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ERRATUM TO: CONGRUENCES AND IDEALS IN A DISTRIBUTIVE LATTICE WITH RESPECT TO A DERIVATION

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The present note is an Erratum for the two theorems of the paper [1]. We assume the reader is familiar with [1] and in particular with the definitions and concepts of Lattice theory.

The proof of [1, Th, 2.9] is wrong. In the end of line 10 of the proof of this theorem the equality $(x)^d \cap (a)^d = Ker\ d$ is not true at all. Also in line 13 the statement $a' \in (x')^d$ iff $a' \in (y')^d$ does not necessarily holds.

Here we have a counterexample to show this theorem is not necessarily true.

Counterexample 1. *Consider the lattice L as follow, $L = \{0, a_1, a_2, a_3, a_{12}, a_{13}, 1\}$ such that 0 and 1 are bottom and top element respectivily, a_1, a_2 and a_3 are attoms, $a_1 \bigvee a_2 = a_{12}, a_1 \bigvee a_3 = a_{13}, a_2 \bigvee a_3 = 1$ and $a_{12} \bigvee a_{13} = 1$. Consider the identity map $d = id_L$ as a derivation on L. So $(a)^d = \{x \in L \mid a \bigwedge d(x) = 0\} = \{x \in L \mid a \bigwedge x = 0\}$. It is clear that $(0)^d = L$, $(a_1)^d = \{0, a_2, a_3\}$, $(a_2)^d = \{0, a_1, a_3, a_{13}\}$, $(a_3)^d = \{0, a_1, a_2, a_{12}\}$, $(a_{12})^d = \{0, a_3\}$, $(a_{13})^d = \{0, a_2\}$, $(1)^d = \{0\}$ and $\mathcal{K}_d = \{1\}$. Thus the congruence $\theta_d = \{(x, y) \mid (x)^d = (y)^d\} = \Delta$ (the identity congruence). Now we introduce a congrunce θ on L, having $\mathcal{K}_d = \{1\}$ as a whole class and properly greater than θ_d. Consider the equivalence relation θ induces by the partition $\{\{0, a_1\}, \{a_2, a_{12}\}, \{a_3, a_{13}\}, \{1\}\}$. It is not difficult to check* *that the equivalence relation θ is a lattice congruence which has a $\mathcal{K}_d = \{1\}$ as a whole class. Clearly θ is properly greater than θ_d.*

Likewise, the Theorem 2.9 of [1] now valid only under the additional assumption with respect to the ideal $I = Ker\ d$. This theorem should be reformulated as:

THEOREM 2. *Let d be a derivation of L. The congruence θ_d is the largest congruence relation having congruence classes $ker\ d$ and $\mathcal{K}_d$, whenever $\mathcal{K}_d \neq \emptyset$.*

PROOF: First we show that $\mathcal{K}_d$ and $ker\ d$ are whole class in which the bottom element in L/θ_d is $ker\ d$ and the top element is $\mathcal{K}_d$ whenever $\mathcal{K}_d \neq \emptyset$.

Let $a \in ker_I d$. For each $b \in ker_I d$, $(a)^d = L = (b)^d$ and hence $a\theta_d b$. Thus $ker_I d \subseteq [a]_{\theta_d}$. For the converse, let $c \in [a]_{\theta_d}$. Then $(c)^d = (a)^d = L$ and $c \in (c)^d$. So $d(c) = d(c \bigwedge c) = c \bigwedge d(c) \in I$ which implies $c \in ker_I d$. Thus $ker_I d = [a]_{\theta_d}$. Since $ker_I d$ is an ideal of L, for each $[y]_{\theta_d} \in L/\theta_d$, we get that $a \bigwedge y \in ker_I d$ and hence $ker_I d = [a]_{\theta_d} = [a \bigwedge y]_{\theta_d} \leq [y]_{\theta_d}$. Therefore $ker_I d$ is the bottom element in L/θ_d. By the similar way and using the fact that if $\mathcal{K}_d \neq \emptyset$, then $\mathcal{K}_d$ is a filter, we can show $\mathcal{K}_d$ is the top element in L/θ_d.

Let θ be any congruence with $\mathcal{K}_d$ and $Ker\ d$ as a congruence classes. Let $x\theta y$. Then $x \in \mathcal{K}_d$ iff $y \in \mathcal{K}_d$. If $x \in \mathcal{K}_d$, then $y \in \mathcal{K}_d$ and hence $(x)^d = ker\ d = (y)^d$. Thus $x\theta_d y$. Now let $x \notin \mathcal{K}_d$ and $a \in (x)^d$. Then $x \bigwedge d(a) = 0$ and $(x \bigwedge d(a))\theta(y \bigwedge d(a))$. So $[y \bigwedge d(a)]_\theta = [0]_\theta = Ker\ d$, which implies that $d(y \bigwedge d(a)) = 0$. Thus $y \bigwedge d(a) = y \bigwedge d(d(a)) = 0$ and hence $a \in (y)^d$. By these conclusions we get $(x)^d = (y)^d$ and therefore $x\theta_d y$. □

Also in line 10 of the proof of [1, Th, 3.4], the equality $d(a \bigvee b) = x_0 = x_0 \bigvee x_0 = d(a) \bigvee d(b)$ is wrong, indeed, $d(a) = a_0, d(b) = b_0$ and $d(a \bigvee b) = (a \bigvee b)_0$ which a_0, b_0 and $(a \bigvee b)_0$ not necessarily equal. The correction should be as follow.

Let $I \cap [a]_\theta = \{a_0\}$ and $I \cap [b]_\theta = \{b_0\}$. Then $(a \bigvee b)\theta(a_0 \bigvee b_0)$ in which $(a_0 \bigvee b_0) \in I$. So $I \cap [a \bigvee b]_\theta = \{a_0 \bigvee b_0\}$ and hence $d(a \bigvee b) = a_0 \bigvee b_0 = d(a) \bigvee d(b)$.

References

[1] M. Sambasiva Rao, *Congruences and ideals in a distributive lattice with respect to a derivation*, **Bulletin of the Section of Logic** 42(1-2) (2013), pp. 1–10.

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