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Chapter 20
Properties of the σ - ideal of microscopic sets

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The investigation of σ-ideals of subsets of the real line has a long tradition. The main motivation for the study of the collection of all microscopic sets, which constitutes a σ-ideal, stems from the fact that whenever one has to prove that a certain property in functional analysis or measure theory is fulfilled for "almost all" elements, the concept of "smallness" of the set of "exceptional points" should be described. The most classical of these concepts are related to Lebesgue nullsets and the sets of first Baire category.

In certain applications, the ideals of measure and category turn out not to be suitable. In these situations it is useful to consider another σ-ideal having some good set-theoretic, algebraic and geometric properties.

What makes microscopic sets interesting is the property that the collection of all microscopic sets constitutes a σ-ideal strictly smaller then the σ-ideal of sets of Lebesgue measure zero and orthogonal to the σ-ideal of sets of first Baire category. Therefore, in cases where it is well-known that a certain property holds everywhere except for a set of Lebesgue measure zero, it is important to check if the set of exceptional points is microscopic. If the answer is positive we get a stronger version of the property in question. In the classical function theory on $\mathbb{R}^n$ the examples of such theorems are given by Fubini's theorem, Kuratowski - Ulam theorem, Steinhaus theorem, Piccard theorem, results of Oxtoby and Ulam concerning homeomorphisms or Sierpiński - Erdös Duality Principle and many others. These results are of importance for various branches of mathematics, such as functional analysis, measure theory, geometric measure theory and descriptive set theory. Their stronger versions will provide a useful subtle tool for mathematicians working on different topics.

20.1 Microscopic sets on the real line

The notion of microscopic set on the real line was introduced at the beginning of 21-st century in the paper [1] by J. Appell.

Definition 20.1. A set $E \subset \mathbb{R}$ is microscopic if for each $\varepsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $E \subset \bigcup_{n\in\mathbb{N}} I_n$, and $\lambda(I_n) \le \varepsilon^n$ for each $n \in \mathbb{N}$.

The family of all microscopic sets will be denoted by $\mathcal{M}$.

Deeper studies of microscopic sets were done by J. Appell, E. D'Aniello and M. Väth in the paper [3] from 2001. They showed that the collection of all microscopic sets constitutes a σ-ideal. It is not trivial to verify that a union of microscopic sets is microscopic, so we repeat it here.

Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of microscopic sets. Let $\varepsilon \in (0,1)$ and define $\varepsilon_k := \varepsilon^{2^k}$ for every $k \in \mathbb{N}$. Then for any $k \in \mathbb{N}$ there exists a sequence of intervals $\{I_{k,n}\}_{n\in\mathbb{N}}$ such that $A_k \subset \bigcup_{n\in\mathbb{N}} I_{k,n}$, and $\lambda(I_{k,n}) \le \varepsilon_k^n$, for each $n \in \mathbb{N}$.

Define a map $\phi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ by

$$\phi(k,n) = 2^{k-1}(2n-1). \tag{20.1}$$

Obviously, it is a bijection.

Now consider the sequence of intervals $\{I_m\}_{m\in\mathbb{N}}$, where $I_m := I_{k,n}$ with $m = \phi(k,n)$.

Then $\bigcup_{k\in\mathbb{N}} A_k \subset \bigcup_{m\in\mathbb{N}} I_m$ and $\lambda(I_m) = \lambda(I_{k,n}) \le \varepsilon_k^n = (\varepsilon^{2^k})^n = \varepsilon^{2^k n} \le \varepsilon^{\phi(k,n)} = \varepsilon^m$, for every $m \in \mathbb{N}$.

Therefore $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{M}$.

Some properties of the considered σ-ideal are straightforward.

If $A \in \mathcal{M}$ and $\alpha \in \mathbb{R}$, then

1. $A + \alpha = \{x + \alpha : x \in A\} \in \mathcal{M}$,

2. $-A = \{-x : x \in A\} \in \mathcal{M}$,
3. $\alpha \cdot A = \{\alpha \cdot x : x \in A\} \in \mathcal{M}$,
4. if $0 \notin A$, then $A^{-1} = \{\frac{1}{x} : x \in A\} \in \mathcal{M}$.

On account of the first two statements we can say that there is no "model set" for this ideal (see [13]). It means there is no set $A \in \mathcal{M}$ such that for each $B \in \mathcal{M}$ there exists $x \in \mathbb{R}$ such that $B \subset A + x$.

Analogously as the σ-ideal of Lebesgue nullsets, $\mathcal{M}$ is G_δ- generated:

Theorem 20.2 ([21], Lemma 2.3). *Every microscopic set is contained in some microscopic set of type G_δ.*

Proof. Let A be a microscopic set. Then for each $j \in \mathbb{N}$ there exists a sequence $\{I_{n,j}\}_{n\in\mathbb{N}}$ of open intervals such that $A \subset \bigcup_{n\in\mathbb{N}} I_{n,j}$ and $\lambda(I_{n,j}) < (\frac{1}{2^j})^n$ for each $n \in \mathbb{N}$. Put $B = \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} I_{n,j}$. Obviously, $A \subset B$ and B is a microscopic set of type G_δ. □

Simultaneously, the set constructed by A. S. Besicovitch in [10] shows that an approximation of Borel sets by F_σ sets with accuracy to microscopic set is impossible. Besicovitch proved that there exists a Borel set $E \subset \mathbb{R}$ such that if $E = A \cup N$ and A is a set of type F_σ, then N is not a set of Hausdorff dimension zero, so also not microscopic, as each microscopic set has Hausdorff dimension zero (compare [3], p. 258-9 or [2], p. 213).

A microscopic set can be also described in another way.

Theorem 20.3 ([15]). *The following conditions are equivalent:*

1) A is microscopic,
2) for every $\eta > 0$ there exists a sequence $\{J_n\}_{n\in\mathbb{N}}$ of intervals such that

$$A \subset \limsup_n J_n \textit{ and } \sum_{k=n}^{\infty} \lambda(J_k) \le \eta^n \textit{ for } n \in \mathbb{N},$$

3) for every $\delta > 0$ there exists a sequence $\{J_n\}_{n\in\mathbb{N}}$ of intervals such that

$$A \subset \limsup_n J_n \textit{ and } \lambda(J_n) \le \delta^n \textit{ for } n \in \mathbb{N}.$$

Proof. 1) $\Rightarrow$ 2) Suppose that E is microscopic set and $\eta \in (0,1)$. Put

$$\theta = \frac{\eta}{1+\eta} \tag{20.2}$$

and $\varepsilon_k = \theta^{2^k}$ for $k \in \mathbb{N}$.

Let k be a fixed positive integer. As E is microscopic, there exists a sequence $\{I_n^k\}_{n\in\mathbb{N}}$ of intervals such that

$$E \subset \bigcup_{n=1}^{\infty} I_n^k \quad \text{and} \quad \lambda(I_n^k) < (\varepsilon_k)^n. \tag{20.3}$$

Let ϕ be a function defined in (20.1) and let $m \in \mathbb{N}$. There exists a unique pair $(k,n) \in \mathbb{N}\times\mathbb{N}$ such that $\phi(k,n) = m$. Put

$$J_m = I_n^k.$$

Then $E \subset \limsup_m J_m$. Let $p \in \mathbb{N}$ and $A_p = \{(k,n) \in \mathbb{N}\times\mathbb{N} : \phi(k,n) \geq p\}$. Using (20.3) and (20.2) we obtain

$$\sum_{m=p}^{\infty} \lambda(J_m) = \sum_{(k,n)\in A_p} \lambda(I_n^k) < \sum_{(k,n)\in A_p} (\varepsilon_k)^n = \sum_{(k,n)\in A_p} \theta^{2^k \cdot n} < \sum_{(k,n)\in A_p} \theta^{2^{k-1}\cdot(2n-1)} =$$
$$\sum_{(k,n)\in A_p} \theta^{\phi(k,n)} = \sum_{m=p}^{\infty} \theta^m = \frac{\theta^p}{1-\theta} \leq \left(\frac{\theta}{1-\theta}\right)^p = \eta^p.$$

The other implications are obvious. □

20.2 Comparison with other σ-ideals

We want to consider families of small subsets of the real line having in mind various concepts of "smallness".

We are going to use the following notation: let $\mathcal{C}$ denote the family of all countable sets, $\mathcal{S}$ - strong measure zero sets, $\mathcal{UMS}$ - universal measure zero sets, $\mathcal{H}_0$ - sets of Hausdorff dimension zero.

We recall here only definitions of sets belonging to $\mathcal{S}$ and $\mathcal{UMS}$.

A set $E \subset \mathbb{R}$ is of strong measure zero if for each sequence of positive reals $\{\varepsilon_n\}_{n\in\mathbb{N}}$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \bigcup_{n\in\mathbb{N}} I_n \; and \; \lambda(I_n) < \varepsilon_n \; for \; n \in \mathbb{N}.$$

A set $E \subset \mathbb{R}$ has universal measure zero if for each Borel measure μ there is a Borel set of μ - measure zero covering E.

Of course each of these families is a σ-ideal. It was observed that the following inclusions hold

$$\mathcal{C} \subsetneq \mathcal{S} \subsetneq \mathcal{M} \subsetneq \mathcal{H}_0 \subsetneq \mathcal{N}.$$

The first one is proper under CH because every Luzin set (an uncountable subset of a real line having countable intersection with every set of the first category, whose existence is proved under CH) is a strong measure zero set ([9], Lemma 8.2.1.) Indeed, assume that A is a Luzin set. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence of all rational numbers and let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of positive real numbers. Then the set

$$\bigcup_{n=1}^{\infty}(r_n-\frac{\varepsilon_{2n}}{3},r_n+\frac{\varepsilon_{2n}}{3})$$

is open and dense, so its complement is a set of the first category. Consequently, the set

$$B=A\setminus\bigcup_{n=1}^{\infty}(r_n-\frac{\varepsilon_{2n}}{3},r_n+\frac{\varepsilon_{2n}}{3})$$

is countable. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of all elements of B. Then

$$A\subset\bigcup_{n=1}^{\infty}(r_n-\frac{\varepsilon_{2n}}{3},r_n+\frac{\varepsilon_{2n}}{3})\cup\bigcup_{n=1}^{\infty}(x_n-\frac{\varepsilon_{2n-1}}{3},x_n+\frac{\varepsilon_{2n-1}}{3}),$$

so A is a strong measure zero set.

An example of a microscopic set which is not a strong measure zero set is given in [15].

The example of a non-microscopic set of Hausdorff dimension zero can be found in [2] and [3].

The classical one-third Cantor set is a Lebesgue nullset but it has a positive Hausdorff dimension. Nevertheless, it is possible to construct a Cantor-type set which is microscopic ([23], Lemma 2), as follows.

We shall define by induction the sequence of open intervals $\{J_{n,i}\}$, $i\in\{1,...,2^{n-1}\}$, $n\in\mathbb{N}$ in a following way. Put $J_{1,1}=(\frac{1}{4},\frac{3}{4})$. Denote by $K_{1,1}$, $K_{1,2}$ successive components of the set $I\setminus J_{1,1}$. Obviously $\lambda(K_{1,i})=\frac{1}{2^{2^1}}$ for $i\in\{1,2\}$. Let $J_{2,1},J_{2,2}$ be two open intervals concentric with $K_{1,1},K_{1,2}$ respectively, such that $\lambda(J_{2,1})=\lambda(J_{2,2})=\lambda(K_{1,1})-2\frac{1}{3^{2^2}}$. Let $K_{2,1},K_{2,2},K_{2,3},K_{2,4}$ denote successive components of the set $I\setminus(J_{1,1}\cup J_{2,1}\cup J_{2,2})$. Notice that $\lambda(K_{2,i})=\frac{1}{3^{2^2}}$ for $i\in\{1,2,3,4\}$.

Let $k\geq 2$. Assume that we have constructed the open, nonempty intervals $J_{l,1},...,J_{l,2^{l-1}}$ concentric with $K_{l-1,1},...,K_{l-1,2^{l-1}}$ respectively, such that

$$\lambda(J_{l,i})=\lambda(K_{l-1,1})-2\frac{1}{(l+1)^{2^l}}$$

for $l \in \{1,...,k\}$ and $i \in \{1,..,2^{l-1}\}$. Let $K_{k,1},...,K_{k,2^k}$ be successive components of the set $I \setminus \bigcup_{l=1}^{k} \bigcup_{i=1}^{2^{l-1}} J_{l,i}$. Notice that $\lambda(K_{k,i}) = \frac{1}{(k+1)^{2^k}}$ for $i \in \{1,...,2^k\}$.
Now let $J_{k+1,1},...,J_{k+1,2^k}$ be open intervals concentric with $K_{k,1},...,K_{k,2^k}$ respectively, such that

$$\lambda(J_{k+1,i}) = \lambda(K_{k,1}) - 2\frac{1}{(k+2)^{2^{k+1}}}$$

for $i \in \{1,..,2^k\}$. Let $K_{k+1,1},...,K_{k+1,2^{k+1}}$ be successive components of the set $I \setminus \bigcup_{l=1}^{k+1} \bigcup_{i=1}^{2^{l-1}} J_{l,i}$. Obviously $\lambda(K_{k+1,i}) = \frac{1}{(k+2)^{2^{k+1}}}$ for $i \in \{1,...,2^{k+1}\}$.

Let us put

$$M = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{2^k} K_{k,i}.$$

Now let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$. Obviously $M \subset \bigcup_{i=1}^{2^{n_0}} K_{n_0,i}$. Moreover

$$\lambda(K_{n_0,i}) = \frac{1}{(n_0+1)^{2^{n_0}}} < \frac{1}{n_0{}^i} < \varepsilon^i$$

for $i \in \{1,...,2^{n_0}\}$. Hence M is a Cantor-type set which is microscopic.

As a perfect set M cannot be a strong measure zero set ([9], Corollary 8.1.5) so $M \in \mathcal{M} \setminus \mathcal{S}$.

A classical result of Marczewski states that every strong measure zero set has universal measure zero ([26], Theorem 5.1), but the σ-ideals $\mathcal{UMS}$ and $\mathcal{M}$ are incomparable. Indeed, by theorem of Marczewski ([26], Theorem 9.1) a set of reals X belongs to $\mathcal{UMS}$ if and only if every set homeomorphic to X has Lebesgue measure zero, so since a microscopic Cantor-type set is homeomorphic to a Cantor set with positive Lebesgue measure, it is not universal measure zero set. On the other hand there exists on the real line a universal measure zero set with Hausdorff dimension one ([32]), it means not microscopic.

The next theorem ensures the existence of a microscopic set which is large in a sense of category, i.e. it is residual.

Theorem 20.4 ([21], Lemma 2.2). *There exists a decomposition of* $\mathbb{R}$

$$\mathbb{R} = A \cup B$$

such that A is of the first category and B is a microscopic set.

Proof. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence of all rational numbers. Let $I_{n,j} = (r_n - \frac{1}{2^{n\cdot j+1}}, r_n + \frac{1}{2^{n\cdot j+1}})$, $n \in \mathbb{N}$, $j \in \mathbb{N}$.

Put

$$G_j = \bigcup_{n=1}^{\infty} I_{n,j}$$

for $j \in \mathbb{N}$,

$$B = \bigcap_{j=1}^{\infty} G_j$$

and $A = \mathbb{R} \setminus B$. For each $\varepsilon > 0$ there exists $j \in \mathbb{N}$ such that $1/2^j < \varepsilon$. Obviously, $B \subset G_j = \bigcup_{n=1}^{\infty} I_{n,j}$ and $\lambda(I_{n,j}) = (\frac{1}{2^j})^n < \varepsilon^n$ for $n \in \mathbb{N}$. Hence B is a microscopic set. Simultaneously, G_j is open dense subset of $\mathbb{R}$ for each $j \in \mathbb{N}$, so B is a residual set and, consequently, A is a set of the first category. $\square$

Consider an equivalent definition of a set of strong measure zero: a set $E \subset \mathbb{R}$ belongs to $\mathcal{S}$ if for each sequence of positive reals $\{\varepsilon_n\}_{n\in\mathbb{N}}$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \limsup_n I_n \text{ and } \sum_{k=n}^{\infty} \lambda(I_k) < \varepsilon_n \text{ for } n \in \mathbb{N}.$$

Looking at this definition of a strong measure zero set and the definition of a microscopic set from Theorem 20.3 we can notice a similarity to the Borel idea ([27], Lemma 14.1) of describing a Lebesgue measure zero set E by existence of a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \limsup_n I_n \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda(I_n) < \infty.$$

Following E. Borel and M. Frechet ([11], [17]) W. Just and C. Laflamme in [20] and [25] classified measure zero sets according to their open covers and considered some σ-ideals of measure zero sets. One of them is σ-ideal of strong measure zero sets, so it is contained in $\mathcal{M}$. We are going to justify that others are imcomparable with $\mathcal{M}$.

Let $\mathcal{H}$ denote a collection of sets $E \subset \mathbb{R}$ with a property that there exists a sequence of positive reals $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to zero such that for all nonincreasing sequences $\{\delta_n\}_{n\in\mathbb{N}}$ if $\delta_n \geq \varepsilon_n$ infinitely often, then there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \limsup_n I_n \quad \text{and} \quad \sum_{k=n}^{\infty} \lambda(I_k) \leq \delta_n \quad \text{for} \quad n \in \mathbb{N}.$$

Then $\mathcal{H}$ is a σ-ideal and any uncountable closed set of measure zero belongs to $\mathcal{H}$ ([25]), so it contains both classical Cantor set and Cantor - type microscopic set.

The next class $\mathcal{L}$ is a collection of sets $E \subset \mathbb{R}$ with a property that there exists a sequence of positive reals $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that for all sequences of intervals $\{I_n\}_{n\in\mathbb{N}}$ if $E \subset \limsup_n I_n$, then the condition $\sum_{k=n}^{\infty} \lambda(I_k) \geq \varepsilon_n$ holds for all but finitely many $n \in \mathbb{N}$.

Since each residual set of measure zero belongs to the $\mathcal{L}$-class ([25]), then by Theorem 20.4 there is a microscopic set in $\mathcal{L}$ and if we consider a union of a microscopic comeager set with a nonmicroscopic set of measure zero (for example a classical one-third Cantor set) we get an example of a nonmicroscopic set from $\mathcal{L}$.

The next collection $\mathcal{U}$ consists of measure zero sets not in $\mathcal{L}$, so $E \subset \mathbb{R}$ belongs to $\mathcal{U}$ if for each sequence of positive reals $\{\varepsilon_n\}_{n\in\mathbb{N}}$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \limsup_n I_n \quad \text{and} \quad \sum_{k=n}^{\infty} \lambda(I_k) < \varepsilon_n \quad \text{for infinitely many } n \in \mathbb{N}.$$

Then $\mathcal{U}$ is a σ-ideal ([25]) and of course $\mathcal{S} \subset \mathcal{U}$. Therefore there are microscopic sets in $\mathcal{U}$ and comeager microscopic sets do not belong to $\mathcal{U}$. Since $\mathcal{H}$ is consistently contained in $\mathcal{U}$ ([25]), there are also nonmicroscopic sets in $\mathcal{U}$. Hence $\mathcal{U}$ is imcomparable with $\mathcal{M}$.

Given a proper σ - ideal $\mathcal{I}$ of subsets of the real line, containing all singletons and G_δ- generated, one can consider the family of all sets which can be covered by F_σ- sets from $\mathcal{I}$ (see for example [8]). If we denote this family by $\mathcal{I}^*$ then $\mathcal{I}^* \subset \mathcal{I}$ and $\mathcal{I}^*$ is a σ-ideal. In particular $\mathcal{M}^*$ is a σ - ideal. The next theorem states the results of the comparison of this family with others.

Theorem 20.5.
1) $\mathcal{C} \subset \mathcal{M}^*$ and $\mathcal{M}^* \setminus \mathcal{C} \neq \emptyset$,
2) $\mathcal{M}^* \subset \mathcal{M}$ and $\mathcal{M} \setminus \mathcal{M}^* \neq \emptyset$,
3) $\mathcal{N}^* \setminus \mathcal{M} \neq \emptyset$ and $\mathcal{M} \setminus \mathcal{N}^* \neq \emptyset$,
4) $\mathcal{M}^* \subset \mathcal{N}^*$ and $\mathcal{N}^* \setminus \mathcal{M}^* \neq \emptyset$,
5) $\mathcal{N} \setminus \left(\mathcal{M} \cup \mathcal{N}^*\right) \neq \emptyset$.

Proof. All the above inclusions follow directly from definitions. We only have to show that the listed differences are really nonempty.

1) Any microscopic Cantor-type set belongs to $\mathcal{M}^* \setminus \mathcal{C}$.

2) Any residual microscopic set belongs to $\mathcal{M} \setminus \mathcal{M}^*$.

3) The classical one-third Cantor set belongs to $\mathcal{N}^* \setminus \mathcal{M}$ and any residual microscopic set belongs to $\mathcal{M} \setminus \mathcal{N}^*$.

4) The family $\mathcal{N}^* \setminus \mathcal{M}^*$ is nonempty since it contains $\mathcal{N}^* \setminus \mathcal{M}$.

5) The union of the classical one-third Cantor set and any residual null set belongs to $\mathcal{N} \setminus \left(\mathcal{M} \cup \mathcal{N}^*\right)$. □

Since $\mathcal{N}^* \subsetneq \mathcal{M} \cap \mathcal{N}$,

$$\mathcal{M}^* \subsetneq \mathcal{M} \cap \mathcal{N},$$

where $\mathcal{M}$ denote a σ-ideal of sets of the first category.

Note that the σ-ideal of σ - porous sets satisfies the analogous inclusion, so it is natural to compare it with $\mathcal{M}^*$.

Let us recall the relevant definitions ([31]). For $A \subset \mathbb{R}$, $x \in \mathbb{R}$ and $\varepsilon > 0$ let

$$\gamma(A,x,\varepsilon) := sup\{r > 0 : \exists_{z \in \mathbb{R}} (z-r, z+r) \subset (x-\varepsilon, x+\varepsilon) \setminus A\},$$

where we put $sup\, \emptyset := 0$.

The *porosity of* A at x is defined by

$$p(A,x) := \limsup_{\varepsilon \to 0^+} \frac{2\gamma(A,x,\varepsilon)}{\varepsilon}.$$

A is called *porous* if $p(A,a) > 0$ for each $a \in A$. It is called σ*-porous* if it belongs to $\mathcal{P}$ – the σ-ideal generated by the porous sets.

As was already mentioned $\mathcal{P} \subsetneq \mathcal{M} \cap \mathcal{N}$ and $\mathcal{M}^* \subsetneq \mathcal{M} \cap \mathcal{N}$. However

Theorem 20.6. *$\mathcal{P}$ and $\mathcal{M}^*$ are incomparable.*

Proof. Note that

$$\mathcal{P} \setminus \mathcal{M}^* \neq \emptyset,$$

since $\mathcal{P} \setminus \mathcal{N}^* \neq \emptyset$ (see [16]).

To show $\mathcal{M}^* \setminus \mathcal{P} \neq \emptyset$ we use the idea from Lemma 0.2 ([19]). We prove that there exists a sequence of sets $\{C_n\}_{n \in \mathbb{N}}$, $C_n \subset (0,1)$ for $n \in \mathbb{N}$, with the following properties:

1. C_n is closed microscopic nowhere dense set for $n \in \mathbb{N}$;
2. $C_n \cap C_m = \emptyset$ for $n,m \in \mathbb{N}$, $n \neq m$;
3. $\bigcup\limits_{n=1}^{\infty} C_n$ is dense in $[0,1]$.

Put $A_1 = (0,1)$. By Theorem 2.10 in [21] there exists a closed nowhere dense microscopic set $C_1 \subset A_1$. Now we proceed by induction. Let $n \in \mathbb{N}$, $n \geq 2$. Suppose that we have defined the pairwise disjoint sets $C_1, C_2, ..., C_{n-1}$ closed, nowhere dense and microscopic such that for each $p \in \{1, ..., n-1\}$

$$C_p \subset \left(\frac{l}{2^k}, \frac{l+1}{2^k}\right) \setminus \bigcup_{i=1}^{p-1} C_i$$

where $k \in \mathbb{N} \cup \{0\}$, $l \in \left\{0, 1, ..., 2^k - 1\right\}$ and the pair k, l fulfills equality $p = 2^k + l$.

There exists the unique pair k, l of integers such that $k \in \mathbb{N} \cup \{0\}$, $l \in \left\{0, 1, ..., 2^k - 1\right\}$ and $n = 2^k + l$. Put

$$A_n = \left(\frac{l}{2^k}, \frac{l+1}{2^k}\right) \setminus \bigcup_{i=1}^{n-1} C_i.$$

Obviously, A_n is an uncountable set of type G_δ, hence, by Theorem 2.10 in [21], there exists a closed, nowhere dense, microscopic set $C_n \subset A_n$, so

$$C_n \cap \bigcup_{i=1}^{n-1} C_i = \emptyset.$$

It is easy to see that the sequence $\{C_n\}_{n \in \mathbb{N}N}$ fulfills the conditions (1)-(3) and $\bigcup_{i=1}^{\infty} C_n \in \mathcal{M}^* \setminus \mathcal{P}$. □

20.3 $\mathcal{B}or \triangle \mathcal{M}$

We consider a σ-field of Borel sets modulo a σ-ideal $\mathcal{M}$:

$$\mathcal{B}or \triangle \mathcal{M} := \{B \triangle M : B \in \mathcal{B}or \text{ and } M \in \mathcal{M}\}.$$

Clearly $\mathcal{B}or \triangle \mathcal{M} \subset \mathcal{B}or \triangle \mathcal{N} = \mathcal{L}$. We shall prove that $\mathcal{B}or \triangle \mathcal{M} \neq \mathcal{L}$. For this purpose we need an auxiliary lemma.

Lemma 20.7. *The following conditions are equivalent:*

1). $E \in \mathcal{B}or \triangle \mathcal{M}$
2). there exist two Borel sets A_1, A_2 such that $A_1 \subset E \subset A_2$ and $A_2 \setminus A_1 \in \mathcal{M}$.

Proof. 1)⇒2). Let $E \in \mathcal{B}or \triangle \mathcal{M}$. Then there exist $B \in \mathcal{B}or$ and $M \in \mathcal{M}$ such that $E = B \triangle M$. Hence $B \setminus M \subset E \subset B \cup M$. There exists a microscopic Borel set N (of type G_δ) such that $M \subset N$. Then $B \setminus N \subset E \subset B \cup N$, $(B \cup N) \setminus (B \setminus N) = N \in \mathcal{M}$ and $B \setminus N$ as well as $B \cup N$ are Borel sets.

2)$\Rightarrow$1). Suppose that there exist two Borel sets A_1, A_2 such that $A_1 \subset E \subset A_2$ and $A_2 \setminus A_1 \in \mathcal{M}$. Then $E = A_1 \cup D$, where $D \subset A_2 \setminus A_1$, so $D \in \mathcal{M}$ and consequently $E \in \mathcal{B}or\triangle\mathcal{M}$. $\square$

Theorem 20.8. *There exists a measurable set E such that $E \notin \mathcal{B}or\triangle\mathcal{M}$.*

Proof. Let C be a classical Cantor set and let B denote the Bernstein set. Put $C_1 = C \cap B$ and $C_2 = C \setminus B$. Then $C = C_1 \cup C_2$, so $C_1 \notin \mathcal{M}$ or $C_2 \notin \mathcal{M}$ (as $C \notin \mathcal{M}$). Suppose that $C_1 \notin \mathcal{M}$ (the case $C_2 \notin \mathcal{M}$ is analogous). Let A_1, A_2 be two arbitrary Borel sets such that $A_1 \subset C_1 \subset A_2$. Suppose that A_1 is uncountable. Then from Alexandroff-Hausdorff theorem it contains some Cantor-type set (uncountable and closed). It gives a contradiction with the fact that both B and $\mathbb{R} \setminus B$ meet every uncountable closed subset of the real line. Hence A_1 is countable, so $C_1 \setminus A_1 \notin \mathcal{M}$. Clearly, $C_1 \setminus A_1 \subset A_2 \setminus A_1$, and $A_2 \setminus A_1 \notin \mathcal{M}$. Put $E = C_1$ Using the previous lemma we get a measurable set E such that $E \notin \mathcal{B}or\triangle\mathcal{M}$. $\square$

Using the notion of a Bernstein set, which is a useful tool to investigate a σ-field of Borel sets modulo a σ-ideal ([7]), we can prove even more.

Let $A \subset \mathbb{R}$ and let A contain a perfect set.

Definition 20.9. A set $B \subset A$ is called a Bernstein set relatively to A if both B and $A \setminus B$ meet each perfect subset of A.

We will use the following:

Proposition 20.10 ([7]). *If a σ-ideal $\mathcal{I}$ has a Borel base and $A \subset X$ is an analytic set such that $A \notin \mathcal{I}$ then there is no set B in $\mathcal{B}or\triangle\mathcal{I}$ which is a Bernstein set relatively to A.*

Applying the last result with A - the classical one-third Cantor set yields:

Corollary 20.11. *There exists a Bernstein set B relatively to a perfect nowhere dense Lebesgue null set, such that $B \notin \mathcal{B}or\triangle\mathcal{M}$.*

According to the above corollary, we have:

Theorem 20.12.

$$(\mathcal{N} \cap \mathcal{ND}) \setminus (\mathcal{B}or\triangle\mathcal{M}) \neq \emptyset,$$

where $\mathcal{ND}$ denotes the family of all nowhere dense sets.

To check the countable chain condition for the σ - field $\mathcal{B}or\triangle\mathcal{M}$ and $\mathcal{M}$ we verify a stronger condition, the "property (D)", which is defined in a more general case. Let X be a perfect Polish space, $(X,+)$ - a metric abelian group, $\mathcal{I}$ - an invariant ideal.

Definition 20.13 ([5]). $\mathcal{I}$ has the property (D) if there exists a set $B \in \mathcal{B}or \setminus \mathcal{I}$ and a perfect set $P \subset X$ such that $\{B+x : x \in P\}$ forms a disjoint family.

Proposition 20.14. *$\mathcal{M}$ has the property (D).*

Proof. If $\mathcal{I}$ and $\mathcal{J}$ are ideals such that $\mathcal{I} \subset \mathcal{J}$ and $\mathcal{J}$ has the property (D) then $\mathcal{I}$ has (D). It was shown in [5] that for every $s \in (0,1)$ the σ-ideal $\mathcal{J}_s$ of s-dimensional Hausdorff measure zero sets has the property (D). So, since $\mathcal{M} \subset \mathcal{H}_0 \subset \mathcal{J}_s$, we are done. □

Corollary 20.15. *$(\mathcal{B}or \triangle \mathcal{M}, \mathcal{M})$ does not satisfy ccc.*

20.4 Studies on the possibility of replacing Lebesgue nullsets by microscopic sets in the classical theorems of measure theory and theory of real functions.

In 2008 A. Karasińska and E. Wagner-Bojakowska in [22] studied how "big" can be a set on which a nowhere monotone continuous function can be injective. They proved that a "typical" (in a sense of Baire category) continuous function on [0,1] is nowhere monotone and injective outside a microscopic set. This result is a strenghtening of the result described in [12] (see Ex. 10: 6.6, p. 471). In the paper [22] one can find an example of a continuous nowhere monotone function with a bounded variation on [0,1] (so not a "typical" continuous function), which is injective outside a microscopic set.

In [21] A. Karasińska, W. Poreda and E. Wagner-Bojakowska proved that the theorem analogous to Sierpiński-Erdös Duality Theorem for the family of microscopic sets and sets of the first category on the real line is valid.

In 1934, W. Sierpiński proved in [29] (assuming CH) that there exists a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f(E)$ is a nullset if and only if E is of the first category. Sierpiński asked whether a stronger theorem is also valid: does there exist a bijection $f : \mathbb{R} \to \mathbb{R}$ that maps each of two classes $\mathcal{M}$ and $\mathcal{N}$ onto the other. The positive answer to this question was given in 1943 by P. Erdös in [14]. Erdös proved (assuming CH) that there exists a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f = f^{-1}$ and $f(E)$ is a nullset if and only if E is a set of the first category. From these properties it follows that $f(E)$ is a set of the first category if and only if E has Lebesgue measure zero. From Erdös result there follows a theorem known as Duality Principle (see [27], Theorem 19.4).

Observe that if we change the notion of set of Lebesgue measure zero by the notion of a microscopic set, the theorem analogous to Duality Principle will also be true.

For this purpose it sufficient to prove that the family $\mathcal{M}$ has the following properties:

(a) $\mathcal{M}$ is a σ-ideal
(b) the union of $\mathcal{M}$ is equal to $\mathbb{R}$
(c) $\mathcal{M}$ has a subfamily $\mathcal{G}$ such that $card(\mathcal{G}) \leq \aleph_1$ and for each $A \in \mathcal{M}$ there exists $B \in \mathcal{G}$ such that $A \subset B$
(d) the complement of each set $A \in \mathcal{M}$ contains a set of cardinality $\aleph_1$ which also belongs to $\mathcal{M}$.

The condition (a) was already justified, (b) is obvious, (c) follows from Theorem 20.2 (assuming CH). We concentrate on (d). For any $A \in \mathcal{M}$ we have $\lambda(A) = 0$, so $\mathbb{R} \setminus A$ contains some uncountable closed subset. Using Theorem 2.10 from [21] we obtain that $\mathbb{R} \setminus A$ contains some microscopic set with cardinality $\aleph_1$.

Using Theorem 19.5 in [27] we obtain

Theorem 20.16 ([21], Theorem 2.12). *(CH). There exists a one-to-one mapping f of the real line onto itself such that $f = f^{-1}$ and $f(E)$ is a microscopic set if and only if E is a set of the first category.*

Consequently, for microscopic sets the theorem analogous to Duality Principle holds:

Theorem 20.17 (CH). *Let P be any proposition involving solely the notions of microscopic set, first category set and notions of pure set theory. Let P^* be the proposition obtained from P by interchanging the terms "microscopic set" and "set of the first category" whenever they appear. Then each of the proposition P and P^* implies the other.*

However, the extended principle, where the notions of measurability and Baire property would be interchanged, is not true.

Among many similarities between σ-ideals $\mathcal{N}$ and $\mathcal{M}$ so called Steinhaus property is worth to be mentioned.

Let A and B be two subsets of the real line. By $A+B$ we denote the algebraic sum of A and B, i.e. $A+B := \{x+y : x \in A, y \in B\}$. In 1920 H. Steinhaus proved in [30] that for arbitrary measurable sets A, B of positive measure, so outside σ-ideal $\mathcal{N}$, $\mathrm{int}(A+B) \neq \emptyset$.

A category analogue of the theorem of Steinhaus was proved by S. Piccard (see [28]). If $A,B \subset \mathbb{R}$ are two sets of the second category having the Baire property, then $\mathrm{int}(A+B) \neq \emptyset$.

Observe that for microscopic sets the analogous property is not true: contrary to the σ-ideal $\mathcal{N}$ of Lebesgue measure zero sets and to the σ-ideal $\mathcal{M}$ of sets of the first category $\mathcal{M}$ has not a Steinhaus property.

Theorem 20.18. *There exists a Borel set $A \subset \mathbb{R}$ such that A is not microscopic and* $\mathrm{int}(A+A) = \emptyset$.

Proof. Let C be a Cantor set and let H be the set of all end-points of component intervals of $[0,1] \setminus C$. Put $A = C \setminus H$ and for each $k \in \mathbb{N}$ let

$$N_k = \bigcup_{i=0}^{3^k} \left\{ \frac{2i}{3^k} \right\}.$$

Then for each $k \in \mathbb{N}$ we have

$$(A \times A) \cap \bigcup_{\alpha \in N_k} \{(x,y) : y = -x + \alpha\} = \emptyset.$$

Hence for each $k \in \mathbb{N}$

$$(A+A) \cap N_k = \emptyset,$$

so

$$(A+A) \cap \bigcup_{k \in \mathbb{N}} N_k = \emptyset.$$

The set $\bigcup_{k \in \mathbb{N}} N_k$ is dense in $[0,2]$, so $\mathrm{int}(A+A) = \emptyset$. □

It is well known that the theorem converse to Steinhaus or Piccard results is not true. There exists a nowhere dense set $A \subset \mathbb{R}$ of measure zero such that $\mathrm{int}(A+A) \neq \emptyset$. This condition holds for example for Cantor set C because $C + C = [0,2]$. Observe, that the analogous property also holds for the σ-ideal $\mathcal{M}$.

Theorem 20.19. *There exists a microscopic set $A \subset \mathbb{R}$ such that* $\mathrm{int}(A+A) \neq \emptyset$.

Proof. Let A be a microscopic set residual in $\mathbb{R}$ (see Theorem 20.4). From theorem of S. Piccard it follows that $A+A$ contains some interval. □

20.5 Extension of the notion of a microscopic set in the Euclidean spaces of higher dimensions.

In the n-dimensional Euclidean space the notion of microscopic set can be introduced using various differentiation bases (as rectangles with sides parallel to coordinate axes, or cubes for example). Hence we can obtain different notions of microscopic sets. The properties of the sets, their invariance with respect to translation, rotation and other algebraic and set-theoretic operations are investigated in [24].

Definition 20.20. We shall say that $A \subset \mathbb{R}^2$ is a microscopic set if for each $\varepsilon > 0$ there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of rectangles with sides which are parallel to coordinate axes such that $A \subset \bigcup_{n\in\mathbb{N}} I_n$ and $\lambda_2(I_n) < \varepsilon^n$ for each $n \in \mathbb{N}$.

Definition 20.21. We shall say that $A \subset \mathbb{R}^2$ is a strongly microscopic set if for each $\varepsilon > 0$ there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of squares with sides which are parallel to coordinate axes such that $A \subset \bigcup_{n\in\mathbb{N}} I_n$ and $\lambda_2(I_n) < \varepsilon^n$ for each $n \in \mathbb{N}$.

Denote by $\mathcal{M}_2$ the family of all microscopic sets in $\mathbb{R}^2$ and by $\mathcal{M}_{2s}$ the family of all strongly microscopic sets in $\mathbb{R}^2$. Obviously, each strongly microscopic set is microscopic, so $\mathcal{M}_{2s} \subset \mathcal{M}_2$.

In the sequel a rectangle with sides which are parallel to coordinate axes will be called an interval.

Analogously as on the real line one can prove the following theorems.

Theorem 20.22 ([24], Theorem 3). *The families $\mathcal{M}_2$ and $\mathcal{M}_{2s}$ are the σ-ideals.*

Theorem 20.23 ([24], Theorem 4). *The following conditions are equivalent:*

(i) A is a microscopic set on the plane.

(ii) For each positive number η there exists a sequence $\{J_n\}_{n\in\mathbb{N}}$ of intervals such that

$$A \subset \limsup_n J_n \text{ and } \sum_{k=n}^{\infty} \lambda_2(J_k) < \eta^n \text{ for each } n \in \mathbb{N}.$$

(iii) For each positive number δ there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of intervals such that

$$A \subset \limsup_n I_n \text{ and } \lambda_2(I_n) < \delta^n \text{ for each } n \in \mathbb{N}.$$

The analogous theorem holds for strongly microscopic sets (the intervals are changed with the squares).

Theorem 20.24 ([24], Theorem 5). *The plane can be represented as the union of two disjoint sets A and B such that A is a set of the first category and B is a strongly microscopic set.*

Corollary 20.25 ([24], Corollary 6). *There exists a strongly microscopic set $B \subset \mathbb{R}^2$ which is residual.*

Theorem 20.26 ([24], Theorem 14). *If $A \in \mathcal{M}_{2s}$ and $(\alpha, \beta) \in \mathbb{R}^2$, then*

(a) $A + (\alpha, \beta) = \{(x+\alpha, y+\beta) : (x,y) \in A\} \in \mathcal{M}_{2s}$,
(b) $-A = \{(-x,-y) : (x,y) \in A\} \in \mathcal{M}_{2s}$,
(c) $(\alpha, \beta) \cdot A = \{(\alpha \cdot x, \beta \cdot y) : (x,y) \in A\} \in \mathcal{M}_{2s}$,
(d) if $A \cap \{(x,y) : x \cdot y = 0\} = \emptyset$, then $A^{-1} = \{(\frac{1}{x}, \frac{1}{y}) : (x,y) \in A\} \in \mathcal{M}_{2s}$.

Clearly the analogous theorem holds for the family $\mathcal{M}_2$.

Let us denote by $\mathcal{N}_2$ the family of all sets of Lebesgue measure zero on the plane and by $\mathcal{C}_2$ - the family of all countable subsets of the plane.

Obviously, each countable set is strongly microscopic and if A is microscopic, then A is of Lebesgue measure zero, so we have

$$\mathcal{C}_2 \subset \mathcal{M}_{2s} \subset \mathcal{M}_2 \subset \mathcal{N}_2.$$

Observe that all these inclusions are proper. It is easy to see that the set

$$A = \{(x,x) : x \in [0,1]\} \tag{20.4}$$

is a set of plane measure zero which is not microscopic on the plane, and the set

$$B = [0,1] \times \{0\} \tag{20.5}$$

is a microscopic set on the plane which is not strongly microscopic. Clearly, each residual strongly microscopic set is uncountable, so

$$\mathcal{C}_2 \subsetneq \mathcal{M}_{2s} \subsetneq \mathcal{M}_2 \subsetneq \mathcal{N}_2.$$

Comparing the sets A and B defined above we see that the family $\mathcal{M}_2$ is not invariant under rotation with respect to the origin. For the family $\mathcal{M}_{2s}$ the situation is quite different.

Theorem 20.27 ([24], Theorem 9). *The set $A \in \mathcal{M}_{2s}$ if and only if for each $\varepsilon > 0$ there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of circles on the plane such that $A \subset \bigcup_{n\in\mathbb{N}} B_n$ and $\lambda_2(B_n) < \varepsilon^n$ for each $n \in \mathbb{N}$.*

Consequently, the family $\mathcal{M}_{2s}$ is invariant under rotation and if $A \in \mathcal{M}_{2s}$, then the projection of A onto any line is a microscopic set.

If $E \subset X \times Y$ and $x \in X$, the set $E_x = \{y \in Y : (x,y) \in E\}$ is called the x-section of E.

Fubini Theorem underlines a close connection between the measure of any plane measurable set and the linear measure of its sections perpendicular to an axis. In [27], Theorem 14.2 one can find an elementary proof of the fact that if E is a plane set of measure zero, then E_x is a linear nullset for all x outside a set of linear measure zero.

Fubini Theorem has a category analogue. Kuratowski and Ulam in 1932 proved (compare [27], Theorem 15.1) that if E is a plane set of the first category, then E_x is a linear set of first category for all x except those belonging to a certain set of the first category.

Note that for strongly microscopic sets the result analogous to Fubini theorem also holds. It is not difficult to observe it because there is a close connection between the area of the square and the length of its side.The result analogous to Fubini theorem for microscopic sets on the plane is also valid.

Theorem 20.28 ([24], Theorem 17). *Let $E \subset \mathbb{R}^2$ be a microscopic set on the plane. Then E_x is a microscopic set on the real line for each $x \in \mathbb{R}$ outside some microscopic set on the real line, i.e. the set*

$$\{x \in \mathbb{R} : E_x \textit{ is not a microscopic set on the real line}\}$$

is microscopic on $\mathbb{R}$.

Using Theorem 20.28 we proved

Theorem 20.29 ([24], Theorem 18). *A product set $A \times B$ is microscopic on the plane if and only if at least one of the sets A or B is microscopic on the real line.*

20.6 Additional remarks

In 2003 G. Horbaczewska and E. Wagner-Bojakowska introduced a definition of convergence of a sequence of functions with respect to the σ-ideal of microscopic sets. The idea comes from the Riesz theorem which states that a sequence of measurable functions $\{f_n\}_{n\in\mathbb{N}}$ is convergent in measure to the function f if and only if for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that the sequence $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ is convergent to f almost everywhere (i.e. outside the set of measure zero). Therefore convergence in measure (in a finite measure space) can be defined using only the notion of a nullset (compare Chapter 6). This enables us to define convergence of the sequence of functions for different σ-ideals. In [18] two kinds of such a convergence, for the σ-ideal of microscopic sets and for the σ-ideal of sets of first Baire category were compared with the convergence in measure and with the convergence introduced by G. Beer using the Hausdorff metric. It was shown that even for continuous functions we have different types of convergence.

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