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Chapter 9
Topological similarity of functions

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9.1 Introduction

In this text we will work with several topological notions that enable us to compare the behavior of functions. So-called relations of continuity and relations of constancy were introduced for the first time in [8]. We show on examples that these relations between functions occur naturally. This approach allows not only to compare functions, but also to generalize some "continuity-preserving" theorems and to generate new ones. Sometimes it allows to replace differentiation by a simpler procedure - manipulation with inequalities - that can be used to examine nondifferentiable functions too. As the reader will see later, comparing functions in this way yields also new insight into the structure of some function spaces. Optimization applications of this new approach will be shown as well.

The most results presented here come from our articles [8], [9], we have slightly improved some of them. A few results are new (e.g. Lemma 9.1). If we use results of other authors, we always cite their works.

9.2 Motivation

New notions mentioned above are defined in the next section. Before defining them, we want the reader to know that these notions did not come just out of blue. Therefore we think it is important to show some preliminary examples of behavior of functions, known by everybody and relevant to our case.

Let us consider the following simple situation. We have two continuous real functions of real variable - f and g - and we are counting a limit of their quotient applying the L'Hospital's rule. Suppose $a \in \mathbb{R}$, f and g have a finite derivative on open intervals $(a-\varepsilon, a)$ and $(a, a+\varepsilon)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ is true. Suppose we obtain

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{t \to a} \frac{f'(t)}{g'(t)} = 3.$$

The equalities above imply that there exists an interval $I = (a-\delta, a+\delta)$ such that $1 < \frac{f'(t)}{g'(t)} < 4$ for all t from $I \setminus \{a\}$. Now we can observe that f and g behave "similarly" on the interval I. Indeed, since none of the derivatives equals zero for a t from $I \setminus \{a\}$, they are both positive or both negative on $(a-\delta, a)$ (and on $(a, a+\delta)$). There are four possibilities for f and g:

- f and g are increasing on I;
- f and g are decreasing on I;
- f and g are both increasing on $(a-\delta, a)$ and decreasing on $(a, a+\delta)$ so they have both a local maximum at a;
- f and g are both decreasing on $(a-\delta, a)$ and increasing on $(a, a+\delta)$ so they have both a local minimum at a.

Could we obtain an information like this without using derivatives? Well, we could argue, that f and g behave similarly because we know that for every b from $I \setminus \{a\}$ the following is true

(*) $1 < \frac{f(b)-f(a)}{g(b)-g(a)} < 4$

Of course, in this particular case we know this is true because we have used the Cauchy mean value theorem ($\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ for a c from $I \setminus \{a\}$). But we can see, that if for all b from $I \setminus \{a\}$ the inequalities from (*) are satisfied, we need not f and g to be differentiable to realize that if f has a local extremum at a then g has a local extremum at a too and vice versa.

Let us consider two more contexts and two other types of inequalities that can assure that two functions behave "similarly".

(I) f and g are real valued functions defined on an arbitrary set X. Suppose there exist two positive constants K and L such that for all points x and y from a set $A \subset X$ if $x \neq y$ then

$$K < \frac{|f(x) - f(y)|}{|g(x) - g(y)|} < L.$$

(II) f and g are defined on a set X. The function f has values in a metric space (Y, d) and the function g has values in a metric space (Z, ρ). Suppose there exist two positive constants K and L such that for all points x and y from a set $A \subset X$,

$$d(f(x), f(y)) < K \cdot \rho(g(x), g(y)) < L \cdot d(f(x), f(y)).$$

If one of the situations described above takes place, we can for example conclude, that if f is bounded on A, so is g. If the set X was endowed with a topology and we would have $X = A$, we could do some predictions about the continuity of f just observing whether g is continuous at a certain point.

The examples mentioned above serve only as a motivation for us. In this paper we examine more general relations between functions. These relations will be described in a purely topological way. But the reader will be able to see that relations and inequalities shown above represent special cases of a more general topological phenomenon.

9.3 Basic notions

In what follows we will use these notions, concerning topological spaces and functions: a net of points, a limit of a net, a net of functions, uniform convergence, pointwise convergence (see e. g. [5] or [6]).

First we introduce the notion of the continuous similarity. The definition was introduced in our article [8] and here we take the liberty to replace the original cumbersome expression " the degree of continuity of g at x is greater or equal than the degree of continuity of f at x " by a simpler expression "g is f-continuous at the point x".

Definition 9.1. Let X, Y, Z be topological spaces, let $f\colon X \to Y$, $g\colon X \to Z$ be functions.

(a) Let x be from X. We say that g is f-continuous at x if for every net $\{x_\gamma\}_{\gamma \in \Gamma}$ of elements from X converging to x the following holds:

If the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y then the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z. We denote this by $c_t^x(g) \geq c_t^x(f)$.

Let A be a subset of X. We say that g is f-continuous on A if for every x from A, $c_t^x(g) \geq c_t^x(f)$ is true. We denote this by $c_t^A(g) \geq c_t^A(f)$. Of course, for a particular x, the expressions $c_t^x(g) \geq c_t^x(f)$ and $c_t^{\{x\}}(g) \geq c_t^{\{x\}}(f)$ describe the same situation. When $c_t^X(g) \geq c_t^X(f)$ is true, we write simply $c_t(g) \geq c_t(f)$ and we say that g is f-continuous.

Let A be a subset of X. We say that f and g are continuously similar on A if $c_t^A(g) \geq c_t^A(f)$ and $c_t^A(f) \geq c_t^A(g)$ is true at the same time. We denote this situation by writing $c_t^A(g) = c_t^A(f)$. If $A = X$ we write also $c_t(g) = c_t(f)$ or, for the sake of simplicity, $f \sim g$ and we say that f and g are continuously similar.

(Of course, the f-continuity of a function g per se does not guarantee, that g will automatically have all nice properties of f. Considering any type of generalized continuity of f, it must be always examined and proven, whether this type of continuity will be inherited by g or not.)

Remark 9.1. To sum up, $f \sim g$ means that for every convergent net $\{x_\gamma\}_{\gamma\in\Gamma}$ from X the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y if and only if the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z.

In general, we can see immediately that if f is continuous on a subset A of X and $c_t^A(g) \geq c_t^A(f)$ is true, then g is continuous on A too. And if g is not continuous at a point x from X and $c_t^x(g) \geq c_t^x(f)$ is true, then f is not continuous at x.

Now the second notion of similarity of functions will be defined. In [8] the notion was defined in a way, that the domain X of f and g was supposed to be a topological space. But the topological structure on X was not used in the definition. Therefore, we will suppose, that X is simply a set.

Definition 9.2. Let X be a nonempty set, let Y, Z be topological spaces. Let $f\colon X \to Y,\ g\colon X \to Z$ be functions.

Let A be a subset of X. We say that g is f-constant on A if for every net $\{x_\gamma\}_{\gamma\in\Gamma}$ of elements from A the following holds

If the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y then the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z. We denote this by $c_s^A(g) \geq c_s^A(f)$. If $A = X$ we write also $c_s(g) \geq c_s(f)$.

Let A be a subset of X. We say that f and g are strongly similar on A if $c_s^A(g) \geq c_s^A(f)$ and $c_s^A(f) \geq c_s^A(g)$ is true at the same time. We denote this situation by writing $c_s^A(g) = c_s^A(f)$. If $A = X$ we write also $c_s(g) = c_s(f)$ or, for the sake of simplicity, $f \approx g$ and we say that f and g are strongly similar.

Remark 9.2. To sum up, $f \approx g$ means that for every net $\{x_\gamma\}_{\gamma\in\Gamma}$ from X $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y if and only if the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z. In our first definition we considered only some nets $\{x_\gamma\}_{\gamma\in\Gamma}$ from X, namely the convergent ones. Now, the same relationship between f and g must be verified - for all nets from X.

In particular, we can see that if X is a topological space and if the function f has values in a complete metric space (Y,d) and g has values in a complete metric space (Z,ρ) and if there exist two positive constants K and L such that for all points t, s from an open neighborhood of x (for all points t and s from a set A),

$$d(f(t),f(s)) \leq K\cdot\rho(g(t),g(s)) \leq L\cdot d(f(t),f(s))$$

is true, then f and g are continuously similar at x (f and g are strongly similar on A).

The following example should help the reader to get a first insight into the new notions.

Example 9.1. a) Let $Y=[0,1]$, $X=Z=(0,1)$. Define $f\colon X\to Y$ and $g\colon X\to Z$ by

for all x from X, $f(x)=x$ and $g(x)=x$.

Although f and g are very similar (only Y and Z differ a little bit), we can see that they are not strongly similar. Indeed, the net $\{f(\frac{1}{n})\}_{n\in\mathbb{N}}$ converges but the net $\{g(\frac{1}{n})\}_{n\in\mathbb{N}}$ does not converge. Only the relation $c_s(f)\geq c_s(g)$ is true. It is easy to check that $f\sim g$ - i. e. f and g are continuously similar.

b) When X,Y,Z are arbitrary topological spaces and $f\colon X\to Y$ and $g\colon X\to Z$ are functions, if g is continuous at a point x from X then $c_t^x(g)\geq c_t^x(f)$ is true. If both f and g are continuous on X, then we can see that $f\sim g$ is true.

c) When X,Y,Z are arbitrary topological spaces and $f\colon X\to Y$ and $g\colon X\to Z$ are functions, if f is constant on X, then $c_s(f)\geq c_s(g)$ holds. If both f and g are constant then $f\approx g$ is true.

Now we present a simple application of the previous notions. First we need the definition of a subcontinuous function. Subcontinuity has been studied by many authors for almost 40 years (see e. g. [7], [16]). A function from a topological space X to a topological space Y is called subcontinuous at x from X if for every net $\{x_\gamma\}_{\gamma\in\Gamma}$ converging to x there is a convergent subnet of $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ ([7]). Now, it is easy to prove the following lemma:

Lemma 9.1. *Let X, Y, Z topological spaces. Let $g\colon X\to Y$, $f\colon X\to Z$ be functions. Let f be subcontinuous at a point x from X. If g is f-continuous at x, g is subcontinuous at x too.*

Proof. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a net converging to x. Since f is subcontinuous at x, there is a convergent subnet of $\{f(x_\gamma)\}_{\gamma\in\Gamma}$. This is equivalent to the fact that for a certain subnet $\{x_\alpha\}_{\alpha\in A}$ of $\{x_\gamma\}_{\gamma\in\Gamma}$ the net $\{f(x_\alpha)\}_{\alpha\in A}$ converges in Z. Since g is f-continuous at x, the net $\{g(x_\alpha)\}_{\alpha\in A}$ converges (in Y) too. But this means the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ has a convergent subnet in Y. So g is subcontinuous at x. □

We have defined some "relations of continuity" and "relations of constancy" between functions and we were using symbols "$\geq$" and "$=$". Of course, the use of these symbols does not automatically turn these relations into relations of order or equality. However, it is easy to see that these relations would be able to create some kind of preorder or some kind of equivalence relations on concrete sets of functions. The following lemma describes the situation.

Lemma 9.2. *Let X, Y, Z, S be topological spaces, let $f\colon X\to Y$, $g\colon Y\to Z$ and $h\colon X\to S$ be functions. Let A be a subset of X. Then the following implications are true:*

(1) If $c_t^A(f)\geq c_t^A(g)$ and $c_t^A(g)\geq c_t^A(h)$ then $c_t^A(f)\geq c_t^A(h)$.
(2) If $c_s^A(f)\geq c_s^A(g)$ and $c_s^A(g)\geq c_s^A(h)$ then $c_s^A(f)\geq c_s^A(h)$.
(3) If $c_t^A(f)=c_t^A(g)$ and $c_t^A(g)=c_t^A(h)$ then $c_t^A(f)=c_t^A(h)$.
(4) If $c_s^A(f)=c_s^A(g)$ and $c_s^A(g)=c_s^A(h)$ then $c_s^A(f)=c_s^A(h)$.
(5) If $f\sim g$ and $g\sim h$ then $f\sim h$.
(6) If $f\approx g$ and $g\approx h$ then $f\approx h$.

Proof. (1) and (2) follow from the definition,(3)-(6) follow from (1) and (2). □

The following two lemmas will be useful when proving some optimization results.

Lemma 9.3. *Let X be a topological space, let Y and Z be Hausdorff topological spaces. Let $f\colon X\to Y$, $g\colon X\to Z$ be functions. Let $f\approx g$. Let x be from X. Then the sets $f^{-1}(f(x))$ and $g^{-1}(g(x))$ are equal.*

Proof. Since the relation between f and g is "symmetrical", it suffices to show that if a point z is from $f^{-1}(f(x))$ then it is from $g^{-1}(g(x))$. Suppose $z\in f^{-1}(f(x))$ is true. Define a sequence $\{a_n\}_{n\in\mathbb{N}}$ by

$$a_n=\begin{cases}x & \text{if } n \text{ is even,}\\ z & \text{if } n \text{ is odd.}\end{cases}$$

Since $f(x)=f(z)$ we can see that the sequence $\{f(a_n)\}_{n\in\mathbb{N}}$ converges. This means the sequence $\{g(a_n)\}_{n\in\mathbb{N}}$ converges too. Since one of its subsequences converges to $g(x)$ and another one converges to $g(z)$, we obtain $g(x) = g(z)$. The point z is proven to be from $g^{-1}(g(x))$. □

The above results imply that the relation "$\approx$" preserves periodicity.

Corollary 9.1. *Let Y, Z be Hausdorff topological spaces, let $f\colon \mathbb{R} \to Y$, $g\colon \mathbb{R} \to Z$ be functions. If $f \approx g$ and f is periodic with a period p, then g is periodic with the same period p.*

The following lemma illustrates some useful properties of the relations "$\approx$" and "$\sim$".

Lemma 9.4. *Let X, Y, Z_1, Z_2 be topological spaces, let $h\colon X \to Y$, $f\colon Y \to Z_1$, $g\colon Y \to Z_2$ be functions. Define $\overline{f}\colon X \to Z_1$, $\overline{g}\colon X \to Z_2$ by*
$\forall x \in X\colon \overline{f}(x) = f(h(x)), \overline{g}(x) = g(h(x))$.
If $f \approx g$ then $\overline{f} \approx \overline{g}$.
If h is continuous and $(f \sim g)$ then $(\overline{f} \sim \overline{g})$.
If $p\colon Z_1 \to Z_1$ is a homeomorphism then $f \approx p(f)$.

Proof. Trivial. □

9.4 Similarity and limits

The following assertion has a very standard proof and it will help us to stop repeating this kind of proof again and again - as it has been done in many classical theorems.

Lemma 9.5. *Let X be a topological space, (Z, ρ) a metric space. Let $f\colon X \to Z$ be a function. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to Z. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ converges uniformly to f. Let $s = \{x_\delta\}_{\delta\in\Delta}$ be a net in X. Let for each $\gamma \in \Gamma$, $\{f_\gamma(x_\delta)\}_{\delta\in\Delta}$ is Cauchy in Z. Then the net $\{f(x_\delta)\}_{\delta\in\Delta}$ is Cauchy in Z.*

Proof. Be ε a positive real number. Denote $t = \frac{\varepsilon}{3}$. Since $\{f_\gamma\}_{\gamma\in\Gamma}$ converges uniformly to f there exists γ from Γ such that for all x from X we have $\rho(f(x), f_\gamma(x)) < t$. The net $\{f_\gamma(x_\delta)\}_{\delta\in\Delta}$ is Cauchy in Z so there exists an index δ_0 such that for all α, β that are greater than δ_0 the inequality

$$\rho(f_\gamma(x_\alpha), f_\gamma(x_\beta)) < t$$

holds. Using the triangle inequality we obtain

$$\begin{aligned}\rho(f(x_\alpha), f(x_\beta)) <& \rho(f(x_\alpha), f_\gamma(x_\alpha)) + \rho(f_\gamma(x_\alpha), f_\gamma(x_\beta)) + \rho(f\gamma(x_\beta), f(x_\beta)) < \\ <& 3t = \varepsilon.\end{aligned}$$

This ends the proof. □

The preceding lemma helps us to prove the following "continuity preserving" theorem. In fact, the theorem says that after a uniform limiting process the degree of continuity and the degree of constancy is preserved, or can become higher. So the limit is never "uglier" than the approaching functions. Of course, the functions f_γ have to be all "equally nice", i. e. h-continuous or h-constant with respect to a function h.

Theorem 9.1. *Let X a topological space, (Y,d) and (Z,ρ) complete metric spaces. Let $h\colon X \to Y$, $f : X \to Z$ be functions. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to Z. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ converges uniformly to f. Let x be a point of X, let A be a subset of X. Then*

(i) If for all γ from Γ, $c_t^A(f_\gamma) \geq c_t^A(h)$ then $c_t^A(f) \geq c_t^A(h)$.

(ii) If for all γ from Γ, $c_t^x(f_\gamma) \geq c_t^x(h)$ and if h is continuous at x then f is continuous at x.

(iii) If for all γ from Γ, $c_s^A(f_\gamma) \geq c_s^A(h)$ then $c_s^A(f) \geq c_s^A(h)$.

(iv) If for all γ from Γ, $c_s^A(f_\gamma) \geq c_s^A(h)$ and if h is constant on A then f is constant on A.

Proof. (i) Take an arbitrary net $s = \{x_\delta\}_{\delta\in\Delta}$ of points of X converging to a point a from A. Suppose the net $\{h(x_\delta)\}_{\delta\in\Delta}$ converges in Y. We have to prove that the net $\{f(x_\delta)\}_{\delta\in\Delta}$ converges in Z. Since for every γ from Γ we have $c_t^A(f_\gamma) \geq c_t^A(h)$ this means that for every γ from Γ the net $\{f_\gamma(x_\delta)\}_{\delta\in\Delta}$ converges in Z, so it is Cauchy. Then, according to the preceding lemma the net $\{f(x_\delta)\}_{\delta\in\Delta}$ is Cauchy in Z, so it is convergent in Z.

(ii) Take an arbitrary net $\{x_\delta\}_{\delta\in\Delta}$ converging to x. The continuity of h at x means that the net $\{h(x_\delta)\}_{\delta\in\Delta}$ converges in Y. According to (i) (just put $A = \{x\}$) the net $\{f(x_\delta)\}_{\delta\in\Delta}$ converges in Z. But this means f is continuous at x.

(iii) The proof is the same as the proof of (i), but instead of a net converging to a point from A we just consider an arbitrary net $s = \{x_\delta\}_{\delta\in\Delta}$ of points of A and we replace the degree of continuity by the degree of constancy.

(iv) First realize that because of (iii) we have $c_s^A(f) \geq c_s^A(h)$. If h is constant, we have $A \subset h^{-1}(h(x))$ for all x from A. To prove that f is constant on A it suffices to prove, that $A \subset f^{-1}(f(x))$ for all x from A. The end of the proof is very similar to the proof of Lemma 9.2 and is omitted. □

In what follows, we are going to work with special nets ("alternate" nets), constructed from other nets. First we will modify the indexed set of a net in the following way:

Let Γ be a directed set. By $\Gamma^{`}$ we will mean a set defined as follows

(*) $\Gamma^{`} = \{(\gamma,1)\colon \gamma\in\Gamma\}\cup\{(\gamma,2)\colon \gamma\in\Gamma\}$ and $\Gamma^{`}$ is equipped with a pre-order defined by

$\forall\alpha,\beta\in\Gamma$ if $\alpha<\beta$ then $(\alpha,1)<(\alpha,2)<(\beta,1)<(\beta,2)$. It is easy to check that $\Gamma^{`}$ is a directed set.

In the proof of the following theorem we need a special kind of net, that we are going to define now. Suppose $\{(x_\gamma)\}_{\gamma\in\Gamma}$ is a net of points of a set X. Be a a point from X. By the symbol $\{x_\gamma,a\}$ we will denote this special net:

$\{x_\gamma,a\} = \{y_{\gamma^{`}}\}_{\gamma^{`}\in\Gamma^{`}}$ where $\Gamma^{`}$ is defined as in (*) and for all γ from Γ we have $y_{(\gamma,1)} = x_\gamma,\ y_{(\gamma,2)} = a$. We can see immediately that the net $\{x_\gamma\}_{\gamma\in\Gamma}$ is a subnet of $\{x_\gamma,a\}$ and that the constant net $\{y_{(\gamma,2)}\}_{\gamma\in\Gamma}$ is a subnet of $\{x_\gamma,a\} = \{y_{\gamma^{`}}\}_{\gamma^{`}\in\Gamma^{`}}$ too.

Moreover, we can see, that if $\{(x_\gamma)\}_{\gamma\in\Gamma}$ converges to a, then $\{x_\gamma,a\}$ converges to a too. And vice versa: if $\{x_\gamma,a\}$ converges to a, then $\{(x_\gamma)\}_{\gamma\in\Gamma}$ (being a subnet of $\{x_\gamma,a\}$) converges to a.

Theorem 9.2. *Let X, Y, Z be Hausdorff topological spaces. Let $f\colon X\to Y$, $g\colon X\to Z$ be functions. Let g be f-constant on X (equivalently expressed: let $c_s(g)\geq c_s(f)$). Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a net in X. Let $A\subset X$ (be a a point from X). Suppose $\lim_{\gamma\in\Gamma} f(x_\gamma)\in f(A)$ ($\lim_{\gamma\in\Gamma} f(x_\gamma) = f(a)$). Then $\lim_{\gamma\in\Gamma} g(x_\gamma)\in g(A)$ ($\lim_{\gamma\in\Gamma} g(x_\gamma) = g(a)$).*

Proof. First of all, since g is f-constant on X the limit $l = \lim_{\gamma\in\Gamma} g(x_\gamma)$ exists in Z. Let $a\in A$ be such that $\lim_{\gamma\in\Gamma} f(x_\gamma) = f(a)$. Consider the net $\{y_{\gamma^{`}}\}_{\gamma^{`}\in\Gamma^{`}} := \{x_\gamma,a\}$. We can see that $\lim_{\gamma^{`}\in\Gamma^{`}} f(y_{\gamma^{`}}) = f(a)$. Since g is f-constant on X this means there exists $m\in Z$ such that $m = \lim_{\gamma^{`}\in\Gamma^{`}} g(y_{\gamma^{`}})$. Now we are going to use the fact that the nets $\{x_\gamma\}_{\gamma\in\Gamma}$ and $\{a\}_{\gamma\in\Gamma}$ (by this we mean the net $\{a_\gamma\}_{\gamma\in\Gamma}$ where for all $\gamma$ from $\Gamma$, $a_\gamma = a$) are both subnets of the net $\{y_{\gamma^{`}}\}_{\gamma^{`}\in\Gamma^{`}}$. This gives us the following equalities: $\lim_{\gamma^{`}\in\Gamma^{`}} g(x_{\gamma^{`}}) = \lim_{\gamma\in\Gamma} g(x_\gamma) = l$ and $\lim_{\gamma^{`}\in\Gamma^{`}} g(x_{\gamma^{`}}) = \lim_{\gamma\in\Gamma} g(a) = g(a)$. So $l = g(a)$ and this means also $l\in g(A)$. (If we put $A = \{a\}$, we see that the "bracket" part of this theorem has been proven too.) □

The proof of the following theorem is very similar to the proof of the preceding theorem. It suffices to use the net $\{x_\gamma,a\}$ again. That is why we omit the proof.

Theorem 9.3. *Let X, Y, Z be Hausdorff topological spaces. Let $f\colon X\to Y$, $g\colon X\to Z$ be functions. Let a be an element of X. Let g be f-continuous at a.*

Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a net in X, that converges to a. Suppose $\lim_{\gamma\in\Gamma} f(x_\gamma) = f(a)$. Then $\lim_{\gamma\in\Gamma} g(x_\gamma) = g(a)$.

9.5 Preserving generalized continuity

The notions defined below were examined for example in [2], [3], [14], [15] and [17].

Definition 9.3. Let (X,T) be topological space. We say, that a set $V \subset X$ is α-open, if and only if there exist an open set $O \in T$ and a nowhere dense set S such that $V = O \setminus S$. The system of all α-open sets in (X,T) is denoted by T_α. T_α defines a new topology on X.

Let (Y,τ) be a topological space. Let x be from X. We say, that a function $f\colon (X,T) \to (Y,\tau)$ is α-continuous at x if for each $W \in \tau$ such that $f(x) \in W$ there exists an $V \in T_\alpha$ such that $x \in V$ and $f(V) \subset W$ is true.

Let X, Y be topological spaces. A function $f\colon X \to Y$ is said to be quasicontinuous at x from X if and only if for any open set V such that $f(x) \in V$ and any open set U such that $x \in U$, there exists a nonempty open set $O \subset U$ such that $f(O) \subset V$.

Let $X = \mathbb{R}$, let Y be a topological space. A function $f\colon X \to Y$ is said to be left (right) hand sided quasicontinuous at a point x from $\mathbb{R}$ if for every $\delta > 0$ and for every open neighborhood V of $f(x)$ there exists an open nonempty set $W \subset (x-\delta, x)$ ($W \subset (x, x+\delta)$) such that $f(W) \subset V$. A function f is bilaterally quasicontinuous at x if it is both left and right hand sided quasicontinuous at this point.

Now we will show that the relation "being continuously similar" preserves the α-continuity of functions. The following theorem shows even more.

Theorem 9.4. *Let X, Y, Z be topological spaces, let $g\colon X \to Y$, $f\colon X \to Z$ be functions. Let g be α-continuous at x. If f is g-continuous at x, then f is α-continuous at x.*

Proof. Denote by T the topology on X and by T_α the α-topology induced by T. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary net that converges in X to x with respect to the topology T_α. To prove the α-continuity of f in x we need to prove, that the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z.

Since the function g is α-continuous at x, the net $\{g(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y (to $g(x)$). Moreover, since the topology T_α is finer than T, the net $\{x_\gamma\}_{\gamma\in\Gamma}$

converges to x also with respect to the topology T. These facts, and the fact that f is g-continuous at x imply, that the net $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Z. It has to be proved that this net converges to $f(x)$. But we can see, that the net $\{g(x_\gamma), g(x)\}$ converges (to $g(x)$). So the net $\{f(x_\gamma), f(x)\}$ converges too because f is g-continuous at x. But the convergence of $\{f(x_\gamma), f(x)\}$ implies the convergence of $\{f(x_\gamma)\}_{\gamma\in\Gamma}$ to $f(x)$. This ends the proof. □

Definition 9.4. ([12]) Let X, Y be topological spaces. Let $\mathscr{A}$ be a system of nonempty subsets of X. A function $f\colon X \to Y$ is said to be $\mathscr{A}$-continuous at x from X if and only if for any open set V such that $f(x) \in V$ and any open set U such that $x \in U$, there exists a set S from $\mathscr{A}$ such that $S \subset U$ and $f(S) \subset V$.

As we will see later, the following theorem says, that many generalized continuity properties are inherited by functions, similar to a nice function. In the proof of this theorem we are going to work with "alternate" nets again.

Theorem 9.5. *Let X, Y, Z be Hausdorff topological spaces. Let $f\colon X \to Y$, $g\colon X \to Z$ be functions. Let x be a point from X. Let $\mathscr{A}$ be a system of nonempty subsets of X and let f be $\mathscr{A}$-continuous at x. If g is f-continuous at x, then g is $\mathscr{A}$-continuous at x too.*

Proof. We proceed by contradiction. Suppose that g is not $\mathscr{A}$-continuous at x. Because of this, there exists an open neighborhood W of $g(x)$ and an open neighborhood U of x such that for any subset S of U such that $S \in \mathscr{A}$ there exists a point s from S such that $g(s) \in Y \setminus W$. In other words no set $A \in \mathscr{A}$ is a subset of the set $g^{-1}(W) \bigcap U$. Denote by

$$\Gamma := \{O\colon\ O \text{ is an open neighborhood of } x,\ O \subset U\},$$
$$A := \{S\colon\ S \text{ is an open neighborhood of } f(x)\}.$$

Define $B = \Gamma \times A$. Define a partial order "$\leq$" on B by:

$$\forall (\gamma_1, \alpha_1), (\gamma_2, \alpha_2) \in B\colon\ (\gamma_1, \alpha_1) \leq (\gamma_2, \alpha_2) \text{ iff } \gamma_2 \subseteq \gamma_1 \text{ and } \alpha_2 \subseteq \alpha_1.$$

It is easy to see that B such equipped is a directed set.

For each $\beta \in B$, $\beta = (\gamma, \alpha)$ the following holds (since γ is an open neighborhood of x and α is an open neighborhood of $f(x)$ and f is $\mathscr{A}$-continuous at x): There exists a set $S_\beta \in \mathscr{A}$ such that $S_\beta \subset \gamma \bigcap U$ and $f(S_\beta) \subset \alpha$. Since S_β cannot be a subset of $g^{-1}(W) \bigcap U$ there exists a point x_β from S_β such that $g(x_\beta) \in Y \setminus W$. At the same time $f(x_\beta) \in \alpha$.

We have just constructed a net of points $\{x_\beta\}_{\beta\in B}$. It is easy to see that this net has the following properties:

(1) $\lim_{\beta\in B} x_\beta = x$;
(2) $\lim_{\beta\in B} f(x_\beta) = f(x)$;
(3) $\forall\beta \in B\colon\ g(x_\beta) \in Y\setminus W$.

Now consider the "alternate" net $\{x_\beta, x\}$ and the corresponding nets $\{f(x_\beta), f(x)\}$ and $\{g(x_\beta), g(x)\}$. Because of (1), the net $\{x_\beta, x\}$ converges to x. Because of (2), the net $\{f(x_\beta), f(x)\}$ converges to $f(x)$. But the convergence of this net and the fact, that g is f-continuous at x imply that the net $\{g(x_\beta), g(x)\}$ is convergent too. Of course, the net $\{g(x_\beta), g(x)\}$ has the same limit as any of its subnets. We see, that it has a constant subnet, with constant values equal to $g(x)$, so the net $\{g(x_\beta), g(x)\}$ converges to $g(x)$. By the same reasoning we obtain that the net $\{g(x_\beta)\}_{\beta\in B}$ converges to $g(x)$.

But this is a contradiction, because for each point x_β we have $g(x_\beta) \in Y\setminus W$, on the other hand W is an open neighborhood of the "would be limit" $g(x)$. □

The previous theorem implies the validity of a result, that was proved for the first time in [8]. That assertion stated, that if X, Y, Z are Hausdorff topological spaces and $f\colon X \to Y,\ g\colon X \to Z$ are functions and x be a point from X such that $c_t^x(f) = c_t^x(g)$ is true, then f is quasicontinuous at x if and only if g is quasicontinuous at x. To see this, it suffices to consider $\mathscr{A}$ as a system of all nonempty open subsets of X.

We are almost ready to give a characterization of some spaces of $\mathscr{A}$-continuous functions. All we need is to combine the result of the preceding theorem with the results of our Theorem 9.1. The following assertion is a corollary of these two theorems.

Corollary 9.2. *Let X be a topological space, let (Y,d), (Z,ρ) be complete metric spaces. Let $h\colon X \to Y,\ f\colon X \to Z$ be functions. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to Z. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ converge uniformly to f. Let x be a point of X, let A be a subset of X. Let h be $\mathscr{A}$-continuous at x ($\mathscr{A}$-continuous at all points from A). Then if for all γ from f_γ is h-continuous at x (f_γ is h-continuous on A) then f is $\mathscr{A}$-continuous at x (f is $\mathscr{A}$-continuous at all points from A).*

9.6 On the structure of some function spaces

Let X be a topological space and let (Y,d) be a Fréchet space. Examining the results from the previous section we can see, that if f from X to Y is a function then for any f_1, f_2 from X to Y such that f_1 and f_2 are f-continuous at a point x, any linear combination of these two function is f-continuous at x too. (This is just because of the "continuous behavior" of linear combination in Fréchet spaces. More concretely, if two nets $\{f_1(x_\gamma)\}_{\gamma\in\Gamma}$ and $\{f_2(x_\gamma)\}_{\gamma\in\Gamma}$ converge in Y, then every net of the form $\{c_1 f_1(x_\gamma)+c_2 f_2(x_\gamma)\}_{\gamma\in\Gamma}$ converges in Y too.)

This means, that if f is $\mathscr{A}$-continuous at a point x, any linear combination of two f-continuous functions f_1 and f_2 is $\mathscr{A}$-continuous at x too. Moreover, as we have proved, the property "being f-continuous at a point x" is preserved under the uniform convergence. This means, that all functions from X to Y, that are f-continuous at x, form a linear subspace of the space of all functions from X to Y. Moreover, this subspace is closed with respect to the operation of the uniform convergence. We will formulate this result in the following theorem.

In what follows, if f is a function from a topological space X into a topological space Y, by $\mathscr{C}_f$ we will denote the set of all f-continuous functions from X to Y.

Theorem 9.6. *Let X be a topological space, let (Y,d) be a Fréchet space.*

(i) Let $f\colon X\to Y$, be a function. Then the set $\mathscr{C}_f$ is a nonempty linear subspace of the linear space of all functions from X to (Y,d). Moreover, $\mathscr{C}_f$ is closed with respect to the uniform convergence.

(ii) Let $\mathscr{A}$ be a system of nonempty subsets of X. Then the set of all functions from X to Y that are $\mathscr{A}$-continuous at x from X (at all points s from a subset $S\subset X$) is a union of a system of nonempty linear spaces of functions such that each of these spaces is containing all continuous functions from X to Y and it is closed with respect to the uniform convergence.

Proof. (i) $\mathscr{C}_f$ is nonempty, because $f\in\mathscr{C}_f$. For the rest see the preceding remark.

(ii) Put

$\mathscr{C}_{\mathscr{A},S}=\{f\colon X\to Y\colon f$ is $\mathscr{A}$-continuous at all points s from $S\}$.

Then $\mathscr{C}_{\mathscr{A},S}=\bigcup_{f\in C_{\mathscr{A},S}}\mathscr{C}_f$. □

Because of the generality of $\mathscr{A}$-continuity we have just characterized a wide range of systems of functions. Concretely, if X,Y are topological spaces and $\mathscr{A}$ a system of nonempty subsets of X, if a function $f\colon X\to Y$ is $\mathscr{A}$-continuous at a point x (at all points of a set $S\subset X$) then it is

(1) continuous at x if $\mathscr{A}=\{U\colon U$ is open in X and $x\in U\}$;

(2) α-continuous at x if $\mathscr{A}=\{O\colon O$ is α-open in X and $x\in O\}$;

(3) quasicontinuous at x (or on (S)) if $\mathscr{A}=\{U\colon U$ is open in $X\}$.

Suppose moreover, that $X=\mathbb{R}$ then f is

(4) left (right) hand sided quasicontinuous at x if
$\mathscr{A}=\{V\colon V=(a,b)$ and $a<b<x\}$ ($\mathscr{A}=\{V\colon V=(a,b)$ and $x<a<b\}$);

(5) bilaterally quasicontinuous at x if

$\mathscr{A} = \{V : V = (a,b) \cup (c,d)$ and $a < b < x < c < d\}$.

Now we can see that our last theorem implies the validity of the following assertions:

Corollary 9.3. *Let X be a topological space, let (Y,d) be a Fréchet space. Let $S \subset X$, $S \neq \emptyset$. Then the set of all functions from X to Y that are quasicontinuous, (α-continuous, left (right) hand sided quasicontinuous, bilaterally quasicontinuous) at all points of S is a union of a system of nonempty linear spaces of functions such that each of these spaces is containing all continuous functions from X to Y and it is closed with respect to the uniform convergence. More concretely, each of these spaces can be of the form $\mathscr{C}_f$, where $f\colon X \to Y$, is a function that is quasicontinuous (α-continuous, left (right) hand sided quasicontinuous, bilaterally quasicontinuous) at all points of S.*

Open question: Does there exist a noncontinuous, quasicontinuous function f from $\mathbb{R}$ to $\mathbb{R}$ such that the linear space $\mathscr{C}_f$ is consisting only of the elements of the form: $c_1 f + c_2 h$ where h (an arbitrary continuous function from $\mathbb{R}$ to $\mathbb{R}$) and $c_1, c_2 \in \mathbb{R}$ are variable?

9.7 Optimization applications

In what follows we are going to use some connectedness properties. We will say that a topological space X is locally arcwise connected at a point x if every neighborhood U of x contains a neighborhood V of x such that any two points a, b from V can be joined by an arc in V, i. e. there exists a function $h\colon [0,1] \to V$ such that $h\colon [0,1] \to h([0,1])$ is a homeomorphism and $h(0) = a$, $h(1) = b$ holds.

The following lemma shows that strongly similar continuous functions defined on an interval attain local extrema at the same points. In a way, it shows a possibility how to investigate a nondifferentiable function for extrema.

Lemma 9.6. *Let $[a,b]$ be an interval in $\mathbb{R}$. Let $f\colon [a,b] \to \mathbb{R}$, $g\colon [a,b] \to \mathbb{R}$ be continuous functions. Let $f \approx g$. Then the following assertions hold*

(1) If $[c,d] \subset [a,b]$ then f is monotonous on $[c,d]$ if and only if g is monotonous on $[c,d]$.

(2) If $[c,d] \subset [a,b]$ then f is strictly monotonous on $[c,d]$ if and only if g is strictly monotonous on $[c,d]$.

(3) A point x from (a,b) is a point of a global (local) extremum of f on $[a,b]$ if and only if x is a point of a global (local) extremum of g on $[a,b]$.

(4) A point x from (a,b) is a point of a strict global (local) extremum of f on $[a,b]$ if and only if x is a point of a strict global (local) extremum of g on $[a,b]$.

Proof. (1) It suffices to show that if g is not monotonous, then f is not monotonous. Suppose g is neither nondecreasing nor nonincreasing on $[c,d]$. Then one of the two following assertions has to be true

(i) There exist t_1, t_2, t_3 from $[c,d]$ such that $t_1 < t_2 < t_3$ and $g(t_1) < g(t_2)$ and $g(t_3) < g(t_2)$ is true.

(ii) There exist t_1, t_2, t_3 from $[c,d]$ such that $t_1 < t_2 < t_3$ and $g(t_1) > g(t_2)$ and $g(t_3) > g(t_2)$ is true.

We are going to work with (i), (ii) can be reduced to (i) by working with the function $-g$, because $-g \approx f$ holds too. Supposing (i) is true, denote $I = g([t_1,t_2])$, $J = g([t_2,t_3])$. Of course I and J are closed intervals and

$$[\max\{g(t_1),g(t_3)\},g(t_2)] \subset I \cap J.$$

Pick an arbitrary point c from $[\max\{g(t_1),g(t_3)\},g(t_2)]$. We can see that there exists two points $o_1 \in (t_1,t_2)$, $o_2 \in (t_2,t_3)$ such that $c = g(o_1) = g(o_2)$. Since $g \approx f$ we obtain $f(o_1) = f(o_2)$ and since $g(t_2) \neq g(o_1)$, we have $f(t_2) \neq f(o_1)$ too. Now remembering that $o_1 < t_2 < o_2$ we see that f is not monotonous on $[c,d]$.

(2) If f is strictly monotonous on $[c,d]$ then it is monotonous on this interval and according to (1) g is monotonous too. Now it suffices to show that g is injective on $[c,d]$. But this has to be true because $f \approx g$ is true and f is injective on $[c,d]$.

Before proving (3) and (4) we should realize that only the case of global extremum on an interval needs to be treated. This is so because a local extremum on an interval is a global extremum on a subinterval.

(3) Suppose x from (a,b) is a point of a global extremum of f. Without loss of generality we are going to assume that f has a global maximum at x. Notice that since $f \approx g$ the sets $f^{-1}(f(x))$ and $g^{-1}(g(x))$ are identical. If f is constant on $[a,b]$, then g is constant on $[a,b]$ too and we are done.

Now we examine the second case - the case when the set $g^{-1}(g(x))$ does not coincide with $]a,b]$. Choose a point t from $[a,b]$ such that $g(t) \neq g(x)$. Suppose $g(t) > g(x)$ (the case $g(t) < g(x)$ is similar and therefore omitted). We will show that for all z from $< a,b >$ we have $g(z) \geq g(x)$. Suppose this is not true. Then there exists a point s from $[a,b]$ such that $g(s) < g(x)$. Suppose $t < x < s$ (other cases, for example $t < s < x$ etc can be treated with the same reasoning that we use for our chosen case). Since the sets $g^{-1}(g(t))$, $g^{-1}(g(s))$ and $g^{-1}(g(x))$ are pairwise disjoint, the sets $f^{-1}(f(t))$, $f^{-1}(f(s))$ and $f^{-1}(f(x))$ are pairwise disjoint too. Examine the case $f(s) < f(t) < f(x)$ (the case $f(t) <$

$f(s) < f(x)$ is similar). Define $c = \inf\{e\colon f([e,s]) \subset (-\infty, f(t)]\}$. Since f is continuous we obtain $x < c < s$. We remind the reader that because of the continuity of f we have $< f(s), f(x) > \subset f([x,s])$. Because of the definition of c and the continuity of f we obtain $f(c) = f(t)$. This means $g(c) = g(t)$. Since g is continuous the set $g([c,s])$ contains the closed interval $[g(s), g(c)]$. Since $g(s) < g(x) < g(t) = g(c)$ is true, there exists a point r from the open interval (c,s) such that $g(r) = g(x)$. This implies $f(r) = f(x)$. But r is from (c,s) and because of the definition of c the point $f(x) = f(r)$ is not from $f([c,s])$. This is a contradiction. We have just proved that for all z from $[a,b]$ the inequality $g(z) \geq g(x)$ holds. The function g is proven to have a global extremum at x.

(4) Suppose f has a strict global maximum at x from (a,b). The (3) proven, we can claim that g has a global extremum at x. If this extremum of g would not be strict, there would exist a point c from $[a,b]$ with the property $g(c) = g(x)$. Since $f \approx g$ this would imply $f(c) = f(x)$, but this is not possible. □

Now we are ready for the main result of this section.

Theorem 9.7. *Let X be a topological space, let x be from X. Let X be locally arcwise connected at x. Let $f\colon X \to \mathbb{R}$, $g\colon X \to \mathbb{R}$ be continuous functions. Let $f \approx g$. Then*

(j) x is a point of a local extremum of f if and only if x is a point of a local extremum of g

(jj) x is a point of a strict local extremum of f if and only if x is a point of a strict local extremum of g

Proof. If x is an isolated point the theorem is true. Suppose x is not isolated.

(j) We will prove that if f has a local extremum at x then g has an extremum at x too. Suppose f has at x a local maximum. This means there exists an arcwise connected open neighborhood U of x such that for all t from U the inequality $f(t) \leq f(x)$ takes place.

Choose an arbitrary point u from $U \setminus \{x\}$. Suppose $g(u) \geq g(x)$. We are going to prove that for all s from U the inequality $g(s) \geq g(x)$. Choose an arbitrary s from U, suppose s is different from x and u. Since U is arcwise connected, there exists an arc connecting the points u, x and s. More concretely there exists a continuous function $h\colon [0,2] \to U$ such that $h(0) = u, h(1) = x$ and $h(2) = s$. Define functions $\overline{f}\colon [0,2] \to \mathbb{R}$ and $\overline{g}\colon [0,2] \to \mathbb{R}$ in the following way

for all z from $[0,2]$, $\overline{f}(z) = f(h(z))$ and $\overline{g}(z) = g(h(z))$.

According to Lemma 9.4 the functions $\overline{f}$ and $\overline{g}$ satisfy $\overline{f} \approx \overline{g}$. Since f has a local extremum on U at the point x, we can see that $\overline{f}$ has a local extremum on $[0,2]$ at the point 1. This means (according to Lemma 9.6) that $\overline{g}$ has a local extremum on $[0,2]$ at the point 1. We know that $\overline{g}(0) = g(u) \leq g(x) = \overline{g}(1)$ so $\overline{g}$ has at 1 a local minimum. Therefore $\overline{g}(2) \geq \overline{g}(1)$ must be true. Since $\overline{g}(2) = g(s)$ and $\overline{g}(1) = g(x)$ we have just proved that for an arbitrary s from U we have $g(s) \geq g(x)$.

The next example shows that without connectedness of X our theorem need not be true.

Example 9.2. Define a subset X of $\mathbb{R}$ by

$$X = \left[-1, -\tfrac{1}{2}\right] \cup \left[-\tfrac{1}{4}, -\tfrac{1}{8}\right] \cup \cdots \cup \{0\} \cup \left[\tfrac{1}{2}, 1\right] \cup \ldots$$

More concretely, $X = \{0\} \cup A \cup B$ where

$$A = \bigcup_{i=0}^{\infty} \left[-\frac{1}{2^{2i}}, -\frac{1}{2^{2i+1}}\right] = \bigcup_{i=0}^{\infty} I_i, \quad B = \bigcup_{i=0}^{\infty} \left[\frac{1}{2^{2i+1}}, \frac{1}{2^{2i}}\right] = \bigcup_{i=0}^{\infty} J_i.$$

We define two functions $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ by

$$f(x) = \begin{cases} -\frac{1}{2^{2i}}, & x \in I_i, \\ 0, & x = 0, \\ -\frac{1}{2^{2i+1}}, & x \in J_i, \end{cases} \qquad g(x) = \begin{cases} -\frac{1}{2^{2i}}, & x \in I_i, \\ 0, & x = 0, \\ \frac{1}{2^{2i+1}}, & x \in J_i. \end{cases}$$

The functions f and g coincide on $A \cup \{0\}$. Globally, it is easy to see that $f \approx g$ is true. The function f has a strict global maximum at 0, but g is nondecreasing on its domain and has no extremum at 0.

The following theorem will enable us to prove easily two optimization theorems.

Theorem 9.8. *Let X be a nonempty set, Y and Z be Hausdorff topological spaces. Let $f\colon X \to Y$, $g\colon X \to Z$ be functions. Let $f \approx g$. Then for any subset A of X the following is true:*

$f(A)$ is closed (compact) in Y if and only if $g(A)$ is closed (compact) in Z.

Proof. First we will prove the "closedness" part of our assertion. It suffices to show that if $f(A)$ is closed in Y then $g(A)$ is closed in Z. Let $\{z_\gamma\}_{\gamma \in \Gamma}$ be a net of points from $g(A)$, which is convergent in Z. Denote its limit by z. We have to prove that there exists a point a in A such that $g(a) = z$ is true. Since each point z_γ is from $g(A)$, for every γ from Γ there exists x_γ from A such that $g(x_\gamma) = z_\gamma$. We see that $\{g(x_\gamma)\}_{\gamma \in \Gamma} = \{z_\gamma\}_{\gamma \in \Gamma}$ converges in Z. Together with $f \approx g$ this implies the net $\{f(x_\gamma)\}_{\gamma \in \Gamma}$ converges in Y. We will denote its limit by y. The set $f(A)$ is closed, so $y \in f(A)$. Consider a point $a \in A$ such that $f(a) = y$. Now let us consider the net $\{p_{\gamma^`}\}_{\gamma^` \in \Gamma^`} := \{x_\gamma, a\}$. We can see that $\lim_{\gamma^` \in \Gamma^`} f(p_{\gamma^`}) = y = f(a)$. Since $f \approx g$ this means there exists $m \in Z$ such that $m = \lim_{\gamma^` \in \Gamma^`} g(p_{\gamma^`})$. But the nets $\{x_\gamma\}_{\gamma \in \Gamma}$ and $\{a\}_{\gamma \in \Gamma}$ (by this we mean the net $\{a_\gamma\}_{\gamma \in \Gamma}$ where for all $\gamma$ from $\Gamma$ $a_\gamma = a$) are both subnets of the net $\{p_{\gamma^`}\}_{\gamma^` \in \Gamma^`}$. This implies: $\lim_{\gamma^` \in \Gamma^`} g(p_{\gamma^`}) = \lim_{\gamma \in \Gamma} g(x_\gamma) = z$ and $\lim_{\gamma^` \in \Gamma^`} g(p_{\gamma^`}) = \lim_{\gamma \in \Gamma} g(a) = g(a)$. So $z = g(a)$ and this means also $z \in g(A)$. The closedness of $g(A)$ is proven.

Now, for "compactness" part of out assertion, again, it suffices to prove, that if $f(A)$ is compact, then $g(A)$ is compact too. Suppose $f(A)$ to be compact. Then it is closed so $g(A)$ is closed too. Consider an arbitrary net $\{g(x_\gamma)\}_{\gamma \in \Gamma}$ in $g(A)$, we have to prove now, that it has a convergent subnet. But the net $\{f(x_\gamma)\}_{\gamma \in \Gamma}$ in $f(A)$ has a convergent subnet, say $\{f(x_\delta)\}_{\delta \in \Delta}$ and because of strong similarity of f and g the net $\{g(x_\delta)\}_{\delta \in \Delta}$ is convergent (in $g(A)$) too.

□

Now, the following theorems will be easy corollaries of the preceding theorem. Let us observe that we do not need any topological structure on the domain set X in the following theorem.

Theorem 9.9. *Let X be a set. Let $f\colon X \to \mathbb{R}$, $g\colon X \to \mathbb{R}$ be functions. Let A be a subset of X such that the set $f(A)$ is closed. Let f achieve its maximum and minimum on A. Let $f \approx g$. Then g achieves its maximum and minimum on A too.*

Proof. Under the conditions of our theorem the set $f(A)$ must be compact. So the set $g(A)$ is a compact too. □

Now we present an optimization result concerning functions, that are strongly similar with Darboux functions.

Theorem 9.10. *Let X be a topological space. Let $f\colon X \to \mathbb{R}$, $g\colon X \to \mathbb{R}$ be functions. Let f be a Darboux function (an image of each connected set under f is connected). Let f achieve its maximum and minimum on a connected subset A of X. Let $f \approx g$. Then g achieves its maximum and minimum on A too.*

Proof. Under the conditions of our theorem the set $f(A)$ must be connected and bounded. Moreover it contains its supremum and infimum. So $f(A)$ is a compact interval. This means the set $g(A)$ is compact too. Therefore g achieves a global maximum and a global minimum on A. □

To conclude this section, let us remark, that all results presented here showed some possibilies how to investigate a nondifferentiable function for extrema. Namely, some nondifferentiable functions are strongly similar to differentiable ones and these can be investigated in a classical way. For the same reason it is also worth investigating which continuous functions defined on convex sets are strongly similar to convex functions.

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