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KNOTS OF IRREDUCIBLE CURVE SINGULARITIES

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Abstract. In the article the relation between irreducible curve plane singularities and knots is described. In these terms the topological classification of such singularities is given.

1. Introduction

Local theory of analytic (algebraic) curves in $\mathbb{C}^2$ i.e. the theory of plane curve singularities is closely related to the theory of knots. If $V = V(f)$, $f \in \mathbb{C}\{x, y\}$, $f \neq \text{const}$, is a local analytic curve described by the equation $f(x, y) = 0$ in a neighbourhood U of the point $0 \in \mathbb{C}^2$, then the intersection $V \cap \mathbb{S}^3_r$ of V with a small 3-dimensional sphere $\mathbb{S}^3_r := \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = r^2\}$ is homeomorphic (even bianalytic) to the unit circle $\mathbb{S}^1$ (if V is irreducible) or to a finite disjoint union of such unit circles (if V is reducible). So this intersection is a knot or a link in $\mathbb{S}^3_r$. Moreover, for all sufficiently small r the knot (link) does not depend on r and uniquely characterizes the topology of V in 4-dimensional ball which boundary is $\mathbb{S}^3_r$. It turns out that knots corresponding to irreducible curve singularities are of very special kind: torus knots of higher orders (also called cable knots). In the article we describe torus knots and relation between irreducible singularities and knots. Due to the form of parameterizations of curve singularities, it is easier to consider the boundary of polycylinders $\{(x, y) \in \mathbb{C}^2 : |x| \leq r, |y| \leq r'\}$ instead of spheres (these both are, of course, homeomorphic sets).

In Section 2 we shortly remember the basics of the knot theory. Section 3 is devoted to the torus knots of the first order. They correspond to the irreducible singularities with one characteristic pair, in particular to singularities $x^n - y^m = 0$, $n, m \in \mathbb{N}$, $\text{GCD}(n, m) = 1$. In Section 4 we will consider the torus knots of higher

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orders. Section 5 describes correspondence between irreducible curve singularities and torus knots. Section 6 is devoted to topological classification of ireducible curve singularities.

2. Basics of the knot theory

The basic sources of this theory are classic textbooks [CF], [R]. Denote $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\} = \{e^{2\pi i\theta} : \theta \in [0,1]\}$ – the *unit circle* in $\mathbb{C}$, and $\mathbb{S}^3$ – the 3-*dimensional sphere* defined as the space $\mathbb{R}^3$ with one point ∞ added, i.e.

$$\mathbb{S}^3 := \mathbb{R}^3 \cup \{\infty\}$$

with the Aleksandrov topology. Recall, open sets in $\mathbb{S}^3$ are: open sets in $\mathbb{R}^3$ and complements of compact sets in $\mathbb{R}^3$ with the point ∞ added. *A knot* is the homeomorphic image of $\mathbb{S}^1$ in $\mathbb{S}^3$ i.e. each subset $W \subset \mathbb{S}^3$ such that $W = \Phi(\mathbb{S}^1)$, where $\Phi : \mathbb{S}^1 \to W$ is a homeomorphism. *A link* is a finite disjoint union of knots (see Fig. 1).

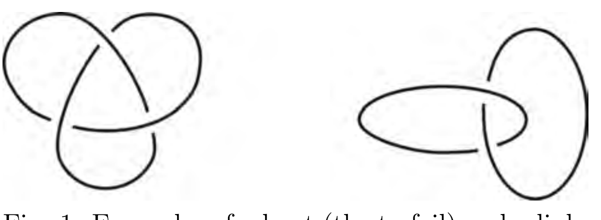

Fig. 1. Examples of a knot (the trefoil) and a link.

Two knots (links) W_1, W_2 are *equivalent* if there exists a homeomorphism $F : \mathbb{S}^3 \to \mathbb{S}^3$ such that $F(W_1) = W_2$. We say then W_1, W_2 have the same *type* and denote $W_1 \sim W_2$. In the sequel we will identify knots with their types. *Trivial knot* (or *unknot*) is the knot

$$\mathbb{S}^1 \ni e^{2\pi i\theta} \mapsto (\cos 2\pi\theta, \sin 2\pi\theta, 0) \in \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}.$$

Remark 2.1. *Studying types of knots in the sphere $\mathbb{S}^3$ is the same as in space $\mathbb{R}^3$ because two knots (links) in $\mathbb{S}^3 \setminus \{\infty\}$ are equivalent in $\mathbb{S}^3$ if and only if they are equivalent in $\mathbb{R}^3$.*

Since in the theory of singularities we will deal only with analytic knots i.e. homeomorphisms $\Phi : \mathbb{S}^1 \to \Phi(\mathbb{S}^1) \subset \mathbb{S}^3$ are analytic functions we will consider only *tame knots* that is knots equivalent to *polygonal knots*. It follows from the fact that each knot of class C^1 (in particular analytic) is equivalent to a polygonal knot

([CF], App. 1). See Figure 2.

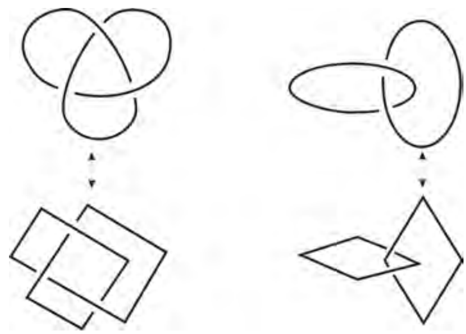

Fig. 2. A polygonal knot and link

A complete invariant of a knot W is its complement $\mathbb{S}^3 \setminus W$, treated as a topological space, because two knots are equivalent if and only if their complements are homeomorphic [GL]. Unforunately it is not true for links [R], p. 49. A weaker invariant of knots and links is the first homotopy group of their complements. We denote it by of $\pi(W)$ and call the *knot (link) group.* So

$$\pi(W) := \pi_1(\mathbb{R}^3 \setminus W; *),$$

where $*$ is an arbitrary point in $\mathbb{R}^3 \setminus W$. Since $\mathbb{R}^3 \setminus W$ is arc connected (remember W is equivalent to polygonal one) $\pi(W)$ does not depend (up to an isomorphism) on the choice of the point $*$. The knot group is not a complete invariant of W. There exist knots having isomorphic groups but not equivalent (see [CF],VIII, 4.8). There are general methods to calculate knot groups (e.g. Wirtinger method, see [CF]) by giving generators of $\pi(W)$ and relations between them. Since for knots related to curve singularities we will describe generators of $\pi(W)$ and relations between them directly, we don't present these methods. We will illustrate this with an example.

Example 2.2. *Two presentations of the knot group of the trefoil W:*

1.

$$\pi(W) = \mathcal{F}(x, y)/ \left(xyxy^{-1}x^{-1}y^{-1}\right)$$

where x, y are loops in Figure 3(a) and $\mathcal{F}(x, y)$ is the free group generated by two elements x, y, and $\left(xyxy^{-1}x^{-1}y^{-1}\right)$ is the smallest normal subgroup in $\mathcal{F}(x, y)$ containing $xyxy^{-1}x^{-1}y^{-1}$,

2.

$$\pi(W) = \mathcal{F}(x, y)/ \left(x^2y^{-3}\right)$$

where x, y are loops in Figure 3(b).

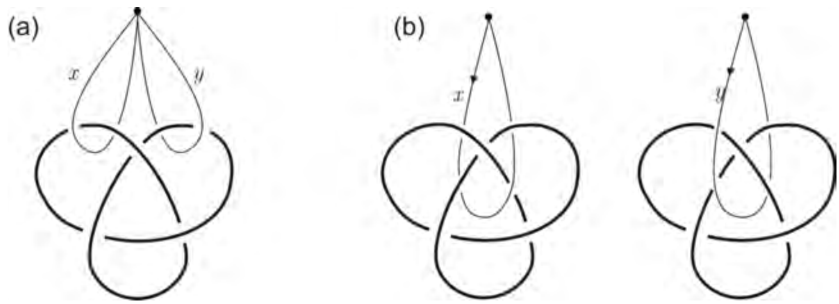

Fig. 3. Generators of the knot group of trefoil.

Although the knot group is not a complete invariant of a knot this is the case for knots (and links) associated to curve singularities. However it is difficult to decide on the basis of knowledge of generators and relations whether two given groups are or are not isomorphic. So we take a weaker invariant of knots to distinguish between torus knots – the Aleksander polynomial of a knot. Its definition is complicated and based on the formal differentation (the free calculus) in knot groups $\pi(W)$ ([CF]). Another approach one can find in [R], p. 206. We recall the first approach.

Let $\mathcal{F} = \mathcal{F}(x_1, \dots, x_n)$ be the free group with n generators $x_1, \dots, x_n$ and $G = \mathcal{F}(x_1, \dots, x_n)/\,(r_1, \dots, r_m)$ an arbitrary group (in general non-abelian) with generators $x_1, \dots, x_n$ and relations $r_1, \dots, r_m \in \mathcal{F}(x_1, \dots, x_n)$ ($(r_1, \dots, r_m)$ denotes the smallest normal subgroup in $\mathcal{F}$ containing $r_1, \dots, r_m$). Adding trivial relations of the type $x_i x_i^{-1}$ we may assume that $m \geq n-1$. In the group ring $\mathbb{Z}[\mathcal{F}]$ we define a formal derivation; first on elements of $\mathcal{F} \subset \mathbb{Z}[\mathcal{F}]$, and next we extend it on the whole ring $\mathbb{Z}[\mathcal{F}]$ in an obvious way. Take any element $\boldsymbol{g} \in \mathcal{F}$. We may represent it in the following way

$$\boldsymbol{g} = x_{i_1}^{\varepsilon_1} \dots x_{i_k}^{\varepsilon_k}, \qquad \varepsilon_j = \pm 1.$$

We define formal partial derivatives of g as follows

$$\frac{\partial \boldsymbol{g}}{\partial x_j} := \varepsilon_1 \delta_{ji_1} x_{i_1}^{(\varepsilon_1 - 1)/2} + x_{i_1}^{\varepsilon_1} \varepsilon_2 \delta_{ji_2} x_{i_2}^{(\varepsilon_2 - 1)/2} + \ldots + x_{i_1}^{\varepsilon_1} \dots x_{i_{k-1}}^{\varepsilon_{k-1}} \varepsilon_k \delta_{ji_k} x_{i_2}^{(\varepsilon_k - 1)/2} \in \mathbb{Z}[\mathcal{F}].$$

In particular $\frac{\partial (xx^{-1})}{\partial x} = 1 - xx^{-1} = 0$, which proves the correctness of the definition of formal differentiation. For illustration, consider the following important examples.

Example 2.3. *In $\mathcal{F}(x, y)$ for $n, m \in \mathbb{N}$ we have:*

1. *If $\boldsymbol{g} = x^n$, then $\frac{\partial \boldsymbol{g}}{\partial x} = 1 + x + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}$.*
2. *If $\boldsymbol{g} = y^{-m}$, then $\frac{\partial \boldsymbol{g}}{\partial y} = -y^{-1} - y^{-2} - \ldots - y^{-m} = -y^{-m} \frac{y^m - 1}{y - 1} = \frac{y^{-m} - 1}{y - 1}$.*

3. If $\boldsymbol{g} = x^n y^{-m}$, *then*

$$\frac{\partial \boldsymbol{g}}{\partial x} = 1 + x + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1},$$

$$\frac{\partial \boldsymbol{g}}{\partial y} = -x^n y^{-1} - x^n y^{-2} - \ldots - x^n y^{-m} = -x^n y^{-m} \frac{y^m - 1}{y - 1} = -\boldsymbol{g} \frac{y^m - 1}{y - 1}.$$

Next for the group $G = \mathcal{F}(x_1, \ldots, x_n)/\left(r_1, \ldots, r_m\right)$, $m \geq n - 1$, we define a matrix over $\mathbb{Z}[\mathcal{F}]$

$$M_G := \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \ldots & \frac{\partial r_1}{\partial x_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial r_m}{\partial x_1} & \ldots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

Example 2.4. *Let* $G = \mathcal{F}(x, y)/\left(x^2 y^{-3}\right)$. *Then* M_G *is a matrix* 1×2.

$$M_G = \left[1 + x,\ x^2(-y^{-1} - y^{-2} - y^{-3})\right] = \left[\frac{x^2 - 1}{x - 1},\ -x^2 y^{-3} \frac{y^3 - 1}{y - 1}\right].$$

Because we will use minors of the matrix M_G and determinants have "good properties" in commutative rings, we abelianize the group G, i.e. we divide G by its commutator $[G : G] := (xyx^{-1}y^{-1} : x, y \in G) \subset G$. We obtain the abelian group

$$G' := G/[G : G].$$

Then the group ring $\mathbb{Z}[G']$ is a commutative ring.

In the case G is the knot group we have

Proposition 2.5. *If* $G = \pi(W)$ *is the knot group of a knot* W *then* $G' \cong \mathbb{Z}$.

This follows from some facts of algebraic topology. As is known, the abelianization of the first homotopy group $\pi_1(X)$ of a "good" topological space X (e.g. topological manifold, and this is the case for knot complement) is isomorphic to the first homology group of X, i.e. $\pi_1(X)' = \pi_1(X)/[\pi_1(X) : \pi_1(X)] \cong H_1(X, \mathbb{Z})$. In the case $X = \mathbb{R}^3 \setminus W$ is the complement of a knot it is easy to show that $H_1(\mathbb{R}^3 \setminus W, \mathbb{Z}) \cong \mathbb{Z}$. Its generator is any loop surrounding one thread of the knot. For instance each loop x and y in Figure 3(a) is a generator. In Figure 3(b) neither x nor y is such a generator. Hence in the case $G = \pi(W)$, choosing one generator in $G' \cong \mathbb{Z}$ we get the isomorphim $\mathbb{Z}[G'] \cong \mathbb{Z}[\mathbb{Z}]$. But the group ring $\mathbb{Z}[\mathbb{Z}]$ is isomorphic to the *ring of Laurent polynomials* $\mathbb{Z}[t, t^{-1}]$. It is easy to see that the ring $\mathbb{Z}[t, t^{-1}]$ has properties:

1. The only invertible elements in $\mathbb{Z}[t, t^{-1}]$ are powers $\pm t^n$, $n \in \mathbb{Z}$,
2. Each element $A(t) \in \mathbb{Z}[t, t^{-1}]$ has a unique representation in the form

$$A(t) = t^n \widetilde{A}(t), \text{ where } n \in \mathbb{Z} \text{ i } \widetilde{A}(t) \in \mathbb{Z}[t],\ \widetilde{A}(0) \neq 0,$$

After the extension of canonical homomorphisms $\mathcal{F} \to G \to G' \simeq \mathbb{Z}$ to homomorphisms of group rings $\mathbb{Z}[\mathcal{F}] \to \mathbb{Z}[G] \to \mathbb{Z}[G'] \simeq \mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[t, t^{-1}]$ and applying this last sequence of homomorphisms to the elements of the matrix M_G we will get the matrix

$$M'_G := \begin{bmatrix} A_{11}(t) & \dots & A_{1n}(t) \\ \dots & \dots & \dots \\ A_{m1}(t) & \dots & A_{mn}(t) \end{bmatrix}$$

which elements are Laurent polynomials. We will call it the *Alexander matrix of* G. This matrix is important in the theory of knots by the following theorem (see [CF]).

Theorem 2.6. *The ideal $E \subset \mathbb{Z}[t, t^{-1}]$ generated by the $(n-1)$ minors of the matrix M'_G does not depend on the choice of the generators $x_1, \dots, x_n$ and the relations $r_1, \dots, r_m$ of group G, so it only depends on the group G .*

Let $M_1(t), \dots, M_k(t)$ be the minors of degree $n-1$ of the matrix M'_G. From the above the greatest common divisor of all M_i depends only, up to invertible elements, on the group G. We call it the *Alexander polynomial of* W and denote by A_W. Hence

$$A_W(t) = \mathrm{GCD}(M_1(t), \dots, M_k(t)) \in \mathbb{Z}[t, t^{-1}].$$

Because A_W is determined up to factors of the type $\pm t^n$, $n \in \mathbb{Z}$, we always choose its normalized form, i.e. one that is an ordinary polynomial in $\mathbb{Z}[t]$ with a non-zero constant term and the highest coefficient $\mathrm{inco}\, A_W$ positive. So, at the end

$$A_W(t) \in \mathbb{Z}[t], \quad A_W(0) \neq 0, \quad \mathrm{inco}\, A_W(t) > 0.$$

For instance the normalized form of the Laurent polynomial $t^{-2} + 2t^{-1} - 3t$ is $-1 - 2t + 3t^3$.

Immediately from theorem 2.6 we obtain

Proposition 2.7. *For knots W_1 and W_2 if $\pi(W_1) \cong \pi(W_2)$ then $A_{W_1} = A_{W_2}$.*

Example 2.8. *For the trefoil knot W we gave two presentations of its group (see Example 2.2):*

I presentation: $\pi(W) = \mathcal{F}(x, y)/\left(xyxy^{-1}x^{-1}y^{-1}\right)$. In this case

$$M_{\pi(W)} = [1 + xy - xyxy^{-1}x^{-1},\ x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}]$$

Since a generator of $\pi(W)' = \pi(W)/[\pi(W) : \pi(W)]$ is the abstract class $[x]$ (it can also be a class $[y] = [x]$), then denoting $t = [x]$ we have

$$M'_{\pi(W)} = [1 + t^2 - t, t - t^2 - 1].$$

Hence

$$A_W(t) = t^2 - t + 1 = \frac{(t^6 - 1)(t - 1)}{(t^2 - 1)(t^3 - 1)}.$$

II presentation: $\pi(W) = \mathcal{F}(x,y)/\left(x^2y^{-3}\right)$. *We have*

$$M_{\pi(W)} = [1+x,\ x^2(-y^{-1}-y^{-2}-y^{-3})]$$

In this case neither $[x]$ *nor* $[y]$ *is a generator of* $\pi(W)'$. *If we take the loop* x *from the previous presentation (for distinction let us denote it by* $\widetilde{x}$ *and we take as before* $t = [\widetilde{x}]$*), then* $[x] = t^3$ *and* $[y] = t^2$. *Then*

$$M'_{\pi(W)} = [1+t^3,\ -t^4-t^2-1] = \left[\frac{t^6-1}{t^3-1},\ -\frac{t^6-1}{t^2-1}\right].$$

Hence

$$A_W(t) = \frac{(t^6-1)(t-1)}{(t^2-1)(t^3-1)} = t^2-t+1.$$

3. Torus knots of the first order

In this section, we will define a particular type of knots, the so-called torus knots. We will start with the simplest type of them - torus knots of the first order.

By T and $\mathbb{T}$ we will denote the *torus* and the *solid torus* in $\mathbb{C}^2$ defined by

$$\begin{aligned} T &:= \{(x,y)\in\mathbb{C}^2 : |x|=1,\ \ |y|=1\} \\ &= \{(x,y)\in\mathbb{C}^2 : x=e^{2\pi i\eta},\ y=e^{2\pi i\theta},\ 0\le\eta,\theta\le 1\},, \\ \mathbb{T} &:= \{(x,y)\in\mathbb{C}^2 : |x|=1,\ \ |y|\le 1\} \\ &= \{(x,y)\in\mathbb{C}^2 : x=e^{2\pi i\eta},\ y=re^{2\pi i\theta},\ 0\le\eta,\theta\le 1,\ 0\le r\le 1\}. \end{aligned}$$

We see that both T and $\mathbb{T}$ lie in the boundary ∂P of the policylinder $P = \{(x,y)\in\mathbb{C}^2 : |x|\le 1,\ |y|\le 1\}$. Because ∂P is homeomorphic to $\mathbb{S}^3 = \mathbb{R}^3\cup\{\infty\}$, any knot in T or $\mathbb{T}$ can be considered as a knot in $\mathbb{S}^3$. However, it depends on the chosen homeomorphism of ∂P on $\mathbb{S}^3$. For calculations and graphical presentation, we will choose such one that this homeomorphism will send T and $\mathbb{T}$ to the standard torus T^{st} and the standard solid torus $\mathbb{T}^{st}$ (see Fig. 4) defined parametrically in $\mathbb{R}^3$ as follows

$$T^{st}:\quad \begin{aligned} x_1 &= (2+\cos 2\pi\theta)\cos 2\pi\eta, \\ x_2 &= (2+\cos 2\pi\theta)\sin 2\pi\eta, \quad 0\le\eta,\theta\le 1 \\ x_3 &= \sin 2\pi\theta \end{aligned}$$

$$\mathbb{T}^{st}:\quad \begin{aligned} x_1 &= (2+r\cos 2\pi\theta)\cos 2\pi\eta, \\ x_2 &= (2+r\cos 2\pi\theta)\sin 2\pi\eta, \quad 0\le\eta,\theta\le 1,\ 0\le r\le 1 \\ x_3 &= r\sin 2\pi\theta \end{aligned}$$

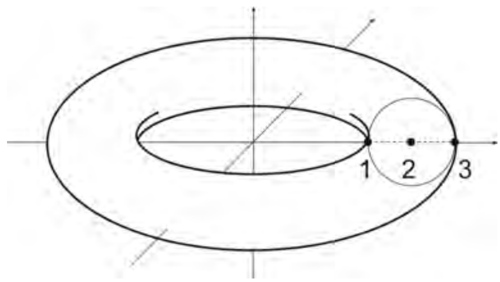

Fig. 4. The standard torus in $\mathbb{R}^3$

This homeomorphism

$$F : \partial P \to \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$$

we define separately on $(\partial P)_1$ and on $(\partial P)_2$, where $\partial P = (\partial P)_1 \cup (\partial P)_2$ and

$$\begin{aligned}(\partial P)_1 &= \{(x,y) \in \mathbb{C}^2 : |x| = 1,\ \ |y| \le 1\}\\ &= \{(x,y) \in \mathbb{C}^2 : x = e^{2\pi i\eta},\ y = re^{2\pi i\theta},\ 0 \le \eta,\theta \le 1,\ 0 \le r \le 1\},\\ (\partial P)_2 &= \{(x,y) \in \mathbb{C}^2 : |x| \le 1,\ \ |y| = 1\}\\ &= \{(x,y) \in \mathbb{C}^2 : x = re^{2\pi i\eta},\ y = e^{2\pi i\theta},\ 0 \le \eta,\theta \le 1,\ 0 \le r \le 1\},\end{aligned}$$

On $(\partial P)_1$ we define F by formula

$$\begin{aligned}F|_{(\partial P)_1}(e^{2\pi i\eta}, re^{2\pi i\theta}) &:= ((2 + r\cos 2\pi\theta)\cos 2\pi\eta, (2 + r\cos 2\pi\theta)\sin 2\pi\eta, r\sin 2\pi\theta)\,,\\ &\quad 0 \le \eta,\theta \le 1,\ 0 \le r \le 1.\end{aligned}$$

It transforms $(\partial P)_1 = \mathbb{T}$ on the standard solid torus $\mathbb{T}^{st}$ and at the same time T on T^{st}.

We define now F on $(\partial P)_2$. It has to transform $(\partial P)_2$ on the complement of solid torus $\mathbb{T}^{st}$ in $\mathbb{R}^3 \cup \{\infty\}$. It is easier to define the inverse homeomorphism

$$\left(F|_{(\partial P)_2}\right)^{-1} : \mathbb{R}^3 \cup \{\infty\} \setminus \mathrm{Int}(\mathbb{T}^{st}) \to (\partial P)_2.$$

$(\partial P)_2$ is homeomorphic to the cartesian product of a disc and a circle $\{x \in \mathbb{C} : |x| \le 1\} \times \{y \in \mathbb{C} : |y| = 1\}$. As the Figure 5 suggests the complement of the solid torus is also homeomorphic to the cartesian product of a disc (a gray disc in the

Figure 5) and a circle.

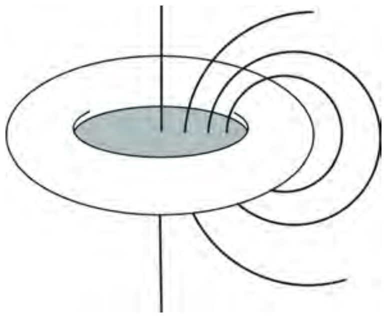

Fig. 5. The homeomorphism of
$\mathbb{R}^3 \cup \{\infty\} \setminus \mathrm{Int}(\mathbb{T}^s t)$ on $(\partial P)_2$

The latter homeomorphism can be chosen so that $F|_{(\partial P)_2}$is identical to $F|_{(\partial P)_1}$ on the common part of their domains, i.e. on the torus T. This completes the construction of homeomorphism F.

For each knot W in ∂P we have the corresponding (by applying F) knot in $\mathbb{S}^3$. In particular to the knot

$$\mathbb{S}^1 \ni e^{2\pi it} \mapsto (e^{2\pi it}, 0) \in \partial P$$

corresponds the trivial knot in $\mathbb{S}^3$. If $W \subset T$ or $W \subset \mathbb{T}$, then we get the knot in $\mathbb{R}^3$ lying in T^{st} or in $\mathbb{T}^{st}$.

Now consider the simplest torus knots. Let $n, m \in \mathbb{N}$ be relatively prime i.e. $\mathrm{GCD}(n, m) = 1$. Then $\Phi : \mathbb{S}^1 \to \partial P$ defined by the formula

$$\Phi(e^{2\pi it}) := (e^{2\pi int}, e^{2\pi imt}), \qquad t \in [0, 1]$$

is one to one (except the ends) by the property of the exponential function, continuous, and thus it is a homeomorphism of the circle on its image, and thus defines a knot in ∂P. We denote it $T_{n,m}$ and the pair (n, m) call the *type* of this knot. Of course $T_{n,m} \subset T$. For each circle $O_\eta := \{(e^{2\pi i\eta}, e^{2\pi i\theta}) : \theta \in [0, 1]\}$, $\eta \in [0, 1]$, the common part $O_\eta \cap T_{n,m}$ consists of n points placed "symmetrically" and similarly for each circle $O_\theta := \{(e^{2\pi i\eta}, e^{2\pi i\theta}) : \eta \in [0, 1]\}$ $\theta \in [0, 1]$, the common part $O_\theta \cap T_{n,m}$ consists of m points placed also "symmetrically". Applying homeomorphism F to $T_{n,m}$ we get a knot in $\mathbb{S}^3$ lying in T^{st}. We denote it with the same symbol $T_{n,m}$ and we call it a *torus knot of the first order of the type* (n, m). Thus it is given in $\mathbb{S}^3$ by the formula

$$F \circ \Phi(e^{2\pi it}) = ((2+\cos 2\pi mt)\cos 2\pi nt, (2+\cos 2\pi mt)\sin 2\pi nt, \sin 2\pi mt), \ t \in [0, 1].$$

The knot $T_{2,3}$ is presented in Figure 6.

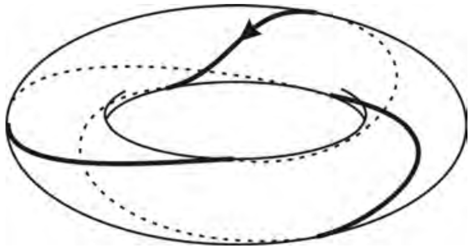

Fig. 6. The torus knot $T_2, 3$

Remark 3.1. *It can be shown that on the torus T there are no torus knots of $T_{n,m}$ for* $\mathrm{GCD}(n,m) > 1$ *(see* [R], *p.19).*

Remark 3.2. *It is easy to prove that the torus knots $T_{n,m}$ for $n = 1$ or $m = 1$ are trivial.*

To describe the knot group of $T_{n,m}$ we recall some properties of T, in particular properties of the *universal covering* of the torus. This is the mapping

$$p : \mathbb{R}^2 \to T,$$
$$p(\eta, \theta) := (e^{2\pi i\eta}, e^{2\pi i\theta}), \quad (\eta, \theta) \in \mathbb{R}^2.$$

Notice that $p(\eta, \theta) = p(\eta', \theta')$ if and only if $(\eta, \theta) - (\eta', \theta') \in \mathbb{Z}^2$. The mapping p has the following known properties:

1. For each $z \in T$ there exists its neighbourhood U such that $p^{-1}(U)$ is a union $\bigcup V_i$ of open and disjoint sets V_i such that $p|_{V_i} : V_i \to U$ is a homeomorphism (this is the definition of a covering),

2. For each continuous curve (in short a curve) $\gamma : [0, 1] \to T$ and arbitrary point $(\eta, \theta) \in p^{-1}(\gamma(0))$ there exists a unique continuous curve $\widehat{\gamma} : [0, 1] \to \mathbb{R}^2$ such that $\widehat{\gamma}(0) = (\eta, \theta)$ and $p \circ \widehat{\gamma} = \gamma$ ($\widehat{\gamma}$ is called the *lifting of the curve γ at (η, θ)*),

3. The first homotopy group $\pi_1(T)$ of the torus T is isomorphic to $\mathbb{Z}^2$. Its generators are the curves $\alpha(t) := (e^{2\pi it}, 1)$ and $\beta(t) := (1, e^{2\pi it})$, $t \in [0, 1]$, called the *longitude* and the *meridian* of the torus.

4. If γ is a closed curve in T and $\gamma = n\alpha + m\beta$, $n, m \in \mathbb{Z}$ in $\pi_1(T)$, then the numbers n, m are characterized via p as follows. Let $\widehat{\gamma}$ be the lifting of γ at $(\eta_0, \theta_0) \in p^{-1}(\gamma(0))$. Since $\gamma(0) = \gamma(1)$ then $\widehat{\gamma}(1) - \widehat{\gamma}(0) \in \mathbb{Z}^2$. Then $\widehat{\gamma}(1) - \widehat{\gamma}(0) = (n, m)$. We say then that the curve γ *circles the torus T n-times along and m-times across.*

5. The first homotopy group $\pi_1(\mathbb{T})$ of the solid torus $\mathbb{T}$ is isomorphic to $\mathbb{Z}$ and its generator is the loop α. For every loop κ in $\mathbb{T}$ with the same origin as α we have $\kappa = \alpha^n$, where n is the index of the curve being projection of κ on $\mathbb{C}$ by pr_1 with respect to the point $0 \in \mathbb{C}$ ($n = \mathrm{Ind}_0\, pr_1 \circ \kappa$).

6. The first homotopy group $\pi_1(\partial P \setminus \mathbb{T})$ of the complement of the solid torus $\mathbb{T}$ is isomorphic also to $\mathbb{Z}$ and its generator is the loop β. Similarly as above for every loop κ in $\partial P \setminus \mathbb{T}$ with the same origin as β we have $\kappa = \beta^m$, where m is the index of the curve being projection of κ on $\mathbb{C}$ by pr_2 with respect to the point $0 \in \mathbb{C}$ ($m = \operatorname{Ind}_0 pr_2 \circ \kappa$).

Now we give some properties of the knot $T_{n,m}$.

Lemma 3.3. *The set $p^{-1}(T_{n,m})$ consists of the family of parallel lines*

$$L_k := \{(\eta,\theta) \in \mathbb{R}^2_{(\eta,\theta)} : \theta = \frac{m}{n}\eta + \frac{k}{n}\}, \quad k \in \mathbb{Z}.$$

Proof. If $(\eta,\theta) \in p^{-1}(T_{n,m})$, then there exists $t \in [0,1]$ such that $(e^{2\pi i\eta}, e^{2\pi i\theta}) = (e^{2\pi int}, e^{2\pi imt})$. From properties of the exponential function we get $\eta = nt + r$, $\theta = mt + s$, $r, s \in \mathbb{Z}$. Hence

$$\theta = \frac{m}{n}\eta + \frac{ns - mr}{n},$$

i.e. $(\eta,\theta) \in L_{ns-mr}$.

Vice versa, if $\theta = \frac{m}{n}\eta + \frac{k}{n}$ for a $k \in \mathbb{Z}$, then putting $t := \frac{\eta}{n} + \frac{ka}{n}$, where $a, b \in \mathbb{Z}$ and $am - bn = 1$ (such a, b always exist because $\mathrm{GCD}(n,m) = 1$), we get

$$e^{2\pi int} = e^{2\pi i(\eta + ka)} = e^{2\pi i\eta},$$
$$e^{2\pi imt} = e^{2\pi i(mt - kb)} = e^{2\pi i(\frac{m}{n}\eta + \frac{k(am-bn)}{n})} = e^{2\pi i(\frac{m}{n}\eta + \frac{k}{n})} = e^{2\pi i\theta}.$$

Then $(\eta,\theta) \in p^{-1}(T_{n,m})$. □

Lemma 3.4. *For every $P, Q \in T \setminus T_{n,m}$ there exists a curve connecting P, Q in $T \setminus T_{n,m}$.*

Proof. Let $p : \mathbb{R}^2_{(\eta,\theta)} \to T$ be the universal covering of the torus. Then by Lemma 3.3 $p^{-1}(T_{n,m})$ is a family of parallel lines L_k, $k \in \mathbb{Z}$, in the plane $\mathbb{R}^2_{(\eta,\theta)}$. Each of the strips lying between adjacent parallel lines is obviously a convex set. Thus, any two of its points can be connected by a segment. Then the image of this segment (via p) will obviously be a curve in T connecting the images of the ends of this segment. Therefore, it is enough to show that the image of each strip (open) is equal to $T \setminus T_{n,m}$. For simplicity we may consider the strip $\boldsymbol{P} := \{(\eta,\theta) : \eta \in \mathbb{R}, \ \frac{m}{n}\eta < \theta < \frac{m}{n}\eta + \frac{1}{n}\}$. Let us take arbitrary point $Q = (e^{2\pi i\eta}, e^{2\pi i\theta}) \in T \setminus T_{n,m}$. We need to show that there is a point $(\eta',\theta') \in \boldsymbol{P}$ such that $(\eta' - \eta, \theta' - \theta) \in \mathbb{Z}^2$. Since $\mathrm{GCD}(n,m) = 1$, there exist $a, b \in \mathbb{N}$ such that

$$am - bn = 1. \tag{1}$$

Put $s := [n\theta - m\eta]$. Then the point $(\eta',\theta') := (\eta + as, \theta + bs)$ satisfies the conditions:

1. $p(\eta',\theta') = p(\eta,\theta) = Q$,
2. $(\eta',\theta') \in \boldsymbol{P}$.

The first condition is obvious and the second follows from the inequalities

$$\begin{aligned} 0 &< n\theta - m\eta - [n\theta - m\eta] < 1, \\ 0 &< n\theta - m\eta - s < 1, \\ 0 &< n\theta - m\eta - s(am - bn) < 1, \\ m\eta + mas &< n\theta + nbs < m\eta + mas + 1, \\ \frac{m}{n}(\eta + as) &< \theta + bs < \frac{m}{n}(\eta + as) + \frac{1}{n}. \end{aligned}$$

The first inequality is obvious because $n\theta - m\eta \notin \mathbb{Z}$ (which follows from the assumption that $(\eta, \theta) \notin T_{n,m}$). □

Consider, in particular, two points Q, R on the torus not belonging to $T_{n,m}$ differing only in the argument $\frac{2\pi}{m}$ of the first coordinate. Therefore

$$Q := (e^{2\pi i\eta}, e^{2\pi i\theta}) \notin T_{n,m},$$
$$R := (e^{2\pi i(\eta - \frac{1}{m})}, e^{2\pi i\theta}) \notin T_{n,m}.$$

They lie on the circle $\{(e^{2\pi it}, e^{2\pi i\theta}),\ t \in [0, 1]\}$, and are separated by "one thread" of the knot $T_{n,m}$ (on this circle lie m points of $T_{n,m}$ arranged symmetrically, see Figure 7).

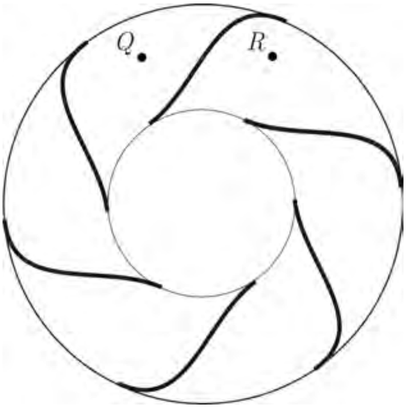

Fig. 7. The points Q and R.

Lemma 3.5. *For the above specified points* $Q, R \notin T_{n,m}$ *a curve connecting* Q *and* R *in* $T \setminus T_{n,m}$ *is the image of the segment* $\overline{[(\eta, \theta), (\eta - \frac{1}{m} + a, \theta + b)]}$ *via* p, *where* $am - bn = 1$.

Proof. Obviously $p(\eta, \theta) = Q$ and $p(\eta - \frac{1}{m} + a, \theta + b) = R$. Moreover the coefficient of the line containing the segment is equal to $\frac{m}{n}$ because

$$\frac{\theta + b - \theta}{\eta - \frac{1}{m} + a - \eta} = \frac{bm}{am - 1} = \frac{bm}{bn} = \frac{m}{n}.$$

Hence the segment is parallel to lines L_k. Therefore it lies in one of the strips. □

Lemma 3.6. *The set of points* (η', θ') *equivalent to* (η, θ) *in* $\mathbb{R}^2$ *(i.e.* $p(\eta', \theta') = p(\eta, \theta)$*) and lying in the same strip as* (η, θ) *is equal to* $\{(\eta + kn, \theta + km) : k \in \mathbb{Z}\}$.

Proof. Obviously the points $(\eta + kn, \theta + km)$, $k \in \mathbb{Z}$, are equivalent to (η, θ). Moreover they lie in the same strip as (η, θ) because the vectors $[km, kn]$ are parallel to L_k.

Take arbitrary point (η', θ') equivalent to (η, θ) w $\mathbb{R}^2$ and lying in the same strip as (η, θ). For simplicity we may assume

$$\frac{m}{n}\eta < \theta < \frac{m}{n}\eta + \frac{1}{n}. \tag{2}$$

Hence $(\eta', \theta') = (\eta + r, \theta + s)$ for some $r, s \in \mathbb{Z}$ and

$$\frac{m}{n}(\eta + r) < \theta + s < \frac{m}{n}(\eta + r) + \frac{1}{n}. \tag{3}$$

By (2) i (3) we get two inequalities

$$0 < n\theta - m\eta < 1,$$
$$0 < (n\theta - m\eta) + (ns - mr) < 1.$$

Since $ns - mr \in \mathbb{Z}$, therefore $ns - mr = 0$. Hence and by the assumption $\mathrm{GCD}(n, m) = 1$ we obtain $r = kn$ and $s = km$ for some $k \in \mathbb{Z}$. □

The last lemma implies a description of the first homotopy group of the complement of the knot $T_{n,m}$ in the torus T.

Proposition 3.7. *For every point* $* \in T \setminus T_{n,m}$

$$\pi_1(T \setminus T_{n,m}; *) \cong \mathbb{Z}. \tag{4}$$

The closed curve κ *which is the image by* p *of the segment* $\overline{(\eta, \theta), (\eta + n, \theta + m)}$, *where* $p(\eta, \theta) = *$, *is a generator of* $\pi_1(T \setminus T_{n,m}; *)$.

Proof. By definition $\kappa(t) := p((\eta + tn, \theta + tm))$, $t \in [0, 1]$, and for every $k \in \mathbb{Z}$, $\kappa^k(t) = p((\eta + tkn, \theta + tkm))$, $t \in [0, 1]$. Take any closed curve ι in $T \setminus T_{n,m}$ at $*$. Its lifting $\widehat{\iota}$ with initial point (η, θ) has the end at a point $(\eta + kn, \theta + km)$ for some $k \in \mathbb{Z}$ (by the previous lemma) and lies in a strip containing (η, θ). Since this strip is a simply connected set the curve $\widehat{\iota}$ is homotopic to the segment joining its ends i.e. to the segment $\overline{(\eta, \theta), (\eta + kn, \theta + km)}$. Hence after composition with p the curve ι is homotopic to κ^k. Then κ is a generator of $\pi_1(T \setminus T_{n,m}; *)$.

To show (4) it suffices to prove that no curve κ^k for $k \in \mathbb{Z} \setminus \{0\}$ is homotopic to the constant curve at $*$. In fact, otherwise for the lifting $\widehat{\kappa^k}$ of κ^k with initial point (η, θ) we would have $\widehat{\kappa^k}(1) = (\eta, \theta)$. On the other hand $\widehat{\kappa^k}(1) = (\eta + kn, \theta + km)$, which implies $k = 0$. □

We will now describe the knot group of $T_{n,m}$. By definition $\pi(T_{n,m}) = \pi_1(\partial P \setminus T_{n,m}; *) = \pi_1(\mathbb{S}^3 \setminus T_{n,m}; *)$, where $* \notin T_{n,m}$. For simplicity we take a point $* \in$

$T \setminus T_{n,m}$. Consider two loops γ, δ as in Figure 8 and 9

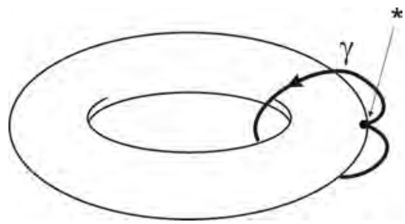

Fig.8. The loop γ.

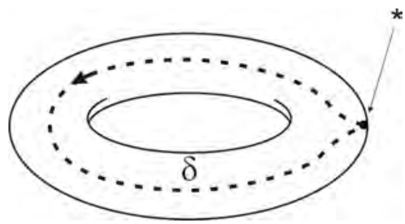

Fig. 9. The loop δ.

We first show

Lemma 3.8. *The loops γ and δ are generators of $\pi(T_{n,m})$.*

Proof. Take arbitrary loop κ in $\mathbb{S}^3 \setminus T_{n,m}$ with the initial and final point at $*$ (in short a loop κ based at $*$). Changing κ homotopically we may assume κ is a broken line. Hence κ has a finite number of common points $A_1, \ldots, A_l$ with T. So, we may represent κ as a finite sum of curves

$$\kappa = \kappa_1 \ldots \kappa_k,$$

where each curve κ_i lies either in $\mathbb{R}^3 \setminus \mathbb{T}$ (i.e. outside the solid torus with exception of ends - see Fig. 10)

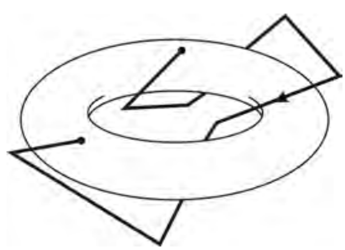

Fig. 10. The loop κ_i outside
$\mathbb{R}^3 \setminus \mathbb{T}$.

or κ_i lies in $\operatorname{Int} \mathbb{T}$ (i.e. in the interior of the solid torus with exception of ends). By Lemma 3.4 each point A_i can be joined with the point $*$ by a curve in $T \setminus T_{n,m}$. Then changing homotopically κ (by moving each point A_i with the entire curve to the point $*$ along such a curve) we may assume that $A_1 = \ldots = A_l = *$. Hence κ is homotopic to a sum of curves $\widetilde{\kappa}_1 \dots \widetilde{\kappa}_k$, where each curve $\widetilde{\kappa}_i$ is a loop in $\mathbb{R}^3 \setminus T_{n,m}$ based at $*$ which lies entirely either in $\mathbb{R}^3 \setminus \mathbb{T}$ or in $\operatorname{Int} \mathbb{T}$. So, $\widetilde{\kappa}_i$ is homotopic to a multiple of γ or a multiple of δ. Hence γ and δ are generators of the group $\pi_1(\overline{\mathbb{R}^3} \setminus T_{n,m}; *)$. □

Now we describe the relation between γ and δ in $\pi(T_{n,m})$. Let $* = p(\eta_0, \theta_0)$. Consider the loop κ_0 which is the image of the segment $\overline{(\eta_0, \theta_0), (\eta_0 + n, \theta_0 + m)}$ by p. It is a loop based at $*$ which lies in $T \setminus T_{n,m}$ and is "parallel" to $T_{n,m}$ in T and which circles the torus T n-times along and m-times across.

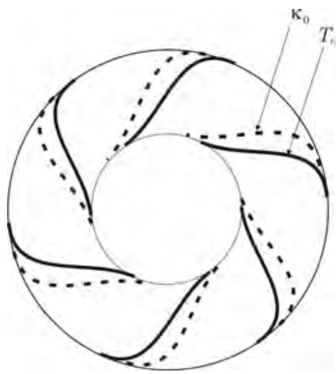

Fig. 12. The loop κ_0.

Let us change κ_0 (leaving the initial and final point fixed) in two ways:

1. "pulling out" the loop κ_0 from T. We get a loop lying in $\mathbb{R}^3 \setminus \mathbb{T}$ circling $\mathbb{T}$ m-times around. Hence κ_0 is homotopic to γ^m.

2. "pushing" the loop κ_0 into the interior of T. We get a loop lying in $\mathbb{T}$ circling $\mathbb{T}$ n-times along. Hence κ_0 is homotopic to δ^n.

In consequence

Lemma 3.9. *For the loops γ and δ in $\pi(T_{n,m})$ we have*

$$\gamma^m = \delta^n.$$

It is a unique non-trivial relation between γ and δ in $\pi(T_{n,m})$. We will use the Seifert-van Kampen theorem (see [CF]) to justify this fact precisely.

Theorem 3.10. *For relatively prime $n, m \in \mathbb{N}$*

$$\pi(T_{n,m}) = \mathcal{F}(\gamma, \delta) / \left(\delta^n \gamma^{-m}\right).$$

Proof. By the canonical homeomorphism $F : \partial P \to \mathbb{S}^3$ we will lead coniderations in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$. Recall $T_{n,m} \subset T^{st} \subset \mathbb{S}^3$. Let's consider two open sets U_1 and U_2 in $\mathbb{S}^3$. The first U_1 is such that its intersection with each half-plane containing the axis Oz is an open disc with a center at the point $(2, 0)$ and radius $3/2$ (notice that intersection T^{st} with such a half-plan is a circle with a center at the point $(2, 0)$

and radius 1 in which n points of $T_{n,m}$ lie) with removed segments of length 1/2 along the radii of this disc with one end at a point $T_{n,m}$ (see Fig. 13)

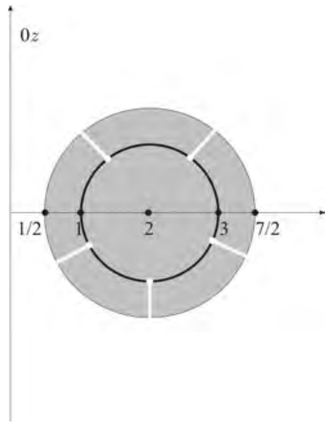

Fig. 13. The set U_1.

The second set U_2 is such that its intersection with each half-plane containing the Oz is the complement (together with the point ∞) of the closed disc with the center at the point $(2, 0)$ and radius 1/2 with removed segments of length 1/2 along the radii of this disc with one end at a point $T_{n,m}$ (see Fig. 14).

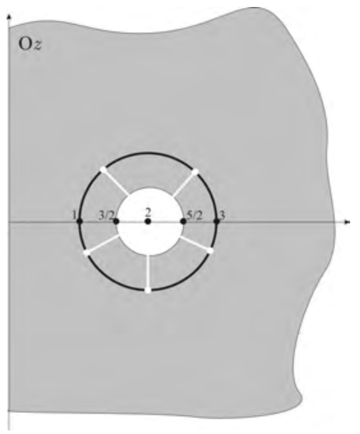

Fig. 14. The set U_2.

The sets U_1, U_2 are connected, arc connected, $U_1 \cup U_2 = \mathbb{S}^3 \setminus T_{n,m}$ and $\pi_1(U_1, *) = \mathcal{F}(\delta)$, $\pi_1(U_2, *) = \mathcal{F}(\gamma)$. The set $U_1 \cap U_2$ is arc connected and its first homotopy group is $\pi_1(T \setminus T_{n,m}; *)$ because each loop in $U_1 \cap U_2$ with beginning and end at $*$ is obviously homotopic to a loop lying in $T \setminus T_{n,m}$. But $\pi_1(T \setminus T_{n,m}; *) = \mathcal{F}(\kappa)$ (see Proposition 3.7), where the loop κ is the image of the segment $\overline{(\eta, \theta), (\eta + n, \theta + m)}$

by p and $p(\eta, \theta) = *$. The homomorphisms

$$\varphi_i : \pi_1(U_1 \cap U_2, *) \to \pi_1(U_i, *), \quad i = 1, 2,$$

are defined on the generator κ by

$$\varphi_1(\kappa) = \delta^n,$$
$$\varphi_2(\kappa) = \gamma^m.$$

In fact, $pr_1 \circ \kappa(t) = pr_1 \circ p(\eta + tn, \theta + tm) = e^{2\pi i(\eta+tn)} = e^{2\pi i\eta}e^{2\pi itn}$ for $t \in [0, 1]$, whence $\mathrm{Ind}_0\, pr_1 \circ \kappa = n$. Then κ, treated as a loop in U_1, is homotopic to δ^n. Similarly we show that κ treated as a loop in U_2 is homotopic to γ^m. Hence by the Seifert-van Kampen theorem

$$\pi(T_{n,m}) = \pi_1(\partial P \setminus T_{n,m}; *) = \mathcal{F}(\gamma, \delta)/\left(\delta^n\gamma^{-m}\right).$$

□

Consider the particular loop t based at $*$ (see Fig. 15) where the point Q differs from

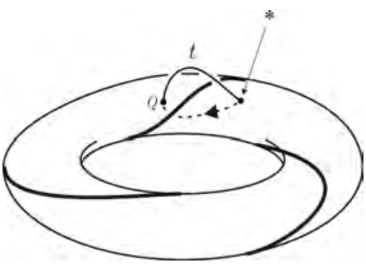

Fig. 15. The loop t.

point $*$ by the argument $\frac{2\pi}{m}$ of the first coordinate. Precisely, if $Q = (e^{2\pi i\eta}, e^{2\pi i\theta})$, then $* = (e^{2\pi i(\eta-1/m)}, e^{2\pi i\theta})$. Firstly we represent t by generators.

Lemma 3.11. *In* $\pi(T_{n,m})$

$$t = \delta^a\gamma^{-b},$$

where $a, b \in \mathbb{N}$ *and* $am - bn = 1$.

Proof. Let $p(\eta, \theta) = Q$. By Lemma 3.5 the point Q can be joined to the point $*$ in $T \setminus T_{n,m}$ with a curve which is the image of the segment $\overline{[(\eta, \theta), (\eta - \frac{1}{m} + a, \theta + b)]}$ via p. By Property 4 of the universal cover p this curve circles the torus T $(a - \frac{1}{m})$-times along and b-times across. Since Q and $*$ differ by the argument $\frac{2\pi}{m}$ of the first coordinate we obtain as in the proof of Lemma 3.8

$$t = \delta^a\gamma^{-b}.$$

□

By Proposition 2.5 it follows that the loop t, and precisely its abstract class $[t]$, is a generator of abelianization $\pi(T_{n,m})' := \pi(T_{n,m})/[\pi(T_{n,m}), \pi(T_{n,m})]$. In particular the classes $[\gamma]$ and $[\delta]$ are generated by $[t]$. In fact

Lemma 3.12. *In* $\pi(T_{n,m})'$

$$[\gamma] = [t]^n,$$
$$[\delta] = [t]^m.$$

Proof. Since $\pi(T_{n,m})'$ is isomorphic to $H_1(\mathbb{S}^3 \setminus T_{n,m}, \mathbb{Z})$, we may lead considerations in the language of homology. Let $p(\eta, \theta) = *$. The loop (= cycle) t is homologic to any cycle t_i lying in the plane $\{e^{2\pi i\eta}\} \times \mathbb{C}$ which circles the one thread of $T_{n,m}$ and the loop (=cycle) γ is homologous to γ_0 which circles all the points of $T_{n,m}$ (see Fig. 16).

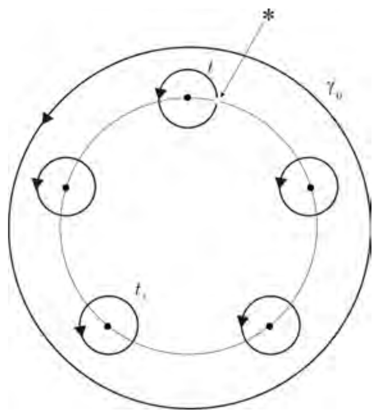

Fig. 16. The cycles t, t_i, γ_0

Since in the plane $\{e^{2\pi i\eta}\} \times \mathbb{C}$ the set $\left(\{e^{2\pi i\eta}\} \times \mathbb{C}\right) \cap T_{n,m}$ has n points the cycle γ_0 is homologous to the sum of cycles t_i, circling these points. Then $[\gamma_0] = n[t]$ in $H_1(\mathbb{S}^3 \setminus T_{n,m}, \mathbb{Z})$. Hence in $\pi(T_{n,m})'$ we have $[\gamma] = [t]^n$. We do the similar reasoning for the loop δ. We obtain in $\pi(T_{n,m})'$, $[\delta] = [t]^m$. □

We can now proceed to calculate the Alexander polynomial of the torus knots $T_{n,m}$.

Theorem 3.13. *For every relatively prime postive integers* n, m *we have*

$$A_{T_{n,m}}(t) = \frac{(t^{nm} - 1)(t - 1)}{(t^n - 1)(t^m - 1)}.$$

Proof. Because $\pi(T_{n,m}) = \mathcal{F}(x, y)/\left(x^n y^{-m}\right)$ then using formal derivatives we obtain

$$\frac{\partial\left(x^n y^{-m}\right)}{\partial x} = 1 + x + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1},$$
$$\frac{\partial\left(x^n y^{-m}\right)}{\partial y} = -x^n y^{-1} - x^n y^{-2} - \ldots - x^n y^{-m} = -x^n y^{-m} \frac{y^m - 1}{y - 1},$$

whence

$$M_{\pi(T_{n,m})} = \left[\frac{x^n - 1}{x - 1}, -x^n y^{-m} \frac{y^m - 1}{y - 1}\right].$$

By Lemma 3.11 $t = x^a y^{-b}$ is a generator of $\pi(T_{n,m})'$ and from Lemma 3.12 $x = t^m$ and $y = t^n$. Then

$$M'_{\pi(T_{n,m})} = \left[\frac{t^{mn}-1}{t^m-1}, -\frac{t^{nm}-1}{t^n-1}\right].$$

Hence

$$A_{T_{n,m}}(t) = \mathrm{GCD}\left(\frac{t^{mn}-1}{t^m-1}, -\frac{t^{mn}-1}{t^n-1}\right) = \frac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)}.$$

The last equality follows from the assumption GCD $(n, m) = 1$ and simple facts about roots of the unity.

This ends the proof. □

In particular for the trivial knot $T_{1,1}$ we obtain $A_{T_{1,1}}(t) \equiv 1$. Hence we get

Corollary 3.14. *If T is a trivial knot then $A_T(t) \equiv 1$.*

Hence we get a topological classification of torus knots of the first order.

Theorem 3.15. *The torus knot of the first order $T_{n,m}$ is trivial if and only if $n = 1$ or $m = 1$. Two torus knots of the first order $T_{n,m}$ and $T_{k,l}$, $n, m, k, l \geq 2$, are equivalent if and only if $(n, m) = (k, l)$ or $(n, m) = (l, k)$.*

Proof. The first part of the theorem follows from Remark 3.2 and the fact that for $n, m \geq 2$ the Aleksander polynomial of $T_{n,m}$ is not constant ($\deg A_{T_{n,m}} > 0$).

Assume now that torus knots $T_{n,m}$ i $T_{k,l}$, $n, m, k, l \geq 2$, are equivalent. Then their groups are isomorphic. Hence $A_{T_{n,m}} = A_{T_{k,l}}$, that is

$$\frac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)} = \frac{(t^{kl}-1)(t-1)}{(t^k-1)(t^l-1)}. \tag{5}$$

This equality implies

$$mn = kl \tag{6}$$

because otherwise, for instance $mn > kl$, some primitive root of unity of degree mn would be a root of left hand side of (5) and not of right hand side, which is impossible. If $mn = kl$ then again from (5) it follows in similar way that

$$m = k \text{ or } m = l. \tag{7}$$

From equalities (6) and (7) we obtain $(n, m) = (k, l)$ or $(n, m) = (l, k)$.

If $(n, m) = (k, l)$ or $(n, m) = (l, k)$, then the identity mapping in the first case and the permutation of coordinates $(x, y) \mapsto (y, x)$ in the second one are homeomorphisms which give the equivalence of knots.

This ends the proof. □

Remark 3.16. *Of course, we can also consider torus knots $T_{n,m}$ for any integers $n, m \in \mathbb{Z} \setminus \{0\}$ which satisfy* GCD $(n, m) = 1$. *In these cases all the above reasoning are analogous with obvious changes. Limiting our considerations to positive n, m results from the fact that we obtain such knots from curves singularities.*

4. Torus knots of higher orders

Recall that by definition $T := \{(x, y) \in \mathbb{C}^2 : |x| = 1,\ |y| = 1\} \subset \partial P$. Let $n_1, m_1 \in \mathbb{N}$ and $\mathrm{GCD}(n_1, m_1) = 1$. Consider the torus knot $T_{n_1,m_1} \subset \partial P$. Then

$$T_{n_1,m_1} = \{(e^{2\pi i n_1 t}, e^{2\pi i m_1 t}),\quad t \in [0, 1]\} \subset T \subset \partial P.$$

Take r_1, $0 < r_1 < 1$. Instead of T_{n_1,m_1} we consider the equivalent to it the torus knot in ∂P

$$\{(e^{2\pi i n_1 t}, r_1 e^{2\pi i m_1 t}),\quad t \in [0, 1]\} \subset \partial P.$$

We will also denote it by T_{n_1,m_1}. We define closed *tubular neighbourhood of* T_{n_1,m_1} contained in ∂P by

$$\mathrm{Tube}(T_{n_1,m_1}) = \bigcup_{(x,y)\in T_{n_1,m_1}} (\{x\} \times \overline{K(y, r_2)}),$$

where r_2 is so small that the closed discs with centers in the points $y_1, \dots, y_{m_1}$ and radius r_2, where

$$\pi_1^{-1}(x) \cap T_{n_1,m_1} = x \times \{y_1, \dots, y_{m_1}\},$$

are contained in the disc $K(0, 1)$ and are pairwise disconnected (see Fig.17).

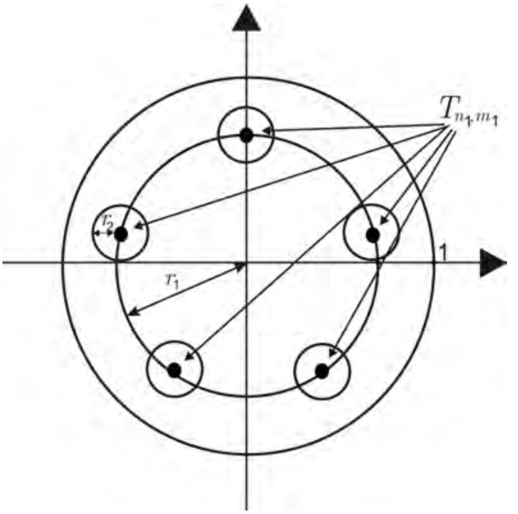

Fig. 17. A tubular neighbourhood of T_{n_1,m_1}.

Then $\mathrm{Tube}(T_{n_1,m_1})$ is given parametrically by

$$\mathrm{Tube}(T_{n_1,m_1}) = \{(e^{2\pi i n_1 t}, r_1 e^{2\pi i m_1 t} + r e^{2\pi i s}),\quad t, s \in [0, 1],\ r \in [0, r_2]\}.$$

Obviously $\mathrm{Tube}(T_{n_1,m_1})$ is contained in ∂P and its boundary $\partial(\mathrm{Tube}(T_{n_1,m_1}))$ is homeomorphic to the torus. We choose the following homeomorphism

$$\Phi_1 : T \to \partial(\mathrm{Tube}(T_{n_1,m_1})),$$
$$\Phi_1(e^{2\pi it}, e^{2\pi is}) = (e^{2\pi in_1t}, r_1e^{2\pi im_1t} + r_2e^{2\pi is}),\ t, s \in [0, 1].$$

Let T_{n_2,m_2} be an arbitrary torus knot of the first order lying in $T \subset \partial P$. Then $n_2, m_2 \in \mathbb{N}$ and $\mathrm{GCD}(n_2, m_2) = 1$. So $\Phi_1(T_{n_2,m_2})$ is a knot in ∂P, and thus (through the homeomorphism F) a knot in $\mathbb{S}^3$. These types of knots are called the *torus knots of the second order* and denote by $T_{(n_1,m_1)(n_2,m_2)}$ (both in ∂P and in $\mathbb{S}^3$). The type of this knot in ∂P does not depend on the choice of the radii r_1, r_2 as long as they satisfy the above assumptions (because there is a homeomorphism transforming the unit disc into oneself, being an identity on the boundary, carrying points $T_{(n_1,m_1)(n_2,m_2)}$ to the points of the same knot with different radii r_1', r_2'). Because the knot T_{n_2,m_2} in T is given by the formula

$$T_{n_2,m_2} = \{(e^{2\pi in_2t}, e^{2\pi im_2t}),\quad t \in [0, 1]\} \subset T,$$

then $T_{(n_1,m_1)(n_2,m_2)}$ is described by the formula (see Fig. 18)

$$T_{(n_1,m_1)(n_2,m_2)} = \{(e^{2\pi in_1n_2t}, r_1e^{2\pi im_1n_2t} + r_2e^{2\pi im_2t}),\quad t \in [0, 1]\}.$$

Fig. 18. The torus knot of the second order.

Higher-order torus knots are defined inductively. For a given torus knot of the k-th order $T_{(n_1,m_1)\cdots(n_k,m_k)} \subset \partial P$, $k \geq 1$, given by the formula

$$[0, 1] \ni t \mapsto (e^{2\pi in_1\ldots n_kt}, r_1e^{2\pi im_1n_2\ldots n_kt} + r_2e^{2\pi im_2n_3\ldots n_kt} + \ldots + r_ke^{2\pi im_kt}) \in \partial P$$

we consider its closed tubular neighbourhood $\mathrm{Tube}(T_{(n_1,m_1)\cdots(n_k,m_k)})$ with sufficiently small radius r_{k+1} (such that this neighbourhood is contained in ∂P_1 and that the closed discs of this neighbourhood in each plane $\{x\} \times \mathbb{C}$ are pairwise disjoined). The boundary $\partial(\mathrm{Tube}(T_{(n_1,m_1)\cdots(n_k,m_k)}))$ is homeomorphic to the torus T. We fix the following homeomorphism

$$\Phi_k : T \to \partial(\mathrm{Tube}(T_{(n_1,m_1)\cdots(n_k,m_k)})),$$
$$\Phi_k(e^{2\pi it}, e^{2\pi is}) = (e^{2\pi in_1\ldots n_kt}, r_1e^{2\pi im_1n_2\ldots n_kt} + r_2e^{2\pi im_2n_3\ldots n_kt} +$$
$$\ldots + r_ke^{2\pi im_kt} + r_{k+1}e^{2\pi is}),\qquad t, s \in [0, 1])\}.$$

Let $T_{n_{k+1},m_{k+1}}$ be any torus knot of the first order lying in $T \subset \partial P$. Then $\Phi_k(T_{n_{k+1},m_{k+1}})$ is a knot in ∂P and thus a knot in $\mathbb{S}^3$. These types of knots are called the *torus knots of the* $(k+1)$*-th order.* It is given by the formula

$$(8) \quad t \mapsto (e^{2\pi i n_1 \ldots n_{k+1} t}, r_1 e^{2\pi i m_1 n_2 \ldots n_{k+1} t} + r_2 e^{2\pi i m_2 n_3 \ldots n_{k+1} t} + \ldots \\ + r_k e^{2\pi i m_k n_{k+1} t} + r_{k+1} e^{2\pi i m_{k+1} t}) \quad \text{for } t \in [0,1].$$

According to Remark 3.2 the torus knot of the first order $T_{n,m}$ is trivial if and only if $n = 1$ or $m = 1$. Due to the lack of symmetry between variables x and y in the definition of higher-order torus knots, this type of theorem only takes place for the first index.

Proposition 4.1. *Let* $T_{(n_1,m_1)\cdots(n_k,m_k)}$ *be a torus knot of the* k*-th order and* $n_i = 1$ *for some* $i \in \{1, \ldots, k\}$. *Then*

$$T_{(n_1,m_1)\cdots(n_k,m_k)} \sim T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(n_{i+1},m_{i+1})\cdots(n_k,m_k)}.$$

Proof. First we show that $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(1,m_i)}$ is equivalent to $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})}$. in ∂P. For any $x = e^{2\pi i t}$ in the plane $\{x\} \times \mathbb{C}$ we have $n_1 \cdot \ldots \cdot n_{i-1}$ points of the knot $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})}$ lying in the unit disc. Around each of these points a circle with a sufficiently small radius r_i is given (such that closed discs with these radii are disjoined and are contained in the interior of unit disc). On each of these circles is given one point of the knot $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(1,m_i)}$ and these points depend continuously on x. Assuming that r_i are small enough, we can include these discs in discs with a greater radii $\widetilde{r}_i > r_i$ and the same centers such that their closures are still disjoined and contained in the open unit disc (see Fig. 19).

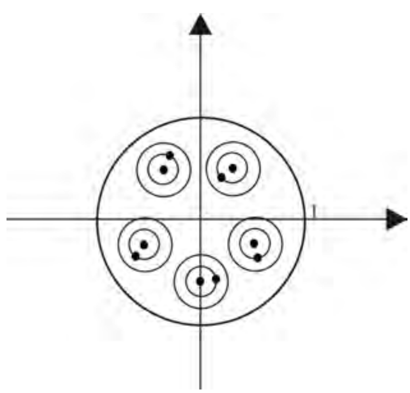

Fig. 19.

It is easy to show that there exists a homeomorphism h_x of the unit disc on itself carrying out the points of the knot $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(1,m_i)}$ on corresponding them points of the knot $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})}$ and being the identity on boundaries of the unit disc and discs of the radii $\widetilde{r}_i$. Moreover, we may choose h_x such that they depend continously on x. Then the mapping

$$\begin{aligned}&F : \partial P \to \partial P,\\ &F|_{\partial P_1}(x, y) := (x, h_x(y)),\\ &F|_{\partial P_2}(x, y) := (x, y)\end{aligned}$$

is a homeomorphism of ∂P transforming $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(1,m_i)}$ on the knot $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})}$. Further constructions of torus knots of successive orders applied to the equivalent ones $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(1,m_i)}$ and $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})}$ lead to equivalent knots. □

Remark 4.2. *There is no similar theorem when $m_i = 1$ for some $i \in \{1, \ldots, k\}$. See Remark 4.8.*

We will now describe the knot group of $T_{(n_1,m_1)\cdots(n_k,m_k)}$. We described the knot groups of torus knots of the first order in Theorem 3.10. For every $n_1, m_1 \in \mathbb{N}$, $\mathrm{GCD}(n_1, m_1) = 1$, we have

$$\pi(T_{n_1,m_1}) = \mathcal{F}(x, y)/\left(x^{n_1}y^{-m_1}\right).$$

We compute now the knot groups of torus knot of the second order $T_{(n_1,m_1)(n_2,m_2)}$. By definition

$$\pi(T_{(n_1,m_1)(n_2,m_2)}) = \pi_1(\partial P \setminus T_{(n_1,m_1)(n_2,m_2)}; *) = \pi_1(\mathbb{R}^3 \setminus \Phi_1(T_{(n_1,m_1)(n_2,m_2)}); *),$$

where the point $* \notin T_{(n_1,m_1)(n_2,m_2)}$. Take the point $*$ lying on the boundary of $\mathrm{Tube}(T_{n_1,m_1})$. As generators we fix the following three loops based at $*$:

1. the loop γ as in the case of torus knot of the first order $T_{n,m}$; call it here γ_0 (see Fig. 20),

2. the loop δ as in the case of torus knot of the first order $T_{n,m}$; call it here δ_1 (see Fig. 20),

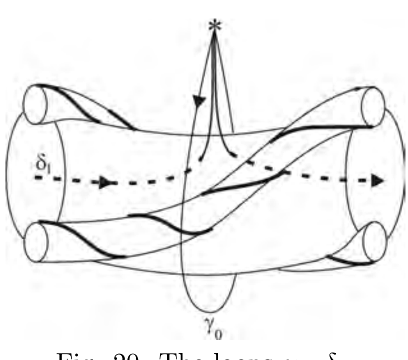

Fig. 20. The loops γ_0, δ_1.

(for a better geometrical representation of these loops, we have drawn a point $*$ outside the torus),

3. the loop being the "axis" of the tubular neighbourhood Tube(T_{n_1,m_1}); call it here δ_2 (see Fig. 21). It is equivalent to T_{n_1,m_1} in ∂P.

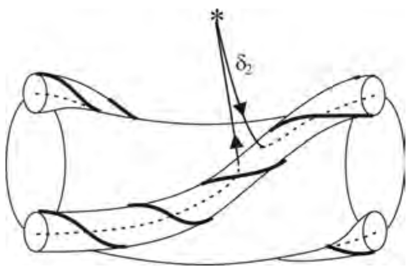

Fig. 21. The loop δ_2.

We show now

Lemma 4.3. *The loops γ_0, δ_1 and δ_2 are generators of $\pi(T_{(n_1,m_1)(n_2,m_2)})$.*

Proof. The proof is similar to the case of the first-order torus knots $T_{n,m}$. By the homeomorphism $F : \partial P \to \mathbb{S}^3$ we move considerations to $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$. Take any loop κ in $\mathbb{R}^3 \setminus T_{(n_1,m_1)(n_2,m_2)}$ based at $*$ (recall we chose the point $*$ in $\partial(\text{Tube}(T_{n_1,m_1}))\setminus T_{(n_1,m_1)(n_2,m_2)}$. Changing κ by a homotopy we may assume κ is a broken line. Hence κ has a finite number of common points with $\partial(\text{Tube}(T_{n_1,m_1}))$. So, we may represent κ as a finite sum of curves

$$\kappa = \kappa_1 \dots \kappa_k,$$

where each curve κ_i lies either in $\mathbb{R}^3 \setminus \text{Tube}(T_{n_1,m_1})$ (except the ends of the curve) or in the interior of Tube(T_{n_1,m_1}) (except the ends of the curve). By Lemma 3.4 each common points of κ with $\partial(\text{Tube}(T_{n_1,m_1})$ can be joined by a curve with the chosen point $*$ in $\partial(\text{Tube}(T_{n_1,m_1})) \setminus T_{(n_1,m_1)(n_2,m_2)}$. Then changing homotopically κ (by moving each common point with the entire curve to the point $*$ along such a curve) we obtain that κ is homotopic to a sum of curves $\widetilde{\kappa}_1 \dots \widetilde{\kappa}_k$, where each curve $\widetilde{\kappa}_i$ is a loop in $\mathbb{R}^3 \setminus T_{(n_1,m_1)(n_2,m_2)}$ based at $*$ and lies entirely either in $\mathbb{R}^3 \setminus \text{Tube}(T_{n_1,m_1})$ or in Int(Tube(T_{n_1,m_1})) (except the ends of the curve). Those running in $\mathbb{R}^3 \setminus \text{Tube}(T_{n_1,m_1})$ are obviously generated by γ_0 i δ_1, and those running in Tube(T_{n_1,m_1}) are a multiple of δ_2. Then γ_0, δ_1 and δ_2 are generators of the group $\pi(T_{(n_1,m_1)(n_2,m_2)})$. □

We describe relations between γ_0, δ_1 and δ_2 in $\pi(T_{(n_1,m_1)(n_2,m_2)})$. Of course, the relationship between γ_0 and δ_1 is the same as in case of torus knot $T_{(n_1,m_1)}$

(R1) $$\gamma_0^{m_1} = \delta_1^{n_1}.$$

To determine the relationship between δ_2 and the pair γ_0, δ_1 we consider an auxiliary loop γ_1 (Fig. 21); it corresponds to the loop t in the case of $T_{n,m}$ from

Lemma 3.11.

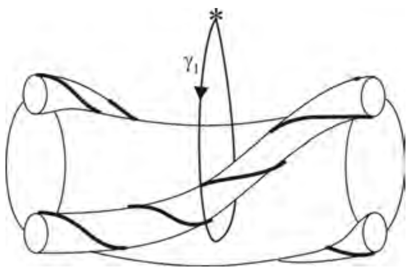

Fig. 21. The loop γ_1.

By this lemma

$$\gamma_1 = \delta_1^a \gamma_0^{-b}, \tag{9}$$

where $a, b \in \mathbb{N}$ and $am_1 - bn_1 = 1$.

Consider the loop κ_0 (Fig. 22) based at $*$, lying in $\partial(\mathrm{Tube}(T_{n_1,m_1}))$ "parallel" to $T_{(n_1,m_1)(n_2,m_2)}$, so circulating $\partial(\mathrm{Tube}(T_{n_1,m_1}))$ n_2-times along and m_2-times across. It means that the homeomorphism Φ_1^{-1} transforms the curve κ_0 into a curve which circles T n_2-times along and m_2-times across. Hence the projection κ_0 on the unit circle (via the projection $pr_1 : \mathbb{C}^2 \to \mathbb{C}$ on the first axis) goes around this circle $n_1 n_2$-times in a positive direction.

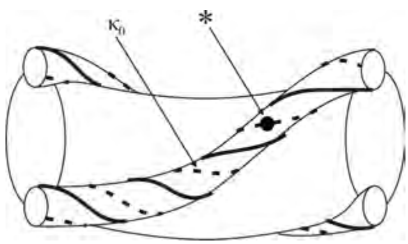

Fig. 22. The loop κ_0.

If we change homotopically κ_0 so that it will lie inside $\mathrm{Tube}(T_{n_1,m_1})$ (with exception of $*$), then obviously

$$\kappa_0 \sim \delta_2^{n_2} \tag{10}$$

(because δ_2 is an "axis" of $\mathrm{Tube}(T_{n_1,m_1})$, and κ_0 circles n_2-times along this tubular neighbourhood).

If we change homotopically κ_0 so that it will lie outside $\mathrm{Tube}(T_{n_1,m_1})$ (with exception of $*$), then

$$\kappa_0 \sim \gamma_1^{m_2 - m_1 n_2} \delta_1^{n_1 n_2}. \tag{11}$$

To justify this, let's first determine the integer s such that the curve $\gamma_1^s \kappa_0$ is homotopic to a multiple of δ_1 and precisely homotopic to $\delta_1^{n_1 n_2}$ (because the projection of κ_0 on the first axis circles the unit circle $n_1 n_2$-times). For one turn κ_0

around Tube(T_{n_1,m_1}) (there are exactly m_2 of them) therefore corresponds $\frac{n_1 n_2}{m_2}$ rotation of the projection on the first axis. So the point P in Figure 23 will do $\alpha := \frac{n_1 n_2}{m_2} \cdot \frac{m_1}{n_1} = \frac{m_1 n_2}{m_2}$ rotation.

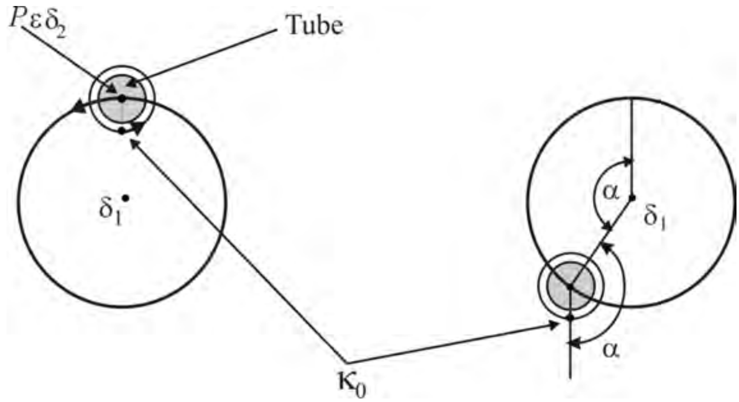

Fig. 23.

Therefore, to obtain a curve being a multiple of δ_1, (it is the axis of T_{n_1,m_1}, so it is represented in the Figure 23 by the center of the circle) $-1+\alpha$ turn should be made.

Because κ_0 turns around Tube(T_{n_1,m_1}), m_2-times then $s = m_2(-1+\alpha) = -m_2 + m_1 n_2$. Hence $\gamma_1^{-m_2+m_1n_2}\kappa_0 \sim \delta_1^{n_1n_2}$. This gives (11).

From (10) and (11) we get the relation

$$\delta_2^{n_2} \sim \gamma_1^{m_2-m_1n_2}\delta_1^{n_1n_2}. \tag{12}$$

In turn, by (9) we get the relation between $\gamma_0, \delta_1, \delta_2$

$$\delta_2^{n_2} \sim \left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}\delta_1^{n_1n_2}. \tag{R2}$$

(R1) and (R2) are the only relations between $\gamma, \delta_1, \delta_2$. To prove this, it is sufficient to use the Seifert-van Kampen theorem (see the proof of Theorem 3.10). Then we get

Theorem 4.4. *For any torus knot of the second order $T_{(n_1,m_1)(n_2,m_2)}$ we have*

$$\pi(T_{(n_1,m_1)(n_2,m_2)}) \cong \mathcal{F}(\gamma_0,\delta_1,\delta_2)/\left(\delta_1^{n_1}\gamma_0^{-m_1}, \left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}\delta_1^{n_1n_2}\delta_2^{-n_2}\right).$$

We can now calculate Alexander's polynomial of torus knots of the second order. For any relatively prime positive integers m, n we define the polynomial

$$W_{n,m}(t) := \frac{(t^{nm}-1)(t-1)}{(t^n-1)(t^m-1)}.$$

These are indeed polynomials by properties of the roots of unity and the assumption that GCD$(m,n) = 1$. Then the Alexander polynomial of the torus knot of the first order $T_{n,m}$ is equal to $W_{n,m}(t)$.

Theorem 4.5. *The Alexander polynomial of the torus knot of the second order* $T^2 := T_{(n_1,m_1)(n_2,m_2)}$ *is given by the formula*

$$A_{T^2}(t) = W_{n_1,m_1}(t^{n_2})W_{n_2,m_2-m_1n_2+m_1n_1n_2}(t).$$

Proof. From Theorem 4.4 we have

$$\pi(T^2) \cong \mathcal{F}(\gamma_0,\delta_1,\delta_2)/\left(\delta_1^{n_1}\gamma_0^{-m_1}, \left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}\delta_1^{n_1n_2}\delta_2^{-n_2}\right).$$

Denoting the first relation by R_1 and the second by R_2 we get through formal differentiation

$$\frac{\partial R_1}{\partial\gamma_0} = -\delta_1^{n_1}\gamma_0^{-m_1}\frac{\gamma_0^{m_1}-1}{\gamma_0-1}, \quad \frac{\partial R_1}{\partial\delta_1} = \frac{\delta_1^{n_1}-1}{\delta_1-1}, \quad \frac{\partial R_1}{\partial\delta_2} = 0.$$

and

$$\begin{aligned}
\frac{\partial R_2}{\partial\gamma_0} &= -\delta_1^a\gamma_0^{-b}\frac{\left(\left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}-1\right)}{\delta_1^a\gamma_0^{-b}-1}\frac{\gamma_0^b-1}{\gamma_0-1},\\
\frac{\partial R_2}{\partial\delta_1} &= \frac{\left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}-1}{\delta_1^a\gamma_0^{-b}-1}\frac{\delta_1^a-1}{\delta_1-1} + \left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}\frac{\delta_1^{n_1n_2}-1}{\delta_1-1},\\
\frac{\partial R_2}{\partial\delta_2} &= -\left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}\delta_1^{n_1n_2}\delta_2^{-n_2}\frac{\delta_2^{n_2}-1}{\delta_2-1}.
\end{aligned}$$

Taking into account the equality $\gamma_1 = \delta_1^a\gamma_0^{-b}$, where $am_1 - bn_1 = 1$, and that in $\pi(T_{(n_1,m_1)(n_2,m_2)})'$ we have $\gamma_0 = \gamma_1^{n_1}$, $\delta_1 = \gamma_1^{m_1}$, we obtain equalities in $\pi(T_{(n_1,m_1)(n_2,m_2)})'$

$$\frac{\partial R_1}{\partial\gamma_0} = -\frac{\gamma_1^{n_1m_1}-1}{\gamma_1^{n_1}-1}, \quad \frac{\partial R_1}{\partial\delta_1} = \frac{\gamma_1^{n_1m_1}-1}{\gamma_1^{m_1}-1}, \quad \frac{\partial R_1}{\partial\delta_2} = 0.$$

and

$$\begin{aligned}
\frac{\partial R_2}{\partial\gamma_0} &= -\gamma_1\frac{\left(\gamma_1^{m_2-m_1n_2}-1\right)}{\gamma_1-1}\frac{\gamma_1^{bn_1}-1}{\gamma_1^{n_1}-1} = -\frac{\left(\gamma_1^{m_2-m_1n_2}-1\right)}{\gamma_1-1}\frac{\gamma_1^{am_1}-\gamma_1}{\gamma_1^{n_1}-1},\\
\frac{\partial R_2}{\partial\delta_1} &= \frac{\gamma_1^{m_2-m_1n_2}-1}{\gamma_1-1}\frac{\gamma_1^{am_1}-1}{\gamma_1^{m_1}-1} + \gamma_1^{m_2-m_1n_2}\frac{\gamma_1^{m_1n_1n_2}-1}{\gamma_1^{m_1}-1},\\
\frac{\partial R_2}{\partial\delta_2} &= -\frac{\gamma_1^{m_2-m_1n_2+m_1n_1n_2}-1}{\delta_2-1}.
\end{aligned}$$

Then minors of the second degree of the Aleksander matrix of the group $\pi(T^2)'$ are

$$M_1 = \begin{vmatrix} \frac{\partial R_1}{\partial \gamma_0} & \frac{\partial R_1}{\partial \delta_1} \\ \frac{\partial R_2}{\partial \gamma_0} & \frac{\partial R_2}{\partial \delta_1} \end{vmatrix} = \frac{\partial R_1}{\partial \gamma_0}\frac{\partial R_2}{\partial \delta_1} - \frac{\partial R_1}{\partial \delta_1}\frac{\partial R_2}{\partial \gamma_0}$$

$$= -\frac{\gamma_1^{n_1 m_1}-1}{\gamma_1^{n_1}-1}\left(\frac{\left(\gamma_1^{m_2-m_1n_2}-1\right)}{\gamma_1-1}\frac{\left(\gamma_1^{am_1}-1\right)}{\gamma_1^{m_1}-1} + \gamma_1^{m_2-m_1n_2}\frac{\gamma_1^{m_1n_1n_2}-1}{\gamma_1^{m_1}-1}\right)$$

$$-\frac{\gamma_1^{n_1m_1}-1}{\gamma_1^{m_1}-1}\left(-\frac{\left(\gamma_1^{m_2-m_1n_2}-1\right)}{\gamma_1-1}\frac{\left(\gamma_1^{am_1}-\gamma_1\right)}{\gamma_1^{n_1}-1}\right)$$

$$= -\frac{\gamma_1^{m_2-m_1n_2}\left(\gamma_1^{n_1m_1}-1\right)\left(\gamma_1^{m_1n_1n_2}-1\right)}{\left(\gamma_1^{n_1}-1\right)\left(\gamma_1^{m_1}-1\right)} - \frac{\left(\gamma_1^{m_2-m_1n_2}-1\right)\left(\gamma_1^{n_1m_1}-1\right)}{\left(\gamma_1^{n_1}-1\right)\left(\gamma_1^{m_1}-1\right)}$$

$$= -\frac{\left(\gamma_1^{n_1m_1}-1\right)\left(\gamma_1^{m_2-m_1n_2+m_1n_1n_2}-1\right)}{\left(\gamma_1^{n_1}-1\right)\left(\gamma_1^{m_1}-1\right)} = -W_{n_1,m_1}(\gamma_1)\frac{\gamma_1^{m_2-m_1n_2+m_1n_1n_2}-1}{\gamma_1-1},$$

$$M_2 = \begin{vmatrix} \frac{\partial R_1}{\partial \gamma_0} & \frac{\partial R_1}{\partial \delta_2} \\ \frac{\partial R_2}{\partial \gamma_0} & \frac{\partial R_2}{\partial \delta_2} \end{vmatrix} = \frac{\partial R_1}{\partial \gamma_0}\frac{\partial R_2}{\partial \delta_2} = \frac{\gamma_1^{n_1m_1}-1}{\gamma_1^{n_1}-1}\frac{\gamma_1^{m_2-m_1n_2+m_1n_1n_2}-1}{\delta_2-1},$$

$$M_3 = \begin{vmatrix} \frac{\partial R_1}{\partial \delta_1} & \frac{\partial R_1}{\partial \delta_2} \\ \frac{\partial R_2}{\partial \delta_1} & \frac{\partial R_2}{\partial \delta_2} \end{vmatrix} = \frac{\partial R_1}{\partial \delta_1}\frac{\partial R_2}{\partial \delta_2} = \frac{\gamma_1^{n_1m_1}-1}{\gamma_1^{m_1}-1}\frac{\gamma_1^{m_2-m_1n_2+m_1n_1n_2}-1}{\delta_2-1}.$$

Because the loop γ_2 (see Figure 24) is a generator of the group $\pi(T^2)'$,

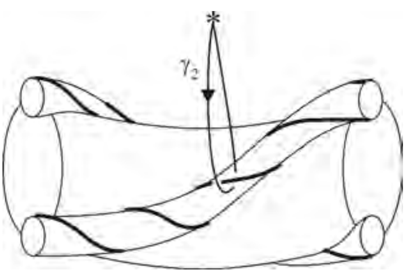

Fig. 24. The loop γ_2.

then in $\pi(T^2)'$ we have $\gamma_1 = \gamma_2^{n_2}$ and $\delta_1 = \gamma_2^{m_1n_2}$. Hence from equality (12) we get $\delta_2 = \gamma_2^{m_2-m_1n_2+m_1n_1n_2}$. Then

$$M_1' = -W_{n_1,m_1}(\gamma_2^{n_2})\frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{n_2}-1},$$

$$M_2' = \frac{\gamma_2^{n_1m_1n_2}-1}{\gamma_2^{n_1n_2}-1}\,\frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{m_2-m_1n_2+m_1n_1n_2}-1},$$

$$M_3' = \frac{\gamma_2^{n_1m_1n_2}-1}{\gamma_2^{m_1n_2}-1}\,\frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{m_2-m_1n_2+m_1n_1n_2}-1}.$$

Since $\mathrm{GCD}(n_2, m_2 - m_1n_2 + m_1n_1n_2) = 1$, we easily show

$$\begin{aligned}
&\mathrm{GCD}(M_1', M_2', M_3') = \mathrm{GCD}(M_1', \mathrm{GCD}(M_2', M_3'))\\
&= \mathrm{GCD}\left(M_1', \frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{m_2-m_1n_2+m_1n_1n_2}-1}\,\mathrm{GCD}\left(\frac{\gamma_1^{n_1m_1n_2}-1}{\gamma_1^{n_1n_2}-1}, \frac{\gamma_1^{n_1m_1n_2}-1}{\gamma_1^{m_1n_2}-1}\right)\right)\\
&= \mathrm{GCD}\left(M_1', \frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{m_2-m_1n_2+m_1n_1n_2}-1} W_{n_1,m_1}(\gamma_2^{n_2})\right)\\
&= W_{n_1,m_1}(\gamma_2^{n_2})\,\mathrm{GCD}\left(\frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{n_2}-1}, \frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{m_2-m_1n_2+m_1n_1n_2}-1}\right)\\
&= W_{n_1,m_1}(\gamma_2^{n_2})W_{n_2,m_2-m_1n_2+m_1n_1n_2}(\gamma_2).
\end{aligned}$$

Putting $t = \gamma_2$ we get the assertion of the theorem. □

We will now discribe the general case of the torus knots of the g-order $T^g := T_{(n_1,m_1)\cdots(n_g,m_g)} \subset \partial P$. We will calculate $\pi(T^g) = \pi_1(\partial P \setminus T^g, *)$, where $* \notin T^g$. We choose the point $*$ on $\partial(\mathrm{Tube}(T^{g-1})) \setminus T^g$. As in the case of the second order torus knots, it can be shown that the generators of $\pi(T^g)$ are γ_0 and the "axes" $\delta_1, \dots, \delta_g$ of consecutive tubular neighbourhoods (see Fig. 25).

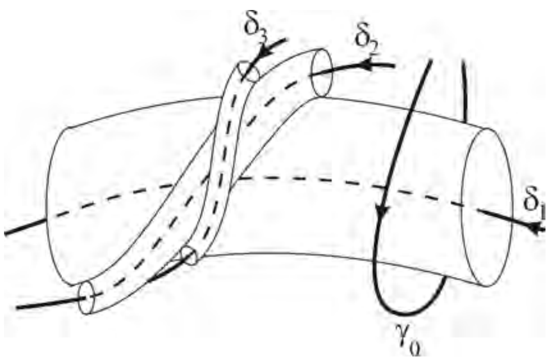

Fig. 25. Generators of $\pi(T^g)$.

Of course $\delta_1 \sim T^0 := T_{1,1}$, $\delta_2 \sim T^1, \dots, \delta_g \sim T^{g-1}$ in ∂P. The same reasoning as in the case of the second-order toru knots (using the Seifert-van Kampen theorem) we will obtain that there are the following relations between these generators

$$\begin{aligned}
&R_1 : \delta_1^{n_1} = \gamma_0^{m_1},\\
(13)\qquad &R_2 : \delta_2^{n_2} = \gamma_1^{m_2-m_1n_2}\delta_1^{n_1n_2},\\
&\dots\dots\dots\dots\dots\dots\\
&R_g : \delta_g^{n_g} = \gamma_{g-1}^{m_g-m_{g-1}n_g}\delta_{g-1}^{n_{g-1}n_g},
\end{aligned}$$

where $\gamma_0, \gamma_1, \ldots, \gamma_{g-1}$ (see Fig. 26) are loops circling one thread of the torus knots $T^0, T^1, \ldots, T^{g-1}$, respectively.

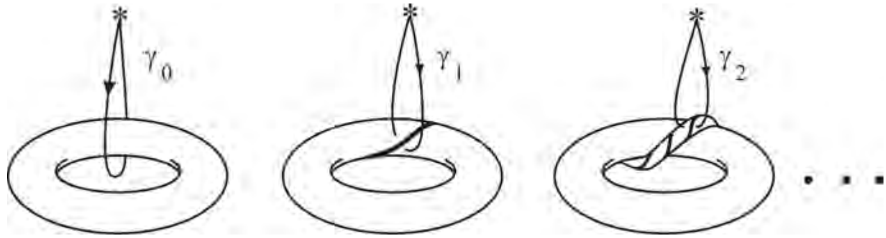

Fig. 26. The loops $\gamma_0, \ldots, \gamma_g - 1$

The loops satisfy the relations

$$\gamma_1 = \delta_1^{a_1}\gamma_0^{-b_1}, \text{ where } a_1, b_1 \in \mathbb{N},\ a_1m_1 - b_1n_1 = 1,$$

$$(14)\qquad \gamma_2 = \delta_2^{a_2}\gamma_1^{-b_2+a_2m_1}\delta_1^{-a_2n_1}, \text{ where } a_2, b_2 \in \mathbb{N},\ a_2m_2 - b_2n_2 = 1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\gamma_{g-1} = \delta_{g-1}^{a_{g-1}}\gamma_{g-2}^{-b_{g-1}+a_{g-1}m_{g-2}}\delta_{g-2}^{-a_{g-1}n_{g-2}}, \text{ where } a_{g-1}, b_{g-1} \in \mathbb{N},$$
$$a_{g-1}m_{g-1} - b_{g-1}n_{g-1} = 1.$$

In fact, we will show this only for γ_2, because the reasoning in the general case is analogous. We have to express γ_2 by $\gamma_1, \delta_1, \delta_2$. Fix point $*$ on $\partial(\mathrm{Tube}(T^1)) \setminus T^2$. Let's denote by Q the point on $\partial(\mathrm{Tube}(T^1)) \setminus T^2$ that differs from the point $*$ by $1/m$ rotation of the projection on the first axis (in coordinates given by the canonical homeomorphism $\Phi_1 : T \to \partial(\mathrm{Tube}(T^1))$) (see Fig. 27).

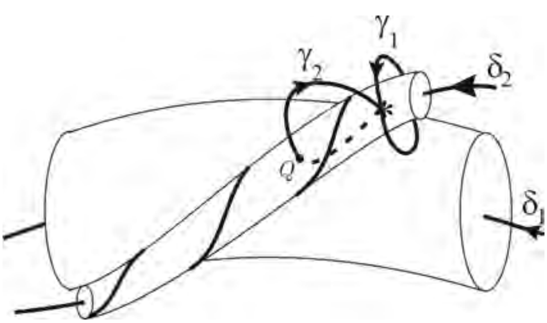

Fig. 27.

By Lemma 3.4 the point Q can be connected to the point $*$ by a curve lying in $\partial(\mathrm{Tube}(T^1))$ "parallel" to T^2. In coordinates given by the canonical homeomorphism $\Phi_1 : T \to \partial(\mathrm{Tube}(T^1))$) this curve circles the torus $\partial(\mathrm{Tube}(T^1))$ $a_2 - 1/m_2$-times along and b_2-times across. Therefore, by moving the point Q along this curve, together with the entire curve γ_2, we get a curve homotopic to curve γ_2 in $\partial P \setminus T^2$ with the initial and final point in $*$, whose the first part lies inside $\mathrm{Tube}(T^1)$ (except the point $*$) and the second one out of $\mathrm{Tube}(T^1)$. Denoting these curves by κ_1 and

κ_2, we have $\gamma_2 = \kappa_1\kappa_2$. We will now express κ_1 and κ_2 by $\gamma_1, \delta_1, \delta_2$. Because the curve connecting Q with $*$ makes $a_2 - 1/m_2$ and the first part of γ_2 (from the point $*$ to Q) makes $1/m_2$ rotation along $\partial(\mathrm{Tube}(T^1))$, and δ_2 is the axis of $\mathrm{Tube}(T^1)$, then

$$\kappa_1 = \delta_2^{a_2}.$$

Determining κ_2 is much more difficult. The curve κ_2 makes $-b_2$ rotations across $\partial(\mathrm{Tube}(T^1))$. We will determine $s \in \mathbb{Z}$ such that $\gamma_1^s\kappa_2$ is a multiple of δ_1, more precisely equal to $\delta_1^{-a_2n_1}$ (because the projection of κ_2 on the first axis makes $-a_2n_1$ rotations; in fact, $-a_2$ rotations in coordinates of Φ_1 but each such rotation corresponds to n_1 rotations of the projection on the first axis in $\mathbb{C}^2$). Let's analyze one rotation around $\partial(\mathrm{Tube}(T^1))$ towards negative orientation (see Fig. 28).

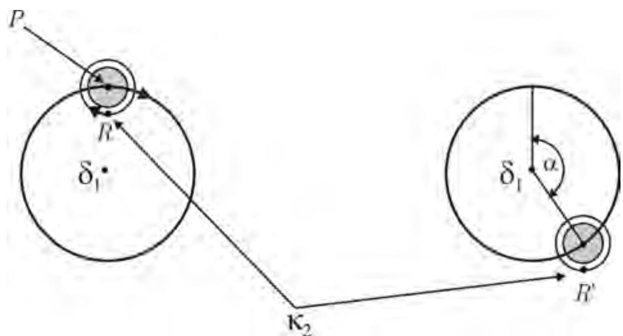

Fig. 28.

After one rotation in the negative direction around $\partial(\mathrm{Tube}(T^1))$ a point R goes to a point R' and the projection of the path made by the point R on the first axis will make of course $n_1\frac{a_2}{b_2}$ rotation (because the rotational speed of R is $\frac{a_2}{b_2}$) towards negative orientation. Then the point P (its rotational speed is $\frac{m_1}{n_1}$) will make rotation $\alpha = \frac{m_1}{n_1}n_1\frac{a_2}{b_2} = \frac{m_1a_2}{b_2}$. Then to get a curve which is a multiple of δ_1 it is not enough to make 1 rotation (for every single rotation of κ_2), but you should also add an α rotation in the negative direction. Because κ_2 makes b_2 rotations so $s = b_2(1-\alpha) = b_2 - m_1a_2$. Hence $\gamma_1^{b_2-m_1a_2}\kappa_2 = \delta_1^{-a_2n_1}$, which gives $\kappa_2 = \gamma_1^{-b_2+m_1a_2}\delta_1^{-a_2n_1}$. Consequently

$$\gamma_2 = \kappa_1\kappa_2 = \delta_2^{a_2}\gamma_1^{-b_2+m_1a_2}\delta_1^{-a_2n_1}.$$

Then we obtain

Theorem 4.6. *For any sequence of pairs of natural numbers* $((n_1, m_1), \dots, (n_g, m_g))$ *such that* $\mathrm{GCD}(n_i, m_i) = 1$, $i = 1, \dots, g$, *we have*

$$\pi(T_{(n_1,m_1)\cdots(n_g,m_g)}) = \mathcal{F}(\gamma_0, \delta_1, \dots, \delta_g)/(R_1, \dots, R_g),$$

where $R_1, \dots, R_g$ *are relations given in (13) and (14).*

We can now give the Aleksander polynomial of the knot $T^g := T_{(n_1,m_1)\cdots(n_g,m_g)}$.

Theorem 4.7.

$$A_{T^g}(t) = W_{n_1,\lambda_1}(t^{n_2\cdots n_g})W_{n_2,\lambda_2}(t^{n_3\cdots n_g})\dots W_{n_{g-1},\lambda_{g-1}}(t^{n_g})W_{n_g,\lambda_g}(t), \tag{15}$$

where the sequence $\lambda_1,\cdots,\lambda_g$ *is defined recursively*

$$\begin{aligned} \lambda_1 &= m_1, \\ \lambda_k &= m_k - m_{k-1}n_k + \lambda_{k-1}n_{k-1}n_k, \quad k \geq 2. \end{aligned} \tag{16}$$

Proof. We proved it for $g = 1$ and $g = 2$. Proof of the general case can be found in [Le]. □

Remark 4.8. *In particular for the knot* $T := T_{(2,1)(2,1)}$ *we have* $A_T(t) = W_{2,3}(t)$, *so it's not trivial knot.*

We will now show that, under additional assumptions, the Alexander polynomial of the torus knots uniquely characterizes it. We will prove it under assumptions

$$n_k > 1, \quad k = 1,\dots,g \tag{17}$$

$$m_k - m_{k-1}n_k > 0, \quad k = 2,\dots,g, \tag{18}$$

This condition is always satisfied for torus knots associated with curves singularities. First, we will prove a lemma.

Lemma 4.9. *If inequalities (17) and (18) holds, then*

$$\lambda_g > \lambda_i n_i \dots n_g, \quad i = 1,\dots,g-1, \tag{19}$$

$$\lambda_g n_g > \lambda_i n_i \dots n_g, \quad i = 1,\dots,g-1. \tag{20}$$

Proof. Because the second inequality follows from the first one, it is enough to prove the first one. From inequalities (17) and (18) we get

$$\begin{aligned} \lambda_g &= m_g - m_{g-1}n_g + \lambda_{g-1}n_{g-1}n_g > \lambda_{g-1}n_{g-1}n_g \\ &= (m_{g-1} - m_{g-2}n_{g-1} + \lambda_{g-2}n_{g-2}n_{g-1})\, n_{g-1}n_g \\ &> \lambda_{g-2}n_{g-2}n_{g-1}^2 n_g \geq \lambda_{g-2}n_{g-2}n_{g-1}n_g \\ &\geq \dots \geq \lambda_i n_i \dots n_g. \end{aligned}$$

□

Theorem 4.10. *Let* $T := T_{(n_1,m_1)\cdots(n_g,m_g)}$ *and* $T' := T_{(n'_1,m'_1)\cdots(n'_h,m'_h)}$ *be two torus knots such that*

$$\begin{aligned} n_i > 1, \quad i = 1,\dots,g, \qquad n'_i > 1, \quad i = 1,\dots,h, \\ m_i - m_{i-1}n_i > 0, \quad i = 2,\dots,g, \\ m'_i - m'_{i-1}n'_i > 0, \quad i = 2,\dots,h. \end{aligned} \tag{21}$$

Assume the Aleksander polynomials $A_T(t)$ *and* $A_{T'}(t)$ *are equal. Then*

$$\begin{aligned} g &= h, \\ n_i &= n'_i, \quad i = 1,\dots,g, \\ m_i &= m'_i, \quad i = 1,\dots,g. \end{aligned}$$

Proof. By assumption $A_T(t) = A_{T'}(t)$. The polynomial $A_T(t)$ is given by formulas (15) and (16). Analogously

$$\begin{aligned} A_{T'}(t) &= W_{n'_1,\lambda'_1}(t^{n'_2\cdots n'_h})W_{n'_2,\lambda'_2}(t^{n'_3\cdots n'_h})\dots W_{n'_{h-1},\lambda'_{h-1}}(t^{n'_h})W_{n'_h,\lambda'_h}(t),\\ \lambda'_1 &= m'_1,\\ \lambda'_k &= m'_k - m'_{k-1}n'_k + \lambda'_{k-1}n'_{k-1}n'_k, \quad k \ge 2. \end{aligned}$$

We will show first that

$$(n_g, \lambda_g) = (n'_h, \lambda'_h). \tag{22}$$

From the form of factors of $A_T(t)$ and $A_{T'}(t)$, namely

$$\begin{aligned} W_{n_i,\lambda_i}(t^{n_{i+1}\cdots n_g}) &= \frac{\left(t^{\lambda_i n_i\cdots n_g}-1\right)\left(t^{n_{i+1}\cdots n_g}-1\right)}{\left(t^{\lambda_i n_{i+1}\cdots n_g}-1\right)\left(t^{n_i\cdots n_g}-1\right)},\\ W_{n_g,\lambda_g}(t) &= \frac{\left(t^{\lambda_g n_g}-1\right)(t-1)}{\left(t^{\lambda_g}-1\right)\left(t^{n_g}-1\right)} \end{aligned}$$

and inequality (20) and analogous for T' it follows that

$$\lambda_g n_g = \lambda'_h n'_h. \tag{23}$$

Indeed, otherwise e.g. if $\lambda_g n_g > \lambda'_h n'_h$, then from the assumption $n_g > 1,\ n'_h > 1$ a primitive root of unity of degree $\lambda_g n_g$ would be a root of the polynomial $A_T(t)$ and would not be a root of $A_{T'}(t)$, which is impossible. From the equality (23) follows in a similar manner to the above that

$$\lambda_g = \lambda'_h. \tag{24}$$

From (23) and (24) we get (22). Hence

$$W_{n_g,\lambda_g}(t) = W_{n'_h,\lambda'_h}(t).$$

Then dividing $A_T(t)$ and $A_{T'}(t)$ by this polynomial and substituting $u = t^{n_g}$ we get the equality of polynomials

$$\begin{aligned} &W_{n_1,\lambda_1}(u^{n_2\cdots n_{g-1}})W_{n_2,\lambda_2}(u^{n_3\cdots n_{g-1}})\dots W_{n_{g-1},\lambda_{g-1}}(u)\\ &\qquad = W_{n'_1,\lambda'_1}(u^{n'_2\cdots n'_{h-1}})W_{n'_2,\lambda'_2}(u^{n'_3\cdots n'_{h-1}})\dots W_{n'_{h-1},\lambda'_{h-1}}(u). \end{aligned}$$

The polynomials on both sides of the equality are the Alexander polynomials of torus knots of $(g-1)$-order $T_{(n_1,m_1)\dots(n_{g-1},m_{g-1})}$ and $T_{(n'_1,m'_1)\dots(n'_{h-1},m'_{h-1})}$. Repeating the above reasoning, we will receive successively

$$(n_{g-1}, \lambda_{g-1}) = (n'_{h-1}, \lambda'_{h-1})$$
$$\dots\dots\dots\dots\dots\dots$$
$$(n_1, \lambda_1) = (n'_1, \lambda'_1)$$

Hence $g = h$ and $n_i = n'_i$ for $i = 1, \dots, g$. Since $\lambda_1 = m_1$ and $\lambda'_1 = m'_1$, we have $m_1 = m'_1$. Further using the formulas for λ_i we easily get tha $m_i = m'_i$ for $i = 2, \dots, g$, too. This ends the proof. □

5. The knot of an irreducible curve

At this section we will recall the known basic properties of analytic curves in the complex plane $\mathbb{C}^2$. Details can be found in many textbooks on complex curves [KP], [W], [BK], [L].

For a given set $V \subset \mathbb{C}^n$ by $\boldsymbol{V}$ or $\widehat{V}$ we denote its germ at $0 \in \mathbb{C}^n$. A *local analytic curve* (for short a *curve*) is any germ $\boldsymbol{V}$ at $0 \in \mathbb{C}^2$ of the zero set of a holomorphic function $f \in \mathbb{C}\{x, y\}$ satisfying the conditions: $f \neq \text{const}$, $f(0,0) = 0$. Then $\operatorname{ord} f > 0$. The curve described by a holomorphic function $f \in \mathbb{C}\{x, y\}$ we denote by $V(f)$. When f is defined in a certain neighbourhood U of point 0, we denote the set of zeros of f in U by $V_U(f)$. Because $\mathbb{C}\{x, y\}$ is the unique factorization domain, we will always assume that the function f describing a curve $\boldsymbol{V}$ is reduced, i.e. there are no multiple factors in the factorization of f in $\mathbb{C}\{x, y\}$ into irreducible factors. Each curve $\boldsymbol{V} = V(f)$ has the unique decomposition into irreducible components

$$\boldsymbol{V} = \boldsymbol{V}_1 \cup \ldots \cup \boldsymbol{V}_k,$$

called *branches* of $\boldsymbol{V}$. The branches uniquely correspond to irreducible factors of f in $\mathbb{C}\{x, y\}$, i.e. if $f = f_1 \ldots f_l$ in $\mathbb{C}\{x, y\}$ and f_i are irreducible and not associated then $k = l$ and after renumbering $\boldsymbol{V}_i = V(f_i)$ for $i = 1, \ldots, k$. Because we are interested in the properties of analytic curves, invariant with respect to biholomorphisms of neighbourhoods of the zero in $\mathbb{C}^2$, we can always assume that the function f describing an analytic curve V satisfies the condition

$$\operatorname{ord} f = \operatorname{ord} f(0, y) \tag{25}$$

(we get this condition by linear change of variables in $\mathbb{C}^2$). Moreover, by the Weierstrass theorem we can additionally assume that f is a distinguished polynomial, i.e. $f \in \mathbb{C}\{x\}[y]$ and has the form

$$f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x), \quad n > 0, \quad \operatorname{ord} a_i \geq i, \quad i = 1, \ldots, n. \tag{26}$$

Then each branch $\boldsymbol{V}_i$ of the curve $V(f)$ has the *Puiseux parameterization*, i.e. there is a holomorphic, one-to-one mapping $\Phi_i(t) = (t^{n_i}, \varphi_i(t))$, $\operatorname{ord} \varphi_i \geq n_i$, defined in a neighbourhood of $0 \in \mathbb{C}$ such that $\boldsymbol{V}_i = \widehat{\operatorname{Im} \Phi_i}$. Moreover, if f is irreducible in $\mathbb{C}\{x\}[y]$, then we may assume that in (26) $\operatorname{ord} a_i > i$, $i = 1, \ldots, n$. Then for a Puiseux parameterization $\Phi(t) = (t^n, \varphi(t))$ of the unique branch of f the inequality $\operatorname{ord} \varphi > n$ holds.

The basic theorem on which the study of the topological structure of $V_U(f)$ in U is based is the theorem on the cone structure of isolated singularity. Before we give this theorem we will define the concept of a cone with a given base. For any $A \subset \mathbb{C}^n \setminus \{\mathbf{0}\}$ the *cone with base* A is the union of segments connecting point 0 with points A (see Fig. 29). We denote it cone(A). Therefore

$$\operatorname{cone}(A) := \{\boldsymbol{z} \in \mathbb{C}^n : \boldsymbol{z} = t\boldsymbol{a},\ t \in [0, 1],\ \boldsymbol{a} \in A\}$$

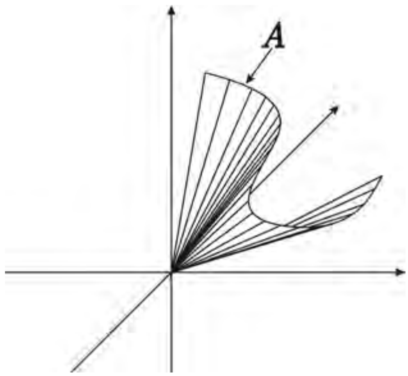

Fig. 29. The cone with a base A.

Note that for any closed polycylinder $P(\varepsilon, \eta) := \{(x, y) \in \mathbb{C}^2 : |x| \leq \varepsilon,\ |y| \leq \eta\} \subset \mathbb{C}^2$ with center at zero and radii $\varepsilon, \eta > 0$ and its boundary

$$\partial P(\varepsilon, \eta) = \{(x, y) \in \mathbb{C}^2 : |x| = \varepsilon,\ |y| \leq \eta\} \cup \{(x, y) \in \mathbb{C}^2 : |x| \leq \varepsilon,\ |y| = \eta\}$$

we have

$$\operatorname{cone}(\partial P(\varepsilon, \eta)) = P(\varepsilon, \eta).$$

Theorem 5.1 (on the cone structure of irreducible curve singularity). *Let* $\boldsymbol{V}$ *be an irreducible curve and* $V := V_U(f)$ *its representative. Suppose* f *is a distinguished polynomial, i.e.* $f \in \mathbb{C}\{x\}[y]$ *has the form (26) and in the Puiseux parameterization* $\Phi(t) = (t^n, \varphi(t))$ *of the unique branch of* f *there is* $\operatorname{ord} \varphi > n$*. Then there exists* $\widetilde{\varepsilon} > 0$ *such that* $P(\widetilde{\varepsilon}, \widetilde{\varepsilon}) \subset U$ *and for every* ε, $0 < \varepsilon < \widetilde{\varepsilon}$, *there exists a homeomorphism of the pairs (see Fig. 30)*

$$(P(\varepsilon, \varepsilon), V \cap P(\varepsilon, \varepsilon)) \overset{\text{top}}{\approx} (P(\varepsilon, \varepsilon), \operatorname{cone}(V \cap \partial P(\varepsilon, \varepsilon))$$

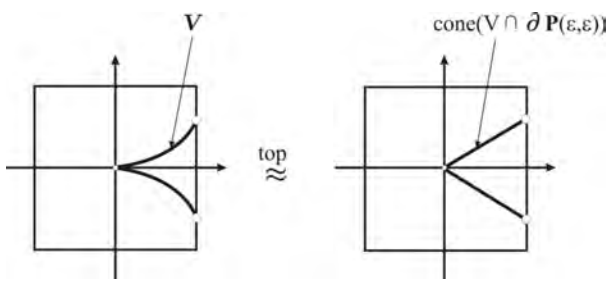

Fig. 30. The cone structure of a local curve.

and for any $\varepsilon, \varepsilon'$, $0 < \varepsilon < \varepsilon' < \widetilde{\varepsilon}$ *there exists a homeomorphism of the pairs*

$$(\partial P(\varepsilon, \varepsilon), V \cap \partial P(\varepsilon, \varepsilon)) \overset{\text{top}}{\approx} (\partial P(\varepsilon', \varepsilon'), V \cap \partial P(\varepsilon', \varepsilon'))\,.$$

A proof can be found in [M], [P], [W]. This theorem says that the immersion of V in P is determined, up to a homeomorphism of P, by the trace of V in the boundary of this polycylinder.

Remark 5.2. *The theorem is usually proven for closed balls. Then assumptions about the form of the function f are superfluous.*

Remark 5.3. *The theorem holds for any isolated singularity (of arbitrary dimension).*

Let $\boldsymbol{V} = V(f)$ be an irreducible curve, where $f \in \mathbb{C}\{x\}[y]$ is a distinguished poynomial of degree n, $n := \operatorname{ord} f = \operatorname{ord} f(0,y)$. Then $\boldsymbol{V}$ has a Puiseux parametrization $\Phi(t) = (t^n, \varphi(t))$, $t \in K$ – a neighbourhood of the origin in $\mathbb{C}$, $\operatorname{ord}\varphi > n$, in a neighbourhood U of zero in $\mathbb{C}^2$, i.e. f is defined in U and

$$V_U(f) = \{\Phi(t) : t \in K\} \ . \tag{27}$$

Denote $V := V_U(f)$. By the theorem on the cone structure there exists $\widetilde{\varepsilon} > 0$ such that $P(\widetilde{\varepsilon}, \widetilde{\varepsilon}) \subset U$ and for every ε, $0 < \varepsilon < \widetilde{\varepsilon}$ the traces of V on boundaries $\partial P(\varepsilon, \varepsilon)$ of policylinders $P(\varepsilon, \varepsilon)$ are topologically equivalent i.e. for every two $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$ there exists a homeomorphism $H : \partial P(\varepsilon_1, \varepsilon_1) \to \partial P(\varepsilon_2, \varepsilon_2)$ such that $H(V \cap \partial P(\varepsilon_1, \varepsilon_1)) = V \cap \partial P(\varepsilon_2, \varepsilon_2)$. Moreover the form of Φ implies that by diminishing $\widetilde{\varepsilon}$ we may assume that $|\varphi(t)| < |t|$ dla $t \in K$.

Hence and again from the form of Puiseux parameterization $\Phi(t) = (t^n, \varphi(t))$ it follows that for ε, $0 < \varepsilon < \widetilde{\varepsilon}$, we have $V \cap \partial P(\varepsilon, \varepsilon) = \Phi(S(\varepsilon^{1/n}))$, where $S(r)$ is the circle in $\mathbb{C}$ with the center at 0 and radius $r > 0$. Thus, the trace of V in the boundary of each of these polycylinders is homeomorphic to S^1. On the other hand, the boundary $\partial P(\varepsilon, \eta)$ is homeomorphic to a three-dimensional sphere $S^3 = \mathbb{R}^3 \cup \{0\}$ and so $V \cap \partial P(\varepsilon, \varepsilon)$ is a knot. It follows from the above that this knot does not depend on the choice of the radius ε. We call it the knot of the curve $\boldsymbol{V}$ and denote it by $K_{\boldsymbol{V}}$. Then, by definition, the knot group $\pi(K_{\boldsymbol{V}})$ is equal to $\pi_1(\partial P(\varepsilon, \varepsilon) \setminus V, *)$, $* \in \partial P(\varepsilon, \varepsilon) \setminus V$. By the theorem on the cone structure we may calculate this group differently.

Lemma 5.4. $\pi(K_{\boldsymbol{V}}) \cong \pi_1(P(\varepsilon, \varepsilon) \setminus V)$.

Proof. Take the point $* \in \partial P(\varepsilon, \varepsilon) \setminus V$ as the base point for both groups $\pi(K_{\boldsymbol{V}})$ and $\pi_1(P(\varepsilon, \varepsilon) \setminus V)$. We have to show that

$$\pi_1(\partial P(\varepsilon, \varepsilon) \setminus V, *) \cong \pi_1(P(\varepsilon, \varepsilon) \setminus V, *).$$

This isomorphism follows from the theorem on cone structure of V in $P(\varepsilon, \varepsilon)$ because each loop in $P(\varepsilon, \varepsilon) \setminus V$ with beginning and end at $*$ is homotopic to a loop lying in $\partial P(\varepsilon, \varepsilon) \setminus V$. □

Let $(\beta_0, \beta_1, \ldots, \beta_h)$ be the characteristic of the curve $\boldsymbol{V}$ and $((m_1, n_1), \ldots, (m_h, n_h))$ be the sequence of characteristic pairs of $\boldsymbol{V}$. Recall that if $\varphi(t) = a_{p_1} t^{p_1} + a_{p_2} t^{p_2} + \ldots$, $a_{p_i} \neq 0$, $i \geq 1$, then

$\beta_0 = n,$

$\beta_i = \min\{p_k : \mathrm{GCD}\,(\beta_0, \beta_1, \ldots, \beta_{i-1}, p_k) < \mathrm{GCD}(\beta_0, \beta_1, \ldots, \beta_{i-1})\},\ i = 1, \ldots, h$

and

$$\frac{\beta_1}{\beta_0} = \frac{m_1}{n_1}, \ \text{GCD}(m_1, n_1) = 1,$$
$$\frac{\beta_2}{\beta_0} = \frac{m_2}{n_1 n_2}, \ \text{GCD}(m_2, n_2) = 1,$$
$$\dots\dots\dots\dots\dots\dots$$
$$\frac{\beta_h}{\beta_0} = \frac{m_h}{n_1 \cdots n_h}, \ \text{GCD}(m_h, n_h) = 1.$$

Since

$$\frac{\beta_1}{\beta_0} < \frac{\beta_2}{\beta_0} < \ldots < \frac{\beta_h}{\beta_0},$$

then

(28) $$m_i < m_{i-1} n_i \text{ dla } i = 2, \ldots, h.$$

Theorem 5.5. *Under the above assumptions on* $\boldsymbol{V}$ *the knot* $K_{\boldsymbol{V}}$ *is the torus knot of the h-order of the type* $(m_1, n_1), \ldots, (m_h, n_h)$, *i.e.*

$$K_{\boldsymbol{V}} \sim T_{(m_1,n_1),\ldots,(m_h,n_h)}.$$

Proof. Fix the above assumptions and notations. Denote by pr_1 the projection of $\mathbb{C}^2$ onto the first axis: $pr_1(x, y) = x$. We will define an auxiliary characteristic sequence $((n'_1, m'_1), \ldots, (n'_{\beta_h}, m'_{\beta_h}))$ of the curve V. Its construction is analogous to the construction of the sequence $((m_1, n_1), \ldots, (m_h, n_h))$ with the difference that we allow equality $n'_i = 1$. If $y(t) = a_{p_1} t^{p_1} + a_{p_2} t^{p_2} + \ldots$, $a_{p_i} \neq 0$, we put

$$\frac{p_1}{n} = \frac{m'_1}{n'_1}, \ \text{GCD}(m'_1, n'_1) = 1,$$
$$\frac{p_2}{n} = \frac{m'_2}{n'_1 n'_2}, \ \text{GCD}(m'_2, n'_2) = 1,$$
$$\dots\dots\dots\dots\dots\dots$$
$$\frac{p_\beta}{n} = \frac{m'_{p_\beta}}{n'_1 \cdots n'_{p_\beta}}, \ \text{GCD}(m'_{p_\beta}, n'_{p_\beta}) = 1,$$

where $p_\beta = \beta_h$. Note that the sequence of characteristic pairs $((m_1, n_1), \ldots, (m_h, n_h))$ is a subsequence of $((m'_1, n'_1), \ldots, (m'_{p_\beta}, n'_{p_\beta}))$ and $(m_h, n_h) = (m'_{p_\beta}, n'_{p_\beta})$. More precisely, it suffices to omit from the sequence $((m'_1, n'_1), \ldots, (m'_{p_\beta}, n'_{p_\beta}))$ the pairs for which $n'_i = 1$. We will show that the knot of $\boldsymbol{V}$ is eqivalent to $T_{(n'_1,m'_1)\ldots(n'_{p_\beta},m'_{p_\beta})}$. Then by Proposition 4.1 $T_{(n'_1,m'_1)\ldots(n'_{p_\beta},m'_{p_\beta})} = T_{(m_1,n_1)\ldots(m_h,n_h)}$, which will give the assertion.

We will show the equality $K_{\boldsymbol{V}} = T_{(n'_1,m'_1)\ldots(n'_{p_\beta},m'_{p_\beta})}$ by approximation of the knot $K_{\boldsymbol{V}}$ by knots received by "truncation" of the parameterization Φ. More specifically, we will prove that for any $i = 1, \ldots, p$ the image of the mapping

$$\Phi_i(t) := (t^n, a_{p_1} t^{p_1} + \ldots + a_{p_i} t^{p_i}), \quad t \in S(\varepsilon^{1/n}),$$

is a knot of the type $T_{(n_1',m_1')\dots(n_i',m_i')}$. In particular for $i = p_\beta$ the image Φ_{p_β} of the circle $S(\varepsilon^{1/n})$ has the type $T_{(n_1',m_1')\dots(n_{p_\beta}',m_{p_\beta}')}$. Then we will notice that the images by Φ_{p_β} and Φ of the circle $S(\varepsilon^{1/n})$ have the same type in $\partial P(\varepsilon,\varepsilon)$, whence we obtain $T_{(n_1',m_1')\dots(n_{p_\beta}',m_{p_\beta}')} \sim K_{\boldsymbol{V}}$, which gives the assertion.

Let's consider first the case $i = 1$, i.e.

$$\Phi_1(t) = (t^n, a_{p_1} t^{p_1}),\ t \in S(\varepsilon^{1/n}).$$

Decreasing ε we may assume $|a_{p_1} t^{p_1}| < \varepsilon$ for $t \in S(\varepsilon^{1/n})$. The image of Φ_1 is obviously equal to the image of the mapping

$$\Phi_1^{\mathrm{red}}(t) := (t^{n_1'}, a_{p_1} t^{m_1'}),\ t \in S(\varepsilon^{1/n_1'}),$$

and this is the first order torus knot of type (m_1', n_1'). This knot lies in the torus $\{(x,y) : |x| = \varepsilon,\ |y| = |a_{p_1}\varepsilon^{m_1'/n_1'}|\}$. Let's denote this knot by T_1.

Consider now the case $i = 2$, i.e.

$$\Phi_2(t) = (t^n, a_{p_1} t^{p_1} + a_{p_2} t^{p_2}),\ t \in S(\varepsilon^{1/n}).$$

Decreasing ε we may assume that $|a_{p_1} t^{p_1} + a_{p_2} t^{p_2}| < \varepsilon$ for $t \in S(\varepsilon^{1/n})$. Notice the image of mapping Φ_2 lies in the boundary of tubular neighbourhood of T_1 with the radius $\left|a_{p_2}\varepsilon^{p_2/n}\right|$. In fact, for every $t \in S(\varepsilon^{1/n})$ we have $\Phi_2(t) = \Phi_1(t) + (0, a_{p_2} t^{p_2})$ and if ε is sufficiently small, then discs with radius $|a_{p_2}|\,\varepsilon^{p_2/n}$ and centers in points of the set $\pi_1^{-1}(x) \cap T_1$, $|x| = \varepsilon$, are contained in the disc $K(0,\varepsilon)$ and they are pairwise disjoined (the latter follows from the fact that the distance of any two different points of T_1 in $\pi_1^{-1}(x)$ is greater or equal to $|a_{p_1}(1-\rho)t^{p_1}| = |a_{p_1}(1-\rho)|\,\varepsilon^{p_1/n}$, where ρ is a primitive root of unity of degree p_1 and the inequality $p_1 < p_2$). Moreover, the image of Φ_2 is of course equal to the image of the mapping

$$\Phi_2^{\mathrm{red}}(t) := (t^{n_1'n_2'}, a_{p_1} t^{m_1'n_2'} + a_{p_2} t^{m_2'}),\ t \in S(\varepsilon^{1/n_1'n_2'}),$$

and this is a torus knot of the second order of the type $(n_1', m_1')\,(n_2', m_2')$. Let's denote this knot by T_2. By repeating this reasoning (decreasing ε each time, if necessary) we will finally get that the image of the circle $S(\varepsilon^{1/n})$ by Φ_{p_β} is a torus knot in $P(\varepsilon,\varepsilon)$ which has the type $(n_1', m_1')\dots(n_{p_\beta}', m_{p_\beta}')$. Moreover it lies in the boundary of $\partial P(\varepsilon,\varepsilon)$. Let's denote this knot by T_{p_β}.

It remains to compare the knot T_{p_β} with the knot $K_{\boldsymbol{V}}$, i.e. the images of the circle $S(\varepsilon^{1/n})$ by Φ_{p_β} and Φ in $\partial P(\varepsilon,\varepsilon)$. Since $p_\beta = \beta_h$ we have to compare the images of Φ_{β_h} and Φ. We have

$$\Phi_{\beta_h}(t) = (t^n, a_{p_1} t^{p_1} + \ldots + a_{\beta_h} t^{\beta_h}),\quad t \in S(\varepsilon^{1/n}),$$
$$\Phi(t) = (t^n, a_{p_1} t^{p_1} + \ldots + a_{\beta_h} t^{\beta_h} + \ldots),\quad t \in S(\varepsilon^{1/n}).$$

For a fixed x, $|x| = \varepsilon$, and every t such that $t^n = x$ we have

$$pr_1^{-1}(x) \cap T_{\beta_h} = \{(t^n, a_{p_1}(\rho t)^{p_1} + \ldots + a_{\beta_h}(\rho t)^{\beta_h}) : \rho \in U(n)\},$$
$$pr_1^{-1}(x) \cap K_{\boldsymbol{V}} = \{(t^n, a_{p_1}(\rho t)^{p_1} + \ldots + a_{\beta_h}(\rho t)^{\beta_h} + \ldots) : \rho \in U(n)\},$$

where $U(n)$ is the set of roots of unity of degree n. Because

$$\mathrm{GCD}(n, p_1, \ldots, \beta_h) = 1,$$

then for sufficiently small ε each of this set has n elements. We show this for the set $pr_1^{-1}(x) \cap T_{\beta_h}$, because the reasoning for the set $pr_1^{-1}(x) \cap K_{\boldsymbol{V}}$ is analogous. If for some $\rho, \widetilde{\rho} \in U(n)$, $\rho \neq \widetilde{\rho}$, and $t \in S(\varepsilon^{1/n})$ it holds

$$a_{p_1} (\rho t)^{p_1} + \ldots + a_{\beta_h} (\rho t)^{\beta_h} = a_{p_1} (\widetilde{\rho} t)^{p_1} + \ldots + a_{\beta_h} (\widetilde{\rho} t)^{\beta_h},$$

then

$$a_{p_1} t^{p_1} (\rho^{p_1} - \widetilde{\rho}^{p_1}) + \ldots + a_{\beta_h} t^{\beta_h} \left(\rho^{\beta_h} - \widetilde{\rho}^{\beta_h}\right) = 0.$$

Then, if this equality hold for an infinite number of $t \to 0$, then

$$\rho^{p_1} - \widetilde{\rho}^{p_1} = 0, \ldots, \rho^{\beta_h} - \widetilde{\rho}^{\beta_h} = 0.$$

Hence

$$\left(\frac{\rho}{\widetilde{\rho}}\right)^{p_1} = 1, \ldots, \left(\frac{\rho}{\widetilde{\rho}}\right)^{\beta_h} = 1.$$

From properties of the roots of unity we conclude $\mathrm{GCD}(n, p_1, \ldots, \beta_h) > 1$, which is contrary to the assumption.

Denote these points as follows

$$pr_1^{-1}(x) \cap T_{\beta_h} = \{P_\rho : \rho \in U(n)\},$$
$$pr_1^{-1}(x) \cap K_{\boldsymbol{V}} = \{\widetilde{P}_\rho : \rho \in U(n)\}.$$

The distance of each two different points P_ρ and $P_{\rho'}$ for $\rho, \rho' \in U(n)$ and ε small enough satisfies the inequality

$$\|P_\rho - P_{\rho'}\| \geq C\varepsilon^{\beta_h/n} \tag{29}$$

for some constant $C > 0$. In fact, since $\rho \neq \rho'$, the condition $\mathrm{GCD}(n, p_1, \ldots, \beta_h) = 1$ implies the existence of p_i such that $\rho^{p_i} - \widetilde{\rho}^{p_i} \neq 0$. Let p_{i_0} be the least such p_i. The for sufficiently small t we have

$$\|P_\rho - P_{\rho'}\| = |a_{p_{i_0}} t^{p_{i_0}} (\rho^{p_{i_0}} - \widetilde{\rho}^{p_{i_0}}) + \ldots + a_{\beta_h} t^{\beta_h} \left(\rho^{\beta_h} - \widetilde{\rho}^{\beta_h}\right)|$$
$$\geq |a_{p_{i_0}} t^{p_{i_0}} (\rho^{p_{i_0}} - \widetilde{\rho}^{p_{i_0}})| - |a_{p_{i_0+1}} t^{p_{i_0+1}} (\rho^{p_{i_0+1}} - \widetilde{\rho}^{p_{i_0+1}}) + \ldots + a_{\beta_h} t^{\beta_h} \left(\rho^{\beta_h} - \widetilde{\rho}^{\beta_h}\right)|$$
$$\geq \widetilde{C}_1 |t^{p_{i_0}}| - \widetilde{C}_2 |t^{p_{i_0+1}}|$$

for some positive constants $\widetilde{C}_1, \widetilde{C}_2$. Since $p_{i_0} < p_{i_0+1}$, for sufficiently small t

$$\widetilde{C}_1 |t^{p_{i_0}}| - \widetilde{C}_2 |t^{p_{i_0+1}}| \geq \frac{\widetilde{C}_1}{2} |t^{p_{i_0}}|.$$

In turn $p_{i_0} \leq \beta_h$, whence

$$\frac{\widetilde{C}_1}{2} |t^{p_{i_0}}| \geq \frac{\widetilde{C}_1}{2} |t^{\beta_h}| = \frac{\widetilde{C}_1}{2} \varepsilon^{\beta_h/n},$$

which gives (29). On the other hand, the distance of points P_ρ and $\widetilde{P}_\rho$ for the same $\rho \in U(n)$ and ε small enough satisfies the inequality

$$\|P_\rho - \widetilde{P}_\rho\| = |a_{\beta_h+1} t^{\beta_h+1} \rho^{\beta_h+1} + \ldots| \leq C' \left|t^{\beta_h+1}\right| = C' \varepsilon^{(\beta_h+1)/n}.$$

Since $\beta_h < \beta_h + 1$, decreasing ε we may assume that points $\widetilde{P}_\rho$ belong to discs with centers at P_ρ and these dics are contained in $K(0,\varepsilon)$ and are pairwise disjoined (see Fig. 31).

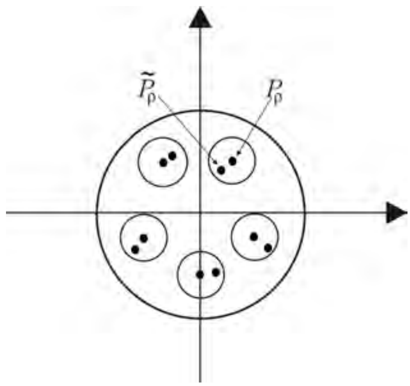

Fig. 31. Points P_ρ and $\widetilde{P}_\rho$.

Of course, points P_ρ and $\widetilde{P}_\rho$ depend continuously on the point x. It is not difficult to prove for each x the existence of homeomorphism h_x of the disc $K(0,\varepsilon)$ on itself, continuously depending on x, transforming points P_ρ into points $\widetilde{P}_\rho$ and being the identity on the boundary.

By extending these homeomorphisms by identity to the rest of the boundary $\partial P(\varepsilon,\varepsilon)$ we get that the knot T_{β_h} is equivalent to the knot $K_{\boldsymbol{V}}$.

Hence $K_{\boldsymbol{V}}$ is a torus knot of the type $((m_1,n_1),\ldots,(m_h,n_h))$. □

6. Topological equivalence of irreducible curves

We can now prove the basic characterization of topological types of irreducible curves. First we will give the necessary definitions. Two curves $\boldsymbol{V} = V(f)$ and $\widetilde{\boldsymbol{V}} = V(\widetilde{f})$ are *topologically equivalent*, if there exist neighbourhoods $U, \widetilde{U}$ of the origin in $\mathbb{C}^2$ such that pairs $(U, V_U(f))$ i $(\widetilde{U}, V_{\widetilde{U}}(\widetilde{f}))$ are homeomorphic, i.e.

$$(U, V_U(f)) \overset{\text{top}}{\approx} (\widetilde{U}, V_{\widetilde{U}}(\widetilde{f})).$$

It means, there exists a homeomorphism $H : U \to \widetilde{U}$ leaving the point 0 fixed, which sends $V_U(f)$ on $V_{\widetilde{U}}(\widetilde{f})$. Of course, this relation is a relation of equivalence in the set of curves. Abstract classes of this relation are called *topological types of curves.*

Theorem 6.1. *Two irreducible curves have the same topological type if and only if they have the same characteristics.*

Proof. 1. $\Leftarrow$. Let $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$ be two irreducible curves with the same characteristic $(\beta_0, \beta_1, \ldots, \beta_h)$. Using a linear change of variables in $\mathbb{C}^2$, we may assume that $\boldsymbol{V} = \widehat{V_U(f)}$ and $\widetilde{\boldsymbol{V}} = \widehat{V_{\widetilde{U}}(\widetilde{f})}$, $f, \widetilde{f} \in \mathbb{C}\{x\}[y]$ are distinguished polynomials, $\operatorname{ord} f =$

$\operatorname{ord} f(0,y) = \operatorname{ord} \widetilde{f} = \operatorname{ord} \widetilde{f}(0,y)$. Denote $V := V_U(f)$, $\widetilde{V} := V_{\widetilde{U}}(\widetilde{f})$. By assumption on common characteristics, it follows that f and $\widetilde{f}$ have the same sequence of characteristic pairs

$$((m_1, n_1), \ldots, (m_h, n_h)).$$

By Theorem 5.5 for their knots $K_{\boldsymbol{V}}$ and $K_{\widetilde{\boldsymbol{V}}}$ we have

$$K_{\boldsymbol{V}} \sim T_{(m_1,n_1),\ldots,(m_h,n_h)},$$
$$K_{\widetilde{\boldsymbol{V}}} \sim T_{(m_1,n_1),\ldots,(m_h,n_h)}.$$

Hence $K_{\boldsymbol{V}} \sim K_{\widetilde{\boldsymbol{V}}}$ as knots in $\partial P(\varepsilon,\varepsilon)$ for sufficiently small ε. Then the pairs $(\partial P(\varepsilon,\varepsilon), K_{\boldsymbol{V}})$ and $(\partial P(\varepsilon,\varepsilon), K_{\widetilde{\boldsymbol{V}}})$ are homeomorphic. Hence, by extending this homeomorphism (along the segments connecting the points of $\partial P(\varepsilon,\varepsilon)$ with 0) we will get a homeomorphism of the pair $(P(\varepsilon,\varepsilon), \operatorname{cone}(K_{\boldsymbol{V}}))$ with the pair $(P(\varepsilon,\varepsilon), \operatorname{cone}(K_{\widetilde{\boldsymbol{V}}}))$. By the theorem on the conic structure these pairs are homeomorphic to $(P(\varepsilon,\varepsilon), V \cap P(\varepsilon,\varepsilon))$ and $(P(\varepsilon,\varepsilon), \widetilde{V} \cap P(\varepsilon,\varepsilon))$, respectively. Then

$$(P(\varepsilon,\varepsilon), V \cap P(\varepsilon,\varepsilon)) \overset{\text{top}}{\approx} (P(\varepsilon,\varepsilon), \widetilde{V} \cap P(\varepsilon,\varepsilon)).$$

Hence, after restriction this homomorphism to the interior of $P(\varepsilon,\varepsilon)$ we get

$$(\operatorname{Int} P(\varepsilon,\varepsilon), V \cap \operatorname{Int} P(\varepsilon,\varepsilon)) \overset{\text{top}}{\approx} (\operatorname{Int} P(\varepsilon,\varepsilon), \widetilde{V} \cap \operatorname{Int} P(\varepsilon,\varepsilon)),$$

i.e. $\boldsymbol{V}$ i $\widetilde{\boldsymbol{V}}$ are topologically equivalent.

2. $\Rightarrow$. Suppose that irreducibles curves $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$ are topologically equivalent. Using a linear change of variables in $\mathbb{C}^2$, we may assume that $\boldsymbol{V} = \widehat{V_U(f)}$, $\widetilde{\boldsymbol{V}} = \widehat{V_{\widetilde{U}}(\widetilde{f})}$ where $f, \widetilde{f} \in \mathbb{C}\{x\}[y]$ are distinguished polynomials and

$$\Phi(t) = (t^n, \varphi(t)), \ \operatorname{ord} \varphi > n, \ t \in K,$$
$$\widetilde{\Phi}(t) = (t^m, \widetilde{\varphi}(t)), \ \operatorname{ord} \widetilde{\varphi} > m, \ t \in \widetilde{K},$$

are their parametrizations in neighbourhoods of U and $\widetilde{U}$, respectively. Let

$$((m_1, n_1), \ldots, (m_g, n_g)),$$
$$((\widetilde{m}_1, \widetilde{n}_1), \ldots, (\widetilde{m}_h, \widetilde{n}_h))$$

be characteristic pairs of $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$ "read off" from the parameterizations Φ and $\widetilde{\Phi}$. By Theorem 5.5

$$K_{\boldsymbol{V}} \sim T_{(m_1,n_1),\ldots,(m_g,n_g)},$$
$$K_{\widetilde{\boldsymbol{V}}} \sim T_{(\widetilde{m}_1,\widetilde{n}_1),\ldots,(\widetilde{m}_h,\widetilde{n}_h)}.$$

We will now show that the knot groups $\pi(K_{\boldsymbol{V}})$ and $\pi(K_{\widetilde{\boldsymbol{V}}})$ are isomorphic. By assumption $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$ are topologically equivalent. Then decreasing U and $\widetilde{U}$, if necessary, there exists a homeomorphism $F : U \to \widetilde{U}$ which maps $V_U(f)$ on $V_{\widetilde{U}}(\widetilde{f})$. Take $\varepsilon > 0$ such that $P(\varepsilon,\varepsilon) \subset U$. Assume $\varepsilon < \widetilde{\varepsilon}$, where $\widetilde{\varepsilon}$ is the radius of a policylinder, appearing in the theorem on the conic structure, "good" for both $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$. Take r_1, r_2, r_3, r_4 such that

$$0 < r_2 < r_1 < \varepsilon, \quad 0 < r_4 < r_3 < \varepsilon$$

and

$$F(P(r_4, r_4)) \subset P(r_2, r_2) \subset F(P(r_3, r_3)) \subset P(r_1, r_1) \subset \widetilde{U}.$$

Hence

$$F(P(r_4, r_4)) \setminus \widetilde{V} \subset P(r_2, r_2) \setminus \widetilde{V} \subset F(P(r_3, r_3)) \setminus \widetilde{V} \subset P(r_1, r_1) \setminus \widetilde{V}.$$

These inclusions induce a sequence of homomorphisms of the first homotopy groups of these sets.

$$\pi_1(F(P(r_4, r_4)) \setminus \widetilde{V}) \xrightarrow{f_1} \pi_1(P(r_2, r_2) \setminus \widetilde{V}) \xrightarrow{f_2} \pi_1(F(P(r_3, r_3)) \setminus \widetilde{V}) \xrightarrow{f_3} \pi_1(P(r_1, r_1) \setminus \widetilde{V}).$$

Of course, the superposition $f_3 \circ f_2$, induced by the embedding $P(r_2, r_2) \setminus \widetilde{V} \hookrightarrow P(r_1, r_1) \setminus \widetilde{V}$, is an isomorphism by Lemma 5.4. Since F is a homeomorphism of U on $\widetilde{U}$ mapping V on $\widetilde{V}$, then

$$\pi_1(F(P(r_4, r_4)) \setminus \widetilde{V}) \cong \pi_1(P(r_4, r_4) \setminus V),$$
$$\pi_1(F(P(r_3, r_3)) \setminus \widetilde{V}) \cong \pi_1(P(r_3, r_3) \setminus V).$$

Hence we get the sequence of homomorphisms

$$\pi_1(P(r_4, r_4) \setminus V) \xrightarrow{\widetilde{f}_1} \pi_1(P(r_2, r_2) \setminus \widetilde{V}) \xrightarrow{\widetilde{f}_2} \pi_1(P(r_3, r_3) \setminus V) \xrightarrow{\widetilde{f}_3} \pi_1(P(r_1, r_1) \setminus \widetilde{V})$$

in which the superposition (induced by emeddings) $\widetilde{f}_3 \circ \widetilde{f}_2$ and $\widetilde{f}_2 \circ \widetilde{f}_1$ are isomorphisms. From here we can easily check that the homomorphism $\widetilde{f}_2$ is also an isomorphism. By Lemma 5.4

$$\pi_1(P(r_3, r_3) \setminus V) \cong \pi(\boldsymbol{V}),$$
$$\pi_1(P(r_2, r_2) \setminus \widetilde{V}) \cong \pi(\widetilde{\boldsymbol{V}}),$$

whence $\pi(\boldsymbol{V}) \cong \pi(\widetilde{\boldsymbol{V}})$. Hence their Aleksander polynomials $A(K_{\boldsymbol{V}})$ and $A(K_{\widetilde{\boldsymbol{V}}})$ are equal. But by the Theorem 5.5 $K_{\boldsymbol{V}} = T_{(m_1,n_1),\dots,(m_g,n_g)}$ and $K_{\widetilde{V}} = T_{(\widetilde{m}_1,\widetilde{n}_1),\dots,(\widetilde{m}_h,\widetilde{n}_h)}$, which implies

$$A(T_{(m_1,n_1),\dots,(m_g,n_g)}) = A(T_{(\widetilde{m}_1,\widetilde{n}_1),\dots,(\widetilde{m}_h,\widetilde{n}_h)}).$$

Because for characteristic pairs $((m_1, n_1), \dots, (m_g, n_g))$, $((\widetilde{m}_1, \widetilde{n}_1), \dots, (\widetilde{m}_h, \widetilde{n}_h))$ of curves $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$, inequalities (28) are satisfied, then by Theorem 4.10 we get $g = h$ and $(m_1, n_1), \dots, (m_g, n_g) = (\widetilde{m}_1, \widetilde{n}_1), \dots, (\widetilde{m}_h, \widetilde{n}_h)$. Then the characterics of $\boldsymbol{V}$ and $\boldsymbol{V}'$ are identical. □

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