http://dx.doi.org/10.18778/7525-971-1.27

Chapter 27
Decompositions of permutations of $\mathbb{N}$ with respect to divergent permutations

ROMAN WITUŁA

2010 Mathematics Subject Classification: 40A05, 05A99.
Key words and phrases: divergent permutations, convergent permutations, sum-preserving permutations.

27.1 Basic technical notions

A bijection of set $A \subset \mathbb{N}$ onto itself is called here a permutation of A. If p is a permutation of $\mathbb{N}$ then the symbol p^k denotes the product (composition is an equivalent term) of k permutations p, i.e., $p^1 := p$, $p^{k+1} := p \circ p^k$, $k \in \mathbb{N}$. A permutation p of $\mathbb{N}$ will be called almost identity on $\mathbb{N}$ if there exists $k = k(p) \in \mathbb{N}$ such that $p(n) = n$ for each $n \in \mathbb{N}$, $n \geqslant k$.

We note that any permutation p of $\mathbb{N}$ can be written as a product of distinct (meaning "disjoint" in this paper) cycles:

– finite cycles: $\big(a = p^n(a), p(a), p^2(a), \ldots, p^{n-1}(a)\big)$ where $a \in \mathbb{N}$ and n is called the length of this cycle,
– infinite cycles: $\big(\ldots, p^{-2}(a), p^{-1}(a), a, p(a), p^2(a), \ldots\big)$ where $a \in \mathbb{N}$.

Also any product of distinct cycles represents a permutation. A cycle of length 2 is called a transposition.

Family of permutations of $\mathbb{N}$ will be denoted by $\mathfrak{P}$. Let $p \in \mathfrak{P}$. The p-order of element $a \in \mathbb{N}$ is defined to be the smallest positive integer k satisfying relation $p^k(a) = a$. If such an integer does not exist we say that p-order of a is infinite. It is obvious that in this case the p-cycle generated by a has the form $(\ldots, p^{-2}(a), p^{-1}(a), a, p(a), p^2(a), \ldots)$, i.e. it is the infinite cycle and $p^k(a) \neq p^l(a)$ for any two different integers k and l. Set $G \subset \mathbb{N}$ will be called a minimal set of generators of p if G is a set of values of any choice function on the family of sets of members of all distinct cycles of p. The p-cycle generated by a will be denoted by $\operatorname{cycle}(p,a)$. Moreover, we say that $(a,b,c,\ldots)$ or $(\ldots,\beta,\alpha,a,b,c,\ldots)$ are p-cycles if all $a,b,c,\alpha,\beta,\ldots$ are positive integers and $b = p(a)$, $c = p^2(a)$, $\alpha = p^{-1}(a)$, $\beta = p^{-2}(a)$ and so on. Finally the order of permutation $p \in \mathfrak{P}$ is defined to be the smallest positive integer k such that $p^k = \operatorname{id}(\mathbb{N})$, where the symbol $\operatorname{id}(A)$ denotes the identity function on A for every nonempty $A \subset \mathbb{N}$. If such an integer does not exist we say that order of p is infinite.

In this paper the inclusion will be denoted by $\subseteq$. Sign $\subset$ is reserved for the proper inclusion, i.e., $A \subset B$ if $A \subseteq B$ and $A \neq B$. Finite set $I \subset \mathbb{N}$ will be called an interval or an interval of $\mathbb{N}$ if there exist $m,n \in \mathbb{N}$, $m \leqslant n$, such that $I = \{k \in \mathbb{N} :\ m \leqslant k \leqslant n\}$. Only this type of intervals will be discussed in the paper.

We say that a nonempty set $A \subset \mathbb{N}$ is a union of n MSI (or of at most n MSI or of at least n MSI) if there exists a family $\mathbb{F}$ of n (or of at most n or of at least n, respectively) intervals of $\mathbb{N}$ forming a partition of A and such that $\operatorname{dist}(I,J) \geqslant 2$ for any two different elements I,J of $\mathbb{F}$. MSI is the abbreviated form of the notion of **m**utually **s**eparated **i**ntervals.

A countable family $\{A_n\}$ of nonempty and finite subsets of $\mathbb{N}$ will be called an increasing sequence if $\max A_n < \min A_{n+1}$ for every $n \in \mathbb{N}$. Let $a,b \in \mathbb{N}$ and $\emptyset \neq A \subset \mathbb{N}$. Then we will write $a < A < b$ if $a < \alpha < b$ for every $\alpha \in A$.

27.2 Main notions and results

Permutation $p \in \mathfrak{P}$ rearranging some convergent real series $\sum a_n$ into divergent series $\sum a_{p(n)}$ is called a divergent permutation. Family of all permutations of that kind will be denoted by $\mathfrak{D}$. Whereas the subset of $\mathfrak{D}$ composed of all permutations p such that p^{-1} is also a divergent permutation, will be denoted by $\mathfrak{D}\mathfrak{D}$ and called, after Kronrod [2] and myself, the family of two-sided divergent permutations.

For contrast, permutation $p \in \mathfrak{C} := \mathfrak{P} \setminus \mathfrak{D}$ is called a convergent permutation because it rearranges each convergent real series $\sum a_n$ into a series which is convergent as well.

In the paper we will use permanently the following combinatorial characterization of divergent permutations (dual to the Agnew's combinatorial characterization of convergent permutations [1], [5], [6]). Let $p \in \mathfrak{P}$. Then $p \in \mathfrak{D}$ if and only if for every $n \in \mathbb{N}$ there exists an interval $I \subset \mathbb{N}$ such that set $p(I)$ is a union of at least n MSI. There exist also many other characterizations of divergent permutations (see [7], [12], [8]) but they will be not used in this paper.

We say that a nonempty family $\mathcal{A} \subset \mathfrak{P}$ is algebraically small if $\mathfrak{P} \setminus G(\mathcal{A}) \neq \emptyset$, where $G(\mathcal{A})$ denotes the group of permutations generated by $\mathcal{A}$. Similarly we say that a family $\mathcal{A} \subset \mathfrak{P}$ is algebraically big if $\mathcal{A} \circ \mathcal{A} := \{p \circ q : \ p, q \in \mathcal{A}\} = \mathfrak{P}$. Henceforward the symbol $\circ$ of composition of permutations will be also used with regard to the superposition of any nonempty sets $A, B \subset \mathfrak{P}$, i.e.

$$A \circ B := \big\{p \circ q : \ p \in A \ \text{ and } \ q \in B\big\}.$$

It is known that $\mathfrak{C}$ is algebraically small. This fact was proven by Pleasants [3], [4].

Remark 27.1. There exist subsets of $\mathfrak{P}$ which are neither algebraically small nor algebraically big. For example, any set G of generators of $\mathfrak{P}$ such that $G \circ G \neq \mathfrak{P}$ possesses this property. In the following example the construction of such set of generators of $\mathfrak{P}$ will be presented.

Example 27.2. Using transfinite induction we can construct a transfinite sequence $\{G_\omega\}_{\omega \in \Omega}$ of subsets of $\mathfrak{P}$ such that

$$G_{end} = \bigcup_{\omega \in \Omega} G_\omega$$

is a set of generators of $\mathfrak{P}$ and for every $\omega_0 \in \Omega$ if G is a group of permutations generated by set $\bigcup_{\omega < \omega_0} G_\omega$ then

$$G_{\omega_0} = \{p\} \cup \bigcup_{\omega < \omega_0} G_\omega$$

and a permutation $p = p(G)$ is chosen from set $\mathfrak{P} \setminus G$.

Hence we can prove that if $p \in G_{end}$ then there exists $n \in \mathbb{N}$ such that

$$p^n \notin G_{end} \circ G_{end}.$$

With this end in view we consider first the following case. Suppose that there exist $p, p^k \in G_{end}$, where

$$k := \min\big\{n \in \mathbb{N} : \ n > 1 \ \text{ and } \ p^n \in G_{end}\big\}.$$

We prove, by definition of sets G_ω, $\omega \in \Omega$, that there exists $n \in \mathbb{N}$, $n \geq 2$ such that both p^{kn} and $p^{k(n+1)}$ do not belong to G_{end}. To this aim let us suppose that for some $n \in \mathbb{N}$ both p^{kn} and $p^{k(n+1)}$ belong to G_{end}. Then by definition of sets G_ω, $\omega \in \Omega$, both p^{kn} and $p^{k(n+1)}$ are predecessors of the chosen p^k which leads to a contradiction since $p^k = p^{k(n+1)} \circ p^{-kn} \notin G_{end}$.

So, the two cases are possible: either all $p^{2nk} \in G_{end}$, which is impossible since the last chosen $p^{2n_0k} = p^{6n_0k} \circ p^{-4n_0k}$ is not in G_{end}, or all $p^{(2n+1)k}$ are in G_{end} which is also impossible, since $p^k = p^{3k} \circ p^{5k} \circ p^{-7k} \notin G_{end}$.

Therefore let us assume that $p^{kn} \notin G_{end}$, $p^{k(n+1)} \notin G_{end}$ and

$$p^{kn} \in G_{end} \circ G_{end},$$

more precisely

$$p^{kn} = \varphi \circ \psi, \qquad \varphi, \psi \in G_{end}, \quad \varphi, \psi \notin \{p^m : \ m \in \mathbb{N}\}.$$

Then both φ and ψ would be chosen before choosing p^k – on the other hand either $\varphi = p^{kn} \circ \psi^{-1} \notin G_{end}$ or $\psi = \varphi^{-1} \circ p^{kn} \notin G_{end}$, which gives a contradiction. Then we would have

$$p^{k(n+1)} \notin G_{end} \circ G_{end},$$

because in opposite case $p^{k(n+1)} = \varphi_1 \circ \psi_1$, $\varphi_1, \psi_1 \in G_{end}$, φ_1, ψ_1 are predecessors of p^k. In consequence $p^k = \varphi_1 \circ \psi_1 \circ \psi^{-1} \circ \varphi^{-1} = p^{k(n+1)} \circ p^{-kn} \notin G_{end}$, which leads to contradiction.

One more important case is left, i.e. when

$$p^{kn} = p^r \circ p^s$$

and $p^r, p^s \in G_{end}$. If also

$$p^{k(n+1)} = p^{r_1} \circ p^{s_1}$$

and $p^{r_1}, p^{s_1} \in G_{end}$, then by definition of the power k we obtain

$$p^k = p^{r_1} \circ p^{s_1} \circ p^{-r} \circ p^{-s} \notin G_{end}$$

which is impossible again.

The remaining cases can be discussed similarly and will be omitted here. Thus indeed

$$G_{end} \circ G_{end} \neq \mathfrak{P}.$$

There exists one more interesting family $\mathfrak{S} \subset \mathfrak{P}$ of permutations preserving the sum of rearranged series. We say that $p \in \mathfrak{S}$ if for each convergent real series $\sum a_n$ the fact that series $\sum a_{p(n)}$ is convergent implies the equality of sums of both series: $\sum a_n = \sum a_{p(n)}$. Certainly $\mathfrak{C} \subset \mathfrak{S}$ but family $\mathfrak{S}$ is already algebraically big which was proven by Kronrod [2] (see also [8] for more details).

Similarly, family $\mathfrak{D}$ and even family $\mathfrak{D}\mathfrak{D}$ are algebraically big. However it seems that this idea with reference to these families quite poorly describes the algebraic nature of these two sets. Aim of this paper is to emphasize "the greatness" and "the internal variety" of families $\mathfrak{D}$ and $\mathfrak{D}\mathfrak{D}$ through more complicated algebraic operations on subfamilies of $\mathfrak{D}$ and $\mathfrak{D}\mathfrak{D}$. These operations are included in the following collection of theorems (proofs of all these results are given in Section 3 of this paper).

For brevity for any $p,q \in \mathfrak{P}$ the composition of permutations p with q will be henceforward denoted by pq (i.e. $pq(n) := p(q(n))$, $n \in \mathbb{N}$).

Theorem 27.3.

(i) Let $k,n \in \{2,3,\ldots\}$. Then each permutation $p \in \mathfrak{P}$ can be expressed in the form $p = q_2^n q_1^k$ for some $q_1, q_2 \in \mathfrak{D}\mathfrak{D}$.

(ii) Every permutation $p \in \mathfrak{P}$ is a composition of two permutations, both of infinite order, belonging to $\mathfrak{D}\mathfrak{D}$.

Corollary 27.4. *We have*

$$\left\{p^k :\ p \in \mathfrak{D}\mathfrak{D}\right\} \circ \left\{p^s :\ p \in \mathfrak{D}\mathfrak{D}\right\} = \mathfrak{P}$$

for every $k,s \in \mathbb{N}$ such that k and s are simultaneously not smaller than two or $k = s = 1$.

We note that Theorem 27.3 (i) results easily from the following proposition and lemma.

Proposition 27.5. *Let $q \in \mathfrak{P}$ and let $\mathbb{G}$ denote a minimal set of generators of q (with respect to inclusion). Let $\mathbb{O} := \{n \in \mathbb{N} :\ q(n) \neq n\}$. If the set $\mathbb{G} \cap \mathbb{O}$ is infinite and q-orders of any two elements $a,b \in (\mathbb{G} \cap \mathbb{O})$ are the same, then for every $k = 2,3,\ldots$, the equation $p^k = q$ has a solution $p \in \mathfrak{D}\mathfrak{D}$ such that $\mathbb{O} = \{n \in \mathbb{N} :\ p(n) \neq n\}$. This means that p is a root of k-th order of q.*

We may suppose that permutation p possesses the following property: for each $n \in \mathbb{N}$ there exist intervals I, J and sets $A \subset p(I)$, $B \subset p^{-1}(J)$, both having the cardinality n, such that

$$|a - a^*| > \operatorname{card} I \qquad \textit{and} \qquad |b - b^*| > \operatorname{card} J$$

for any two different $a, a^* \in A$ and $b, b^* \in B$, respectively.

Lemma 27.6. *Each permutation $p \in \mathfrak{P}$ is a product of two permutations q_1 and q_2 of $\mathbb{N}$ having the following form*

$$\prod_{n\in\mathbb{N}} \big(a_{2n-1}, a_{2n}\big), \tag{27.1}$$

where $\{a_n\}$ is some one-to-one sequence of all positive integers. In other words, both q_1 and q_2 are products of infinitely many distinct transpositions.

Theorem 27.7. *Let $p \in \mathfrak{P}$.*

(i) If there exists an infinite set of generators of p, which is minimal with respect to inclusion, then for each $k = 2, 3, \ldots$ there exist permutations $\phi_i = \phi_i(k) \in \mathfrak{DD}$ and $\psi_i = \psi_i(k) \in \mathfrak{DD}$, $i = 1, 2$, all having infinite order, such that $\phi_2\phi_1^k = \psi_2^k\psi_1 = p$.

(ii) If there exist a finite set of generators of p, then for every $k = 2, 3, \ldots$ there exist permutations $\phi_i = \phi_i(k) \in \mathfrak{CC}$ and $\psi_i = \psi_i(k) \in \mathfrak{P}$ for $i = 1, 2$, such that $\psi_1^k\phi_1 = \phi_2\psi_2^k = p$.
Moreover, if p belongs to $\mathfrak{D}$ or $\mathfrak{DD}$ then ψ_1 and ψ_2 can be chosen to belong also to $\mathfrak{D}$ or $\mathfrak{DD}$, respectively. This result follows at once from the relations (see [8], [10], [11]):

$$\mathfrak{C} \circ \mathfrak{C} = \mathfrak{C}, \quad \mathfrak{DC} \circ \mathfrak{DC} = \mathfrak{DC}, \quad \mathfrak{DC} \circ \mathfrak{CC} = \mathfrak{CC} \circ \mathfrak{DC} = \mathfrak{DC}.$$

Theorem 27.8. *Let us denote by $\mathfrak{I}$ the family of all almost identity permutations on $\mathbb{N}$. Then for any $k = 2, 3, \ldots, \infty$ there exists a group of permutations $\mathfrak{G}_k \subset \mathfrak{I} \cup \mathfrak{DD}$ with the following properties:*

(i) q^k is the identity function of $\mathbb{N}$ for every element $q \in \mathfrak{G}_k$,

(ii) the set of all elements $q \in \mathfrak{G}_k \cap \mathfrak{DD}$, whose order is precisely equal to k, has the cardinality of the continuum,

(iii) if k is a prime number or $k = \infty$ then the order of any element $q \in \mathfrak{G}_k \cap \mathfrak{DD}$ is precisely equal to k.

Final remark

The following relation holds as well

$$\bigcup_{k=2}^{\infty} \mathfrak{DD}^k \neq \mathfrak{P}, \tag{27.2}$$

where $\mathfrak{D}\mathfrak{D}^k := \{p^k : p \in \mathfrak{D}\mathfrak{D}\}$, $k = 2,3,\ldots$. A reason for this relation is given by the following fact.

If $p \in \mathfrak{D}\mathfrak{D}^k$, where $k \geqslant 2$ and in the decomposition of permutation p into cycles there are only finitely many infinite cycles , then number of these cycles is divisible by k. Whereas we know that there exist permutations $p \in \mathfrak{D}\mathfrak{D}$ which are infinite cycles (see Example 27.9). Thus

$$\mathfrak{D}\mathfrak{D} \setminus \bigcup_{k=2}^{\infty} \mathfrak{D}\mathfrak{D}^k \neq \emptyset. \tag{27.3}$$

Simultaneously it means that relations (27.2) and (27.3) are of algebraic nature.

By the way we would like to notice that we do not know whether

$$\bigcup_{k=1}^{\infty} \mathfrak{D}\mathfrak{D}^k = \mathfrak{P}.$$

We do not know either if there exists a permutation $p \in \mathfrak{D}\mathfrak{C}$ which is an infinite cycle.

Example 27.9. In this example we present a permutation $q \in (\mathfrak{D}\mathfrak{D} \cap \mathfrak{S})$ which is an infinite cycle.

Let $\{I_n\}$ be an increasing sequence of intervals of positive integers forming a partition of $\mathbb{N}$ and satisfying the condition

$$\operatorname{card} I_{2n-1} = \operatorname{card} I_{2n} = 3n, \qquad n \in \mathbb{N}.$$

Then permutation q is given by relation

$$\big(\ldots, c_5, c_3, c_1, c_2, c_4, c_6, \ldots\big),$$

where c_{γ_n} is "a finite cycle" of the form

$$c_{\gamma_n} = \big(i_{3n}^{\gamma_n}, i_n^{\gamma_n}, i_{3n-2}^{\gamma_n}, i_{n-1}^{\gamma_n}, i_{3n-4}^{\gamma_n}, i_{n-2}^{\gamma_n}, \ldots \\ \ldots, i_{n+2}^{\gamma_n}, i_1^{\gamma_n}, i_{3n-1}^{\gamma_n}, i_{3n-3}^{\gamma_n}, i_{3n-5}^{\gamma_n}, \ldots i_{n+1}^{\gamma_n}\big),$$

whereas $G_{\gamma_n} = \{i_s^{\gamma_n} : s = 1,2,\ldots,3n\}$ – here γ_n denotes the upper index – is the increasing sequence of all elements of interval I_{γ_n} for $\gamma_n \in \{2n-1, 2n\}$ for each $n \in \mathbb{N}$.

Since each of two following sets

$$q([i_1^{\gamma_n}, i_n^{\gamma_n}]) \qquad \text{and} \qquad q^{-1}([i_1^{\gamma_n}, i_n^{\gamma_n}])$$

is a union of n MSI for each $n \in \mathbb{N}$, therefore $q \in \mathfrak{D}\mathfrak{D}$.

Next, from the relations given below

$$\begin{aligned} q(G_{2n+1}) &= \big(G_{2n+1} \setminus \{i_{3n+3}^{2n+1}\}\big) \cup \{i_{3n}^{2n-1}\}, \\ q(G_1) &= \big(G_1 \setminus \{i_3^1\}\big) \cup \{i_3^2\}, \\ q(G_{2n}) &= \big(G_{2n} \setminus \{i_{3n}^{2n}\}\big) \cup \{i_{3n+3}^{2n+2}\}, \end{aligned}$$

which hold for any $n \in \mathbb{N}$, we get that $q \in \mathfrak{S}$.

27.3 Proofs

Proof of Proposition 27.5. Let us fix $k \in \mathbb{N}$, $k > 1$. Suppose that the sets $\{a_1^{(n)}, a_2^{(n)}, \ldots, a_{kn}^{(n)}\}$, $n \in \mathbb{N}$, form the partition of $\mathbb{G} \cap \mathbb{O}$. We can assume that

$$|u - v| > \max\big\{a_i^{(n)} :\ i = 1, 2, \ldots, n\big\} \tag{27.4}$$

for any two different $u, v \in \{a_{i+n}^{(n)} :\ i = 1, 2, \ldots, n\}$. Let r be the q-order common for each element of $\mathbb{O}$ and let the permutation p be given by following formula

$$p = \prod_{n \in \mathbb{N}} \prod_{i=1}^{n} p(i, n)$$

in case when $r = \infty$ and by the formula

$$p = \prod_{n \in \mathbb{N}} \prod_{i=1}^{n} q(i, n)$$

in the case when $r < \infty$. Notations $p(i, n)$ and $q(i, n)$ designate here the cycles defined in the following way

$$\begin{aligned} p(i,n) := \Big(\ldots, q^{-1}\big(b_i^{(n)}\big), q^{-1}\big(b_{i+n}^{(n)}\big), q^{-1}\big(a_{i+2n}^{(n)}\big), \ldots, q^{-1}\big(a_{i+(k-1)n}^{(n)}\big), \\ b_i^{(n)}, b_{i+n}^{(n)}, a_{i+2n}^{(n)}, \ldots, a_{i+(k-1)n}^{(n)}, \\ q\big(b_i^{(n)}\big), q\big(b_{i+n}^{(n)}\big), q\big(a_{i+2n}^{(n)}\big), \ldots, q\big(a_{i+(k-1)n}^{(n)}\big), \ldots\Big) \end{aligned}$$

and

$$q(i,n) := \Big(b_i^{(n)}, b_{i+n}^{(n)}, a_{i+2n}^{(n)}, \ldots, a_{i+(k-1)n}^{(n)},$$
$$q\big(b_i^{(n)}\big), q\big(b_{i+n}^{(n)}\big), q\big(a_{i+2n}^{(n)}\big), \ldots, q\big(a_{i+(k-1)n}^{(n)}\big), \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$q^{r-1}\big(b_i^{(n)}\big), q^{r-1}\big(b_{i+n}^{(n)}\big), q^{r-1}\big(a_{i+2n}^{(n)}\big), \ldots, q^{r-1}\big(a_{i+(k-1)n}^{(n)}\big)\Big),$$

where $b_i^{(n)} = \tau^n(a_i^{(n)})$, $b_{i+n}^{(n)} = \tau^n(a_{i+n}^{(n)})$ and $\tau = \tau(i,n)$ denotes the transposition of elements $a_i^{(n)}$ and $a_{i+n}^{(n)}$ for every $i = 1,\ldots,n$ and $n \in \mathbb{N}$. Then we easily verify that $p^k = q$ and $\mathbb{O} = \{n \in \mathbb{N} : \ p(n) \neq n\}$. As a result we have the following inclusion

$$\{a_{1+n}^{(n)}, a_{2+n}^{(n)}, \ldots, a_{2n}^{(n)}\} \subset \gamma(I_n),$$

where $I_n = \big[1, \max\{a_i^{(n)} : \ i = 1,\ldots,n\}\big]$ and $\gamma = p$ if $n \in 2\mathbb{N}$ and $\gamma = p^{-1}$ if $n \in 2\mathbb{N}-1$. By (27.4) we conclude that each of the sets $p(I_n)$, $n \in 2\mathbb{N}$, and $p^{-1}(I_n)$, $n \in 2\mathbb{N}-1$, is a union of at least n MSI. Thus $q \in \mathfrak{D}\mathfrak{D}$ as desired. □

Proof of Theorem 27.3 (i). This assertion follows immediately from Proposition 27.5 applied to Lemma 27.6. □

Proof of Lemma 27.6. Let $p \in \mathfrak{P}$. If p is a finite cycle having one of the following forms

$$p = \big(b_{-n}, b_{-n+1}, \ldots, b_{-1}, b_0, b_1, \ldots, b_{n-1}, b_n\big)$$

or

$$p = \big(b_{-n}, b_{-n+1}, \ldots, b_{-1}, b_1, \ldots, b_{n-1}, b_n\big),$$

then

$$\begin{cases} p = q_2 q_1, & \text{where} \\ q_1 = \prod_{k=i}^{n-1} (b_k, b_{-k-1}) & \text{and} \qquad q_2 = \prod_{k=1}^{n} (b_k, b_{-k}) \end{cases} \tag{27.5}$$

for $i = 0$ or 1, respectively.

Next, if p is an infinite cycle of the form

$$p = \big(\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots\big),$$

then

$$\begin{cases} p = q_2 q_1, & \text{where} \\ q_1 = \prod_{k \in \mathbb{N}_0} (b_k, b_{-k-1}) & \text{and} \qquad q_2 = \prod_{k \in \mathbb{N}} (b_k, b_{-k}). \end{cases} \tag{27.6}$$

In general case, if set $\{n \in \mathbb{N} : \ p(n) \neq n\}$ is infinite then we can apply decompositions (27.5) and (27.6) to all the finite and infinite p-cycles, respectively. On the other hand, if p is almost identity on $\mathbb{N}$, then by applying decomposition (27.5) to all the nontrivial p-cycles and, additionally, by using the following decomposition

$$\text{the identity function of } f \text{ is equal to } qq,$$

where $f := \{f_n : \ n \in \mathbb{N}\}$ is a sequence of all fixed points of p and $q = \prod_{k\in\mathbb{N}}(f_{2k-1}, f_{2k})$, we may express p as a composition of two permutations of form (27.1). This completes the proof of lemma. □

Proof of Theorem 27.3 (ii). Let us fix $p \in \mathfrak{P}$. In the sequel we will construct the permutations $q_1, q_2 \in \mathfrak{D}\mathfrak{D}$, both having infinite order, and such that $q_2 q_1 = p$. First we choose inductively an increasing sequence $\{I_n\}$ of intervals satisfying the conditions

$$\operatorname{card} I_n = 5n, \tag{27.7}$$

$$\text{sets } \bigcup_{n\in\mathbb{N}} I_{2n-1} \text{ and } \bigcup_{n\in\mathbb{N}} p^{-1}(I_{2n}) \text{ are disjoint,} \tag{27.8}$$

$$\begin{array}{c}\text{complements of the following two sets: } \bigcup_{n\in\mathbb{N}} I_n \text{ and} \\ \bigcup_{n\in\mathbb{N}} I_{2n-1} \cup p^{-1}(I_{2n}) \text{ in } \mathbb{N} \text{ are infinite.}\end{array} \tag{27.9}$$

Next we define permutation ϕ of set $\bigcup_{n\in\mathbb{N}} I_n$ by the following formula

$$\begin{cases} \phi(i+a) = 2i+1+a, \\ \phi(i+n+a) = 2i+a, \end{cases} \tag{27.10}$$

$$\begin{cases} \phi(2i+1+2n+a) = i+2n+a, \\ \phi(2i+2n+a) = i+3n+a, \end{cases} \tag{27.11}$$

$$\phi(i+4n+a) = (i+1)(\operatorname{mod} n) + 4n + a, \tag{27.12}$$

where $a = \min I_n$ for every $i = 0, 1, \ldots, n-1$ and $n \in \mathbb{N}$. Hence $\phi(I_n) = I_n$ for every $n \in \mathbb{N}$. Now we can define the permutations q_1 and q_2.

Let q_1 be an increasing map of the complement of set $\bigcup_{n\in\mathbb{N}} I_{2n-1} \cup p^{-1}(I_{2n})$ in $\mathbb{N}$ onto the complement of set $\bigcup_{n\in\mathbb{N}} I_n$ in $\mathbb{N}$. Suppose also that q_1 is equal to the restriction of ϕ to $\bigcup_{n\in\mathbb{N}} I_{2n-1}$. On the other hand, let q_2 be equal to the restriction of ϕ to $\bigcup_{n\in\mathbb{N}} I_{2n}$. The values of q_1 and q_2 corresponding to all the other elements of $\mathbb{N}$ are defined by the equation $p = q_2 q_1$.

The main properties of q_1 and q_2, required to be verified, are as follows:

$$q_1, q_2 \in \mathfrak{D}\mathfrak{D}, \tag{27.13}$$

$$\text{orders of } q_1 \text{ and } q_2 \text{ are infinite.} \tag{27.14}$$

To check (27.13) we observe that, by (27.10) and (27.11), each of the following sets

$$\phi\big([\min I_n, n-1+\min I_n]\big) \quad \text{and} \quad \phi^{-1}\big([2n+\min I_n, 3n-1+\min I_n]\big)$$

is a union of n MSI. Now, if we use the definitions of q_1 and q_2, the assertion follows.

To prove (27.14) it is sufficient to use the definitions of q_1 and q_2 in the same manner as above, together with an observation that, by (27.12), for every $n \in \mathbb{N}$ the permutation ϕ has a cycle of length n and the domain of which is contained in I_n. □

Proof of Theorem 27.7 (i). Let $k \in \mathbb{N}$, $k > 1$. We aim to construct permutations $\phi_i, \psi_i \in \mathfrak{D}\mathfrak{D}$, $i = 1, 2$, satisfying condition $\phi_2\phi_1^k = \psi_2^k\psi_1 = p$ and all having the infinite order. We shall distinguish two cases.

First, let us suppose that p has infinitely many infinite cycles. Let $\mathbb{G} \subset \mathbb{N}$ denote the family of generators of all infinite p-cycles which is minimal with respect to inclusion. Next, suppose that the infinite sets $\mathbb{G}_1$ and $\mathbb{G}_2$ form a partition of $\mathbb{G}$ and in turn that the sets $\{a_1^{(n)}, a_2^{(n)}, \dots, a_n^{(n)}\}$, $n \in \mathbb{N}$, form a partition of $\mathbb{G}_1$. Let $s(i,n)$ be positive integers for every $i = 1, \dots, n$, chosen so that

$$|u - v| > \max\{a_i^{(n)} : i = 1, \dots, n\} \tag{27.15}$$

for every two different u and v from the set

$$\{p^{\varepsilon s(i,n)}(a_i^{(n)}) : i = 1, \dots, n \text{ and } \varepsilon = \pm 1\}.$$

Then the cycle $\big(\dots, p^{-1}(a_i^{(n)}), a_i^{(n)}, p(a_i^{(n)}), \dots\big)$ can be written in the form

$$\xi_i^{(n)}\big(\zeta_i^{(n)}\big)^k,$$

where $\xi_i^{(n)}$ and $\zeta_i^{(n)}$ are the cycles defined as follows

$$\xi_i^{(n)} = \big(\dots, p^{-2-s}(a), p^{-1-s}(a), p^{-s}(a), a, p^s(a), p^{2s}(a), \dots,$$
$$p^{(k-1)s}(a), p^{1+(k-1)s}(a), p^{2+(k-1)s}(a), \dots\big)$$

and

$$\zeta_i^{(n)} = \big(p^{-s}(a), a, p^s(a), p^{2s}(a), \ldots, p^{(k-2)s}(a),\\ p^{1-s}(a), p(a), p^{1+s}(a), p^{1+2s}(a), \ldots, p^{1+(k-2)s}(a),\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ p^{-2}(a), p^{s-2}(a), p^{2s-2}(a), p^{3s-2}(a), \ldots, p^{(k-1)s-2}(a),\\ p^{-1}(a), p^{s-1}(a), p^{2s-1}(a), p^{3s-1}(a), \ldots, p^{(k-1)s-1}(a)\big).$$

Here we have $s = s(i,n)$ and $a = a_i^{(n)}$ for $i = 1, \ldots, n$ and $n \in \mathbb{N}$.

Let us put

$$\phi_1 = q \prod_{n\in\mathbb{N}} \prod_{i=1}^{n} \zeta_i^{(n)}$$

and

$$\phi_2 = \Big(\prod_{a\in\mathbb{G}_3} \text{cycle}(p,a)\Big) \circ \Big(\prod_{n\in\mathbb{N}} \prod_{i=1}^{n} \xi_i^{(n)}\Big),$$

where $\mathbb{G}_3 \subset \mathbb{N}$ is a family of generators of all finite p-cycles, which is minimal with respect to the inclusion, and q denotes a permutation of $\mathbb{N}$ such that

$$q^k = \prod_{a\in\mathbb{G}_2} \big(\ldots, p^{-1}(a), a, p(a), \ldots\big)$$

and

$$\big\{n \in \mathbb{N} :\ q(n) \neq n\big\} = \big\{p^m(a) :\ m \in \mathbb{Z} \text{ and } a \in \mathbb{G}_2\big\}. \tag{27.16}$$

The existence of q results from Proposition 27.5. Additionally, we require that q has the property from Proposition 27.5. This and (27.16) yield that $\phi_1 \in \mathfrak{DD}$. From the definition of ϕ_2 it follows that

$$\big\{p^{\varepsilon s(i,n)}\big(a_i^{(n)}\big) :\ i = 1, \ldots, n\big\} \subset \phi_2^{\varepsilon}(I_n),$$

where $I_n := [1, \max\{a_i^{(n)} :\ i = 1, \ldots, n\}]$, for $\varepsilon = \pm 1$ and for every $n \in \mathbb{N}$. Henceforth and from (27.15) we conclude that any of the sets $\phi_2^{\varepsilon}(I_n)$, $\varepsilon = \pm 1$, is a union of at least n MSI, so that $\phi_2 \in \mathfrak{DD}$, as required. A trivial verification shows that $\phi_2 \phi_1^k = p$ and that the orders of ϕ_1 and ϕ_2 are infinite.

Construction of permutations ψ_1 and ψ_2 may be carried out in the similar way. Then it is sufficient to set $\psi_1 = \phi_2$ and $\psi_2 = \phi_1$ and to define cycles $\zeta_i^{(n)}$ in the following way

$$\zeta_i^{(n)} = \big(p^{1-s}(a), p(a), p^{s+1}(a), p^{2s+1}(a), \ldots, p^{(k-2)s+1}(a),$$
$$p^{2-s}(a), p^2(a), p^{s+2}(a), p^{2s+2}(a), \ldots, p^{(k-2)s+2}(a),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$p^{-1}(a), p^{s-1}(a), p^{2s-1}(a), p^{3s-1}(a), \ldots, p^{(k-1)s-1}(a),$$
$$a, p^s(a), p^{2s}(a), p^{3s}(a), \ldots, p^{(k-1)s}(a)\big).$$

Definition of $\xi_i^{(n)}$ is the same as above.

Now we consider the case in which permutation p has infinitely many of finite cycles. Let $\mathbb{F} \subset \mathbb{N}$ denote a family of generators of all finite p-cycles but such that any two different elements $a, b \in \mathbb{F}$ generate different p-cycles.

Suppose that the infinite sets $\mathbb{F}_n$, $n \in \mathbb{N}$, form a partition of $\mathbb{F}$. Now we fix a one-to-one sequence $a_i^{(n)}$, $i \in \mathbb{Z}$, of all elements of family $\mathbb{F}_n$ for each $n \in \mathbb{N}$. The choice of these sequences is such that the following inequality holds

$$|u-v| > \max\big\{\gamma\big(a_{2i}^{(n)}\big) : \ i \in \mathbb{Z} \ \text{ and } \ |i| \leqslant n+1\big\} \tag{27.17}$$

for any two different elements u and v of set

$$\big\{\gamma\big(a_{2i-1}^{(n)}\big) : \ i \in \mathbb{Z} \ \text{ and } \ |i| \leqslant n+1\big\},$$

where $\gamma = p$ or γ is the identity function on $\mathbb{N}$, for every $n \in \mathbb{N}$. Let us denote by $s(i,n)$ p-order of element $a_i^{(n)}$ for all indices $i \in \mathbb{Z}$ and $n \in \mathbb{N}$. Now we define the auxiliary cycles

$$\sigma_n = \big(\ldots, p\big(a_2^{(n)}\big), p\big(a_1^{(n)}\big), p\big(a_0^{(n)}\big), p\big(a_{-1}^{(n)}\big), p\big(a_{-2}^{(n)}\big), \ldots\big),$$

$$\delta_n = \big(\ldots, p\big(a_{-1}^{(n)}\big), p^2\big(a_{-1}^{(n)}\big), \ldots, p^{s(-1,n)}\big(a_{-1}^{(n)}\big),$$
$$p\big(a_0^{(n)}\big), p^2\big(a_0^{(n)}\big), \ldots, p^{s(0,n)}\big(a_0^{(n)}\big),$$
$$p\big(a_1^{(n)}\big), p^2\big(a_1^{(n)}\big), \ldots, p^{s(1,n)}\big(a_1^{(n)}\big), \ldots\big),$$

$$\zeta_n = \big(\ldots, p\big(a_1^{(n)}\big), p^2\big(a_1^{(n)}\big), \ldots, p^{s(1,n)}\big(a_1^{(n)}\big),$$
$$p\big(a_0^{(n)}\big), p^2\big(a_0^{(n)}\big), \ldots, p^{s(0,n)}\big(a_0^{(n)}\big),$$
$$p\big(a_{-1}^{(n)}\big), p^2\big(a_{-1}^{(n)}\big), \ldots, p^{s(-1,n)}\big(a_{-1}^{(n)}\big), \ldots\big),$$

and

$$\xi_n = \big(\ldots, a_{-2}^{(n)}, a_{-1}^{(n)}, a_0^{(n)}, a_1^{(n)}, a_2^{(n)}, \ldots\big),$$

for every $n \in \mathbb{N}$.

Since $p^{s(i,n)}\big(a_i^{(n)}\big) = a_i^{(n)}$ for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, a trivial verification shows that

$$\sigma_n \delta_n = \zeta_n \xi_n = \prod_{i \in \mathbb{Z}} \Big(p\big(a_i^{(n)}\big), p^2\big(a_i^{(n)}\big), \ldots, p^{s(i,n)}\big(a_i^{(n)}\big)\Big) \tag{27.18}$$

for every $n \in \mathbb{N}$. Let us define

$$\phi_2 = \prod_{n \in \mathbb{N}} \sigma_n \qquad \text{and} \qquad \psi_1 = \prod_{n \in \mathbb{N}} \xi_n.$$

Then the following inclusions follow immediately

$$p\big(A_1^{(n)}\big) \subseteq \phi\Big(\big[1, \max p\big(A_2^{(n)}\big)\big]\Big) \qquad \text{and} \qquad A_1^{(n)} \subseteq \psi\Big(\big[1, \max A_2^{(n)}\big]\Big),$$

where $\phi \in \{\phi_2, \phi_2^{-1}\}$, $\psi \in \{\psi_1, \psi_1^{-1}\}$, $A_1^{(n)} = \big\{a_{2i-1}^{(n)} :\ i \in \mathbb{Z} \text{ and } |i| \leqslant n\big\}$ and $A_2^{(n)} = \big\{a_{2i}^{(n)} :\ i \in \mathbb{Z} \text{ and } |i| \leqslant n+1\big\}$, $n \in \mathbb{N}$. This forces, by (27.17), that any of the following sets $\phi\big(\big[1, \max p\big(A_2^{(n)}\big)\big]\big)$ and $\psi\big(\big[1, \max A_2^{(n)}\big]\big)$ is a union of at least $(2n+1)$ MSI for every $\phi \in \{\phi_2, \phi_2^{-1}\}$, $\psi \in \{\psi_1, \psi_1^{-1}\}$ and for every $n \in \mathbb{N}$. Thus $\phi_2, \psi_1 \in \mathfrak{D}\mathfrak{D}$.

It remains to define the permutations ϕ_1 and ψ_2. To this aim let us observe that, by Proposition 27.5, there exist solutions $\phi, \psi \in \mathfrak{D}\mathfrak{D}$ of the following equations

$$\phi^k = \prod_{\omega \in \Gamma_1} \omega \qquad \text{and} \qquad \psi^k = \prod_{\omega \in \Gamma_2} \omega,$$

where

$$\Gamma_1 = \big\{\omega :\ \omega = \delta_n \ \text{ for some } n \in \mathbb{N} \text{ or } \omega \text{ is an infinite } p\text{-cycle}\big\},$$
$$\Gamma_2 = \big\{\omega :\ \omega = \zeta_n \ \text{ for some } n \in \mathbb{N} \text{ or } \omega \text{ is an infinite } p\text{-cycle}\big\}.$$

Put $\phi_1 = \phi$ and $\psi_2 = \psi$. Hence, from the fact that any of permutations ϕ_2 and ψ_1 has an infinite cycle, we see that all four permutations ϕ_i, ψ_i, $i = 1, 2$, are of infinite order. Moreover, relation (27.18) makes it obvious that

$$\phi_2 \phi_1^k = \psi_2^k \psi_1 = p.$$

This completes the proof. □

Proof of Theorem 27.7 (ii). Let us fix $k \in \{2, 3, \ldots\}$. Let $\mathbb{G} \subset \mathbb{N}$ be a minimal set of generators of p (with respect to inclusion). Suppose that $\mathbb{H}$ is the subset of $\mathbb{G}$ of all elements having the infinite p-order. Since, by hypothesis, $\mathbb{G}$ is finite, therefore set $\mathbb{H}$ is nonempty. Let $a \in \mathbb{H}$ and let $n(s)$, $s \in \mathbb{Z}$, be an increasing sequence of integers corresponding to such choice of a. The required properties of $n(s)$, $s \in \mathbb{Z}$, on this occasion are the following

$$n(s) < 0 \text{ iff } s < 0,\ s \in \mathbb{Z}, \text{ and } n(0) = 0, \tag{27.19}$$

$$n(s+1) \equiv n(s) (\operatorname{mod} k),\ s \in \mathbb{Z}, \tag{27.20}$$

$$p^{n(s)}(a) < \big\{p^t(a) :\ t \in \mathbb{Z} \text{ and } n(s+1) \leqslant t \leqslant n(s+2)\big\} < p^{n(s+3)}(a), \tag{27.21}$$

for every $s \in \mathbb{Z}$, $s \geqslant 0$, and

$$p^{n(s-3)}(a) > \big\{p^t(a) :\ t \in \mathbb{Z} \text{ and } n(s-2) \leqslant t \leqslant n(s-1)\big\} > p^{n(s)}(a), \tag{27.22}$$

for every $s \in \mathbb{Z}$, $s \leqslant 0$.

Now define three auxiliary cycles. We put

$$\begin{aligned} \xi_1^{(a)} = \prod_{s\in\mathbb{Z}} \big(&p^{n(s)-w(s)+1}(a), p^{n(s)-2w(s)+1}(a), \dots, p^{n(s)-kw(s)+1}(a),\\ &p^{n(s)-w(s)+2}(a), p^{n(s)-2w(s)+2}(a), \dots, p^{n(s)-kw(s)+2}(a),\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &p^{n(s)-1}(a), p^{n(s)-w(s)-1}(a), \dots, p^{n(s)-(k-1)w(s)-1}(a),\\ &p^{n(s)}(a), p^{n(s)-w(s)}(a), \dots, p^{n(s)-(k-1)w(s)}(a)\big), \end{aligned}$$

$$\begin{aligned} \xi_2^{(a)} = \prod_{s\in\mathbb{Z}} \big(&p^{n(s)}(a), p^{n(s)+v(s)}(a), \dots, p^{n(s)+(k-1)v(s)}(a),\\ &p^{n(s)+1}(a), p^{n(s)+v(s)+1}(a), \dots, p^{n(s)+(k-1)v(s)+1}(a),\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &p^{n(s)+v(s)-1}(a), p^{n(s)+2v(s)-1}(a), \dots, p^{n(s)+kv(s)-1}(a)\big), \end{aligned}$$

and

$$\begin{aligned} \zeta^{(a)} = \big(&\dots, p^{n(-1)}(a), p^{n(-1)+v(-1)}(a), \dots, p^{n(-1)+(k-1)v(-1)}(a),\\ &p^{n(0)}(a), p^{n(0)+v(0)}(a), \dots, p^{n(0)+(k-1)v(0)}(a),\\ &p^{n(1)}(a), p^{n(1)+v(1)}(a), \dots, p^{n(1)+(k-1)v(1)}(a), \dots\big) \end{aligned}$$

for every $a \in \mathbb{G}$, where $v(s) = k^{-1}\big(n(s+1) - n(s)\big)$ and $w(s) = v(s-1)$, $s \in \mathbb{Z}$. Since the sequence $n(s)$, $s \in \mathbb{Z}$, is increasing, we obtain from (27.20) that all indices $v(s)$, $s \in \mathbb{Z}$, are positive integers. Verification of the following equalities may be then carried out immediately. We have

$$\big(\xi_1^{(a)}\big)^k \zeta^{(a)} = \zeta^{(a)} \big(\xi_2^{(a)}\big)^k = \big(\dots, p^{-1}(a), a, p(a), \dots\big)$$

for each $a \in \mathbb{H}$, and consequently

$$\psi_1^k \phi_1 = \phi_2 \psi_2^k = p,$$

where

$$\phi_i := \Big(\prod_{a \in \mathbb{G}\setminus\mathbb{H}} \text{cycle}(p,a)\Big) \circ \Big(\prod_{a\in\mathbb{H}} \zeta^{(a)}\Big) \quad \text{and} \quad \psi_i := \prod_{a\in\mathbb{H}} \xi_i^{(a)}$$

for $i = 1,2$. To see that $\phi_1, \phi_2 \in \mathfrak{CC}$ we just have to show that $\zeta^{(a)} \in \mathfrak{CC}$ for each $a \in \mathbb{H}$. For this the following suffices.

Let $a \in \mathbb{H}$ and let I be an interval such that $a \notin I$ and $J := I \cap \{p^n(a) : \ n \in \mathbb{Z}\} \neq \emptyset$. Set $n(s) = \min J$ and $n(t) = \max J$. Then, in view of conditions (27.19), (27.21) and (27.22), the following inclusion is fulfilled

$$\big(\zeta^{(a)}\big)^{\varepsilon}(I) \supset I \setminus \big\{p^l(a) : l = n(\tau) - iw(\tau) \text{ or } l = n(\tau) + iv(\tau) \\ \text{where } \tau = s \text{ or } t \text{ and for } i = 0,1,\dots,k\big\},$$

where $\varepsilon = -1$ or 1. Hence we check at once that set $\zeta^{(a)}(I)$ is a union of at most $4(k+1)$ MSI. Thus we have $\zeta^{(a)} \in \mathfrak{CC}$ as claimed.

Let us notice additionally that if p belongs to $\mathfrak{D}$ or to $\mathfrak{DD}$ then the above constructions of permutations ψ_1 and ψ_2 imply that ψ_1 and ψ_2 can be then selected such that they belong also to $\mathfrak{D}$ or $\mathfrak{DD}$, respectively. This result follows at once from the relations (see [8], [10], [11]):

$$\mathfrak{C} \circ \mathfrak{C} = \mathfrak{C}, \quad \mathfrak{DC} \circ \mathfrak{DC} = \mathfrak{DC}, \quad \mathfrak{DC} \circ \mathfrak{CC} = \mathfrak{CC} \circ \mathfrak{DC} = \mathfrak{DC}.$$

This completes also the proof of theorem. □

Proof of Theorem 27.8. Let us fix $k \in \{2,3,\dots,\infty\}$. Let p_n, $n \in \mathbb{N}$, be a sequence of prime numbers whose range is infinite. This sequence does not necessarily contain all prime numbers and may not be a one-to-one sequence. Assume that the increasing sequence $\{I_n\}$ of intervals is a partition of $\mathbb{N}$ and that we have

$$\text{card}\, I_n = \begin{cases} (2k-1)n & \text{for every } n \in \mathbb{N} \text{ whenever } k \in \mathbb{N}, \\ (2p_n - 1)n & \text{for every } n \in \mathbb{N} \text{ when } k = \infty. \end{cases}$$

Now we define an auxiliary permutation q_n of I_n for each $n \in \mathbb{N}$. We set

$$q_n\big(s_n + t\big) = s_n + n + 2t,$$
$$q_n\big(s_n + (2i-1)n + 2t\big) = s_n + (2i+1)n + 2t,$$
$$q_n\big(s_n + (2l-3)n + 2t\big) = s_n + t,$$

for $t = 0, 1, \ldots, n-1$, and for $i = 1, 2, \ldots, l-2$, where $l = k$ whenever $k \in \mathbb{N}$, or $l = p_n$ when $k = \infty$. For the remaining $t \in I_n$ we put $q_n(t) = t$. Then a trivial verification shows that

$$q_n^l = \mathrm{id}(I_n), \tag{27.23}$$

$$q_n^i\big([s_n, s_n + n - 1]\big) = \big\{s_n + (2i-1)n + 2t : \ t = 0, 1, \ldots, n-1\big\} \tag{27.24}$$

and

$$\big(q_n^i\big)^{-1}\big([s_n, s_n + n - 1]\big) = \big\{s_n + (2(l-i)+1)n + 2t : \ t = 0, 1, \ldots, n-1\big\}, \tag{27.25}$$

i.e. each of two sets $q_n^i([s_n, s_n + n - 1])$ and $(q_n^i)^{-1}([s_n, s_n + n - 1])$ is a union of n MSI for each $i = 1, 2, \ldots, l-1$, where $l = k$ whenever $k \in \mathbb{N}$, or $l = p_n$ when $k = \infty$. Let $M(k)$ denote the family of all sequences $\{a_n\}$ such that

$$a_n \in \{1, 2, \ldots, k\} \quad \text{for all } n \in \mathbb{N} \text{ whenever } k \in \mathbb{N}$$

or

$$a_n \in \{1, 2, \ldots, p_n\} \quad \text{for all } n \in \mathbb{N} \text{ when } k = \infty.$$

Now we are ready to define family $\mathfrak{G}_k$. We put

$$\mathfrak{G}_k = \Big\{q : \ q = \prod_{n \in \mathbb{N}} q_n^{a_n} \ \text{ and } \ \{a_n\} \in M(k)\Big\},$$

where permutation $q = \prod_{n \in \mathbb{N}} q_n^{a_n}$ is defined as follows

$$q(t) = q_n^{a_n}(t) \ \text{ for every } t \in I_n \text{ and } n \in \mathbb{N}.$$

Observe that intervals I_n, $n \in \mathbb{N}$, are pairwise disjoint and hence this definition is correct. It is obvious that if k is finite then $q^k = \mathrm{id}(\mathbb{N})$. Let $q \in \mathfrak{G}_k$, $q = \prod_{n \in \mathbb{N}} q_n^{a_n}$. If the inequality $a_n < l_n$ holds for infinitely many indices $n \in \mathbb{N}$, where

$$l_n := \begin{cases} k & \text{for all } n \in \mathbb{N} \text{ whenever } k \in \mathbb{N}, \\ p_n & \text{for all } n \in \mathbb{N} \text{ when } k = \infty, \end{cases}$$

then, by (27.24) and (27.25), we get $q \in \mathfrak{D}\mathfrak{D}$. Furthermore, if k is a prime number or $k = \infty$ then the order of this q is precisely equal to k as required. □

Acknowledgement

Author would like to thank the Referee for the constructive comments and suggestions.

References

[1] R. P. Agnew, *Permutations preserving convergence of series*, Proc. Amer. Math. Soc. 6 (1955), 563–564.
[2] A. S. Kronrod, *On permutation of terms of numerical series*, Mat. Sbornik 60(2) (1946), 237–280 (in Russian).
[3] P. A. B. Pleasants, *Rearrangements that preserve convergence*, J. London Math. Soc. 15, II. Ser. (1977), 134–142.
[4] P. A. B. Pleasants, *Addendum: Rearrangements that preserve convergence*, J. Lond. Math. Soc. 18, II. Ser. (1978), 576.
[5] P. Schaefer, *Sum-preserving rearrangemments of infinite series*, Amer. Math. Monthly 88 (1981), 33–40.
[6] P. Schaefer, *Sums of rearranged series*, College Math. J. 17 (1986), 66–70.
[7] R. Wituła, *On the set of limit points of the partial sums of series rearranged by a given divergent permutation*, J. Math. Anal. Appl. 362 (2010), 542–552.
[8] R. Wituła, *Permutations Preserving Sums of Rearranged Real Series*, Cent. Eur. J. Math. 11 (2013), 956–965.
[9] R. Wituła, *The family* $\mathfrak{F}$ *of permutations of* $\mathbb{N}$, Math. Slovaca (in press).
[10] R. Wituła, *Algebraic and set-theoretical properties of some subsets of families of convergent and divergent permutations*, Tatra Mountains Math. Publ. 55 (2013), 27–36.
[11] R. Wituła, *Algebraic properties of the convergent and divergent permutations*, Filomat (in review).
[12] R. Wituła, D. Słota, R. Seweryn, *On Erdős' theorem for monotonic subsequences*, Demonstratio Math. 40 (2007), 239–259.

Roman Wituła
Institute of Mathematics, Silesian University of Technology
ul. Kaszubska 23, 44-100 Gliwice, Poland
E-mail: roman.witula@polsl.pl