\Rightarrow$ (1).

To prove this implication, we are going to argue by contradiction. So we assume the existence of some $\rho > 0$ and an infinite sequence $\{T_n\}$ in the set $\mathcal{M}$ such that

$$\|T_n - T_m\| > \rho, \quad \forall n, m, \quad \text{with} \quad n \neq m,$$

We consider $\mathcal{Y} := l^\infty(Y)$ the Banach space of bounded Y-valued sequences,

$$l^\infty(Y) := \{\tilde{y} = (y_n) : \, y_n \in Y, \text{ for all}\, n \in \mathbb{N}, \text{ and } \sup_n \|y_n\| < \infty\},$$

endowed with the norm of uniform convergence.

We consider the mapping Ψ defined from X to $\mathcal{X}$ by the following.

$$\Psi : X \longrightarrow \mathcal{Y}$$
$$x \longmapsto \Psi(x) := (T_n(x))_n.$$

By (a) the mapping Ψ is well defined. It is clear that Ψ is linear. By the uniform boundedness principle, one can see that Ψ is norm continuous from the Banach space $(X, \|.\|)$ to the Banach space $(\mathcal{Y}, \|.\|_\infty)$

By the assumption (b), the map $\Psi : B_X \longrightarrow \mathcal{Y}$ is sequentially weak-norm continuous. We set $Q := \psi(B_X)$. Since X does not contain a copy of l^1 then by using the Rosenthal-Dor Theorem (or see Lemma 2.3 above), Q is precompact. So the norm closure $\mathcal{S} := \overline{Q}$ of Q is compact in the Banach space $\mathcal{Y}$. Now, we apply Proposition 2.2 to the compact $\mathcal{S}$ in the Banach space $\mathcal{Y}$. We denote $\mathcal{B} := B_{\mathcal{Y}^*}$ the closed unit ball of $\mathcal{Y}^*$.

We denote $(C(\mathcal{S}, \mathbb{K}), \|.\|_\infty)$ the Banach space of continuous functions from $\mathcal{S}$ to the scalar field $\mathbb{K}$, endowed with the uniform norm.

By Proposition 2.2, the restriction map

$$\Theta : (\mathcal{B}, w^*) \longrightarrow (C(\mathcal{S}, \mathbb{K}), \|.\|_\infty)$$
$$\sigma \longmapsto \Theta(\sigma) := \sigma \mid_{\mathcal{S}} .$$

is continuous. By Alaoglu's Theorem, we know that $(\mathcal{B}, w^*)$ is compact under the w^*-topology, therefore the range $\Theta(\mathcal{B})$ of $\mathcal{B}$ is compact in $C(\mathcal{S}, \mathbb{K})$.

Let φ be any linear continuous functional in B_{Y^*} (the closed unit ball of the topological dual space Y^* of Y) and let n be any arbitrary non-negative integer. We consider the map $\sigma_{n,\phi}$ defined by the following.

$$\sigma_{n,\phi} : \mathcal{Y} \longrightarrow \mathcal{Y}$$
$$\tilde{y} = (y_m)_m \longmapsto \varphi(y_n).$$

Then $\sigma_{n,\varphi}$ is linear, moreover for all $\tilde{y} = (y_m)_m \in \mathcal{Y}$, we have

$$\mid \sigma_{n,\varphi}(\tilde{y}) \mid = \mid \varphi(y_n) \mid \leq \|y_n\| \leq \|\tilde{y}\|_\infty.$$

Therefore $\sigma_{n,\varphi}$ belongs to the set $\mathcal{B}$.

We observe that for all $x \in B_X$, we have

$$\sigma_{n,\varphi}(\Psi(x)) = \varphi(T_n(x)).$$

We conclude that the set of maps

$$\{\sigma_{n,\varphi} \mid_{\mathcal{S}} : \ n \in \mathbb{N}, \varphi \in B_{Y^*}\}$$

is precompact in $(C(\mathcal{S}, \mathbb{K}), \|.\|_\infty)$.

Now, if we take two non-negative integers n, m with $n \neq m$, then, by assumption, we have $\|T_n - T_m\| > \rho$. We can find at least an element $x \in B_X$ such that $\|T_n(x) - T_m(x)\| > \rho$. We can also find at least a continuous functional $\omega \in B_{Y^*}$ such that

$$\mid \omega(T_n(x)) - \omega(T_m(x)) \mid > \rho.$$

Therefore, we have

$$\|\sigma_{n,\omega} \mid_{\mathcal{S}} - \sigma_{n,\omega} \mid_{\mathcal{S}} \| > \rho \quad \forall n, m, \quad \text{with} \quad n \neq m, \tag{3.4}$$

(3.4) shows that the set $\{\sigma_{n,\omega} \mid_{\mathcal{S}} : \ n \in \mathbb{N}\}$ is an infinite ρ-discrete subset of the compact subset $\Theta(\mathcal{B})$ of the Banach space $(C(\mathcal{S}, \mathbb{K}), \|.\|_\infty)$, which is a contradiction. This completes the proof of Theorem 1.1.

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