

**EXTREMAL PROPERTIES OF LINE ARRANGEMENTS
IN THE COMPLEX PROJECTIVE PLANE**

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ABSTRACT. In the present note we study some extreme properties of point-line configurations in the complex projective plane from a viewpoint of algebraic geometry. Using Hirzebruch-type inequalities we provide some new results on r -rich lines, symplial arrangements of lines, and the so-called free line arrangements.

1. INTRODUCTION

In the present note we study some classical questions in the theory of point-line configurations in the complex projective plane using tools from algebraic geometry. This path is rather classical, and it dates back to the famous Hirzebruch's inequality [9] which can be viewed as a main tool in the subject. Let us recall that if $\mathcal{L} = \{\ell_1, \dots, \ell_d\} \subset \mathbb{P}_{\mathbb{C}}^2$ is an arrangement of $d \geq 6$ lines such that there is no point where all the lines meet and there is no point where $d - 1$ lines meet simultaneously, then

$$t_2 + t_3 \geq d + \sum_{r \geq 5} (r - 4)t_r,$$

where t_r denotes the number of r -fold points, i.e., points where exactly r -lines from the arrangement meet. Hirzebruch's inequality can be found in many papers devoted to combinatorics, and one of the most important problems is to prove Hirzebruch's inequality using only combinatorial methods [3, p. 315; Problem 7]. This problem is motivated mostly due to Hirzebruch's approach, namely he used the theory of Hirzebruch-Kummer covers of the complex projective plane branched along line arrangements. Moreover, Hirzebruch's inequality is (only) a very strong by-product of the whole story since the main aim was to construct new examples of

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complex compact ball-quotient surfaces, i.e., minimal complex compact algebraic surfaces of general type such that the universal cover is the complex unit ball. The very first observation which comes from Hirzebruch's inequality is that every complex line arrangement has always double or triple intersection points. The real counterpart of Hirzebruch's inequality is the classical Melchior's result [10] which tells us that for a real line arrangement \mathcal{A} (defined over the real numbers) which is not a pencil of lines one always has

$$t_2 \geq 3 + \sum_{r \geq 4} (r-3)t_r,$$

and the equality holds if and only if \mathcal{A} is a simplicial line arrangement. Melchior's inequality provides an alternative proof of the dual orchard problem – every real line arrangement which is not a pencil has at least one double intersection point.

It is worth emphasizing that Hirzebruch's inequality is proved using, in the final step, the Bogomolov-Miyaoka-Yau inequality [11] which tells us that for a smooth complex projective surface with Kodaira dimension ≥ 0 one always has

$$K_X^2 \leq 3e(X),$$

where K_X is the canonical divisor and $e(X)$ denotes the topological Euler characteristic. It was very desirable to have meaningful generalizations of the Bogomolov-Miyaoka-Yau inequality to the case of pairs (X, D) , where X is a normal complex projective surface and D is a boundary divisor, and now we have several choices – the most powerful is the orbifold Euler characteristic. It turns out that using it we can show the following result which is due to Bojanowski [2].

Theorem 1.1 (Bojanowski). *Let $\mathcal{L} = \{\ell_1, \dots, \ell_d\}$ be a finite set of lines in the complex projective plane. Assume that $t_r = 0$ for $r \geq \frac{2d}{3}$, then*

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{r \geq 5} \left(\frac{r^2}{4} - r \right) t_r.$$

The main aim of the present note is to apply Bojanowski's result in the context of certain questions, extremal in their nature, for point-line configurations. The note is inspired mostly by F. de Zeeuw's paper [6], and we are going to follow his path in the context of r -rich lines.

2. ON r -RICH LINES

Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite set of mutually distinct points in the complex projective plane (some of our results should be also formulated over the reals where obtained bounds are usually much better). Then we denote by ℓ_r the number of r -rich lines, i.e., those lines in the plane containing exactly r -points from the configuration \mathcal{P} . We are going to use the dual version of Bojanowski's inequality.

Theorem 2.1 (Bojanowski). *Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite set of mutually distinct points in the complex projective plane. Assume that there is no subset of $\frac{2n}{3}$ points which are collinear, then*

$$\ell_2 + \frac{3}{4}\ell_3 \geq n + \sum_{r \geq 5} \left(\frac{r^2}{4} - r \right) \ell_r.$$

Using Bojanowski’s inequality, we can derive very strong bounds on r -rich lines, namely

- a) $f_1 := \sum_{r \geq 2} r \ell_r \geq \frac{n(n+3)}{3}$;
- b) $f_2 := \sum_{r \geq 2} r^2 \ell_r \geq \frac{4n^2}{3}$.

The first result is (strong) Beck’s theorem on point configurations in the complex projective plane which was proved by de Zeeuw [6].

Theorem 2.2. *Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite set of mutually distinct points in the complex projective plane. Assume that there is no subset of $\frac{2n}{3}$ points which is collinear, then*

$$\sum_{r \geq 2} \ell_r \geq \frac{n^2 + 6n}{12}.$$

Now we are ready to give our proof of Beck’s theorem.

Proof. Using (dual) Hirzebruch’s inequality we see that

$$4f_0 - f_1 \geq n + \ell_2,$$

where $f_0 := \sum_{r \geq 2} \ell_r$. Then

$$4f_0 - f_1 + f_1 \geq n + \ell_2 + f_1 \geq n + \frac{n^2 + 3n}{3} \geq \frac{n^2 + 6n}{3},$$

so we arrive at

$$f_0 \geq \frac{n^2 + 6n}{12}.$$

□

Looking at Hirzebruch’s inequality, we see that for point configurations (except the case when all the points are collinear or all but one point are collinear) one has

$$\ell_2 + \ell_3 \geq n.$$

Taking into account that Bojanowski’s inequality is more accurate, we can formulate the following conjecture as it was suggested by de Zeeuw [7, Conjecture 4.5].

Conjecture 2.3. *For point configurations in \mathbb{C}^2 which do not have large pencils as subconfigurations (i.e., not too many points are collinear) one has*

$$\ell_2 + \ell_3 \geq c \cdot n^2$$

for a positive constant c .

If we restrict our attention to a real point configuration, one can show that if $\mathcal{P} \subset \mathbb{P}_{\mathbb{R}}^2$ is a finite set of n points such that at most $\alpha \cdot n$ are collinear, where $\alpha = \frac{6+\sqrt{3}}{9}$, then

$$(\Delta) \quad \ell_2 + \ell_3 \geq \frac{1}{18}n^2.$$

This bound follows from an improvement on Beck's theorem on two extremes proved by de Zeeuw [6, Corollary 2.3].

Theorem 2.4 (Beck's theorem on two extremes). *Let \mathcal{P} be a finite set of n points in $\mathbb{P}_{\mathbb{R}}^2$, then one of the following is true:*

- *There is a line which contains more than $\alpha \cdot n$ points of \mathcal{P} , where $\alpha = \frac{6+\sqrt{3}}{9}$.*
- *There are at least $\frac{n^2}{9}$ lines spanned by \mathcal{P} .*

Now we are ready to show (Δ) .

Proof. If \mathcal{P} is a finite set of points, then we have

$$\ell_2 \geq 3 + \sum_{r \geq 4} (r-3)t_r.$$

Adding $\ell_2 + 2\ell_3$ on both sides we obtain

$$2\ell_2 + 2\ell_3 \geq 3 + \ell_2 + 2\ell_3 + \sum_{r \geq 4} (r-3)t_r \geq 3 + \sum_{r \geq 2} \ell_r.$$

If at most $\alpha \cdot n$ points from \mathcal{P} are collinear with $\alpha = \frac{6+\sqrt{3}}{9}$, then

$$2\ell_2 + 2\ell_3 \geq 3 + \sum_{r \geq 2} \ell_r \geq \frac{n^2}{9},$$

which completes the proof. \square

Over the complex numbers, we can only show the following bound, which takes into account also quadruple points.

Theorem 2.5. *Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a point configuration in the complex projective plane such that no subset of $\frac{2n}{3}$ is collinear. Then*

$$\ell_2 + \ell_3 + \ell_4 \geq \frac{n(n+15)}{18}.$$

Proof. Using Bojanowski's inequality we have

$$\ell_3 + \frac{3}{4}\ell_3 \geq n + \sum_{r \geq 5} \frac{r^2 - 4r}{4} \ell_r.$$

Now we need to observe that for $r \geq 5$ one has

$$\frac{r^2 - 4r}{4} \geq \frac{1}{8} \cdot \frac{r^2 - r}{2},$$

and using the combinatorial count

$$\binom{n}{2} = l_2 + 3l_3 + 6l_4 + \sum_{r \geq 5} \binom{r}{2} l_r$$

we obtain that

$$l_2 + \frac{3}{4}l_3 \geq n + \frac{1}{8} \left(\binom{n}{2} - l_2 - 3l_3 - 6l_4 \right).$$

Simple manipulations give

$$\frac{9}{8}(l_2 + l_3 + l_4) \geq \frac{9}{8}l_2 + \frac{9}{8}l_3 + \frac{6}{8}l_4 \geq \frac{n(n+15)}{16},$$

so finally we obtain

$$l_2 + l_3 + l_4 \geq \frac{n(n+15)}{18}.$$

□

3. SIMPLICIAL LINE ARRANGEMENTS

Definition 3.1. Let $\mathcal{A} = \{H_1, \dots, H_d\}$ be a central arrangement of $d \geq 3$ hyperplanes in \mathbb{R}^3 (so it provides an arrangement of lines in the real projective plane). We say that \mathcal{A} is simplicial if each connected components of the complement of \mathcal{A} in \mathbb{R}^3 is a simplicial cone.

It is well-known, by Melchior’s result, that \mathcal{A} is a simplicial line arrangement if and only if the following equality holds

$$t_2 = 3 + \sum_{r \geq 4} (r - 3)t_r.$$

We will also need the following folklore result on the multiplicity of an *irreducible* simplicial line arrangement in the real projective plane (i.e., the maximal multiplicity of singular points).

Definition 3.2. Let \mathcal{A}_1 and \mathcal{A}_2 be central arrangements in \mathbb{K}^ℓ and \mathbb{K}^m , where \mathbb{K} is any field, with defining polynomials $Q_1(x_1, \dots, x_\ell)$ and $Q_2(x_1, \dots, x_m)$, respectively. The product arrangement $\mathcal{A}_1 \times \mathcal{A}_2$ is the arrangement in $\mathbb{K}^{\ell+m} = \mathbb{K}^\ell \times \mathbb{K}^m$ with defining polynomial

$$Q(x_1, \dots, x_{\ell+m}) = Q_1(x_1, \dots, x_\ell) \cdot Q_2(x_{\ell+1}, \dots, x_{\ell+m}).$$

We say that a central arrangement \mathcal{A} is *irreducible* if \mathcal{A} cannot be expressed as a product arrangement.

Theorem 3.3 (Folklore). *Let $\mathcal{A} \subset \mathbb{P}_{\mathbb{R}}^2$ be an irreducible simplicial line arrangement, then the multiplicity of \mathcal{A} is $\leq \frac{d}{2}$.*

An interested reader might want to consult [8, Proposition 2.1] for a modern proof of the above result.

We would like to add the following observation to the above list of constraints.

Proposition 3.4. *Let \mathcal{A} be an irreducible simplicial arrangement in the real projective plane, then*

$$t_3 + t_4 + t_5 \geq d - 3.$$

Proof. By Melchior’s result,

$$t_2 = 3 + \sum_{r \geq 3} (r - 3)t_r$$

and we can plug it into Bojanowski’s inequality obtaining

$$3 + \sum_{r \geq 4} (r - 3)t_r + \frac{3}{4}t_3 \geq d + \sum_{r \geq 4} \left(\frac{r^2 - 4r}{4} \right) t_r.$$

It leads to

$$3t_3 \geq 4(d - 3) + \sum_{r \geq 4} (r^2 - 8r + 12)t_r = 4(d - 3) - 4t_4 - 3t_5 + \sum_{r \geq 6} (r^2 - 8r + 12)t_r.$$

Then we have

$$3t_3 + 4t_4 + 3t_5 \geq 4(d - 3),$$

which completes the proof. □

4. COMBINATORICS AND THE FREENESS OF LINE ARRANGEMENTS

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential and central hyperplane arrangement in \mathbb{C}^3 , it means that $H_i = V(\ell_i)$ for homogeneous linear form ℓ_i and $\bigcap_{i=1}^n H_i = 0 \in \mathbb{C}^3$ – the last condition tells us that \mathcal{A} also defines an arrangement of lines in $\mathbb{P}_{\mathbb{C}}^2$. The main combinatorial object which can be associated with \mathcal{A} is the intersection lattice $L_{\mathcal{A}}$ – it consists of the intersections of the elements of \mathcal{A} , ordered by reverse inclusion. In this setting, \mathbb{C}^3 is the lattice element $\tilde{0}$ and the rank one elements of $L_{\mathcal{A}}$ are the planes. In this section we denote by S the polynomial ring $\mathbb{C}[x, y, z]$,

Definition 4.1. The Möbius function $\mu : L_{\mathcal{A}} \rightarrow \mathbb{Z}$ is defined as

$$\begin{aligned} \mu(\tilde{0}) &= 1, \\ \mu(t) &= - \sum_{s < t} \mu(s), \text{ if } \tilde{0} < t. \end{aligned}$$

Definition 4.2. The Poincaré and the characteristic polynomials of \mathcal{A} are defined as

$$\pi(\mathcal{A}, t) = \sum_{x \in L_{\mathcal{A}}} \mu(x) \cdot (-t)^{\text{rank}(x)}, \text{ and } \chi(\mathcal{A}, t) = t^{\text{rank}(\mathcal{A})} \pi\left(\mathcal{A}, \frac{-1}{t}\right).$$

Definition 4.3. The module of \mathcal{A} -derivations is the submodule of $\text{Der}_{\mathbb{C}}(S)$ consisting of vector fields tangent to \mathcal{A} , namely

$$D(\mathcal{A}) = \{\theta \in \text{Der}_{\mathbb{C}}(S) \mid \theta(\ell_i) \in \langle \ell_i \rangle \text{ for all } \ell_i \text{ with } \text{Zeros}(\ell_i) \in \mathcal{A}\}.$$

Definition 4.4. An arrangement is free when $D(\mathcal{A})$ is a free S -module.

Theorem 4.5 (Terao’s factorization). *If $D(\mathcal{A})$ is free, then*

$$\pi(\mathcal{A}, t) = (1 + t)(1 + a_1t)(1 + a_2t).$$

Now we would like to present the main result for this section.

Theorem 4.6. *Let $\mathcal{L} \subset \mathbb{P}_{\mathbb{C}}^2$ be an arrangement of d lines with $t_r = 0$ for $r > \frac{2d}{3}$. Assume that \mathcal{L} is free, then*

$$\sum_{r \geq 2} (r - 4)^2 t_r \geq 12.$$

Proof. Let us recall that for an arrangement of lines $\mathcal{L} \subset \mathbb{P}_{\mathbb{C}}^2$ the Poincaré polynomial has the following form

$$\pi(\mathcal{L}, t) = 1 + dt + \left(\sum_{r \geq 2} (r - 1)t_r \right) t^2 + \left(\sum_{r \geq 2} (r - 1)t_r + 1 - d \right) t^3,$$

which follows from simple calculations using the Möbius function – for each line $\ell_i \in \mathcal{L}$ we have that $\mu(\ell) = -1$, and for each point $P \in L(\mathcal{L})$ of multiplicity r we have $\mu(P) = r - 1$. Since \mathcal{L} is central, then $(1 + t)$ divides $\pi(\mathcal{L}, t)$, which follows from the fact that the Euler derivation is always an element of $D(\mathcal{L})$ [5, Section 8.1], and it leads to the following presentation

$$\pi(\mathcal{L}, t) = (1 + t) \left(1 + (d - 1)t + \left(\sum_{r \geq 2} (r - 1)t_r + 1 - d \right) t^2 \right).$$

Now the freeness of \mathcal{L} implies that

$$(d - 1)^2 - 4 \cdot \left(\sum_{r \geq 2} (r - 1)t_r - d + 1 \right) = d^2 + 2d - 3 - 4 \sum_{r \geq 2} (r - 1)t_r \geq 0.$$

By the standard combinatorial count

$$d(d - 1) = \sum_{r \geq 2} r(r - 1)t_r$$

one obtains

$$(\star) \quad 3d + \sum_{r \geq 2} \left(r^2 - 5r + 4 \right) t_r \geq 3.$$

Using Bojanowski’s inequality, we get

$$- \sum_{r \geq 2} \left(\frac{r^2}{4} - r \right) t_r \geq d$$

and this leads us to

$$-3 \sum_{r \geq 2} \left(\frac{r^2}{4} - r \right) t_r + \sum_{r \geq 2} \left(r^2 - 5r + 4 \right) t_r \geq 3,$$

and we finally obtain

$$\sum_{r \geq 2} (r-4)^2 t_r \geq 12.$$

□

Our result gives us some insights in the context of free line arrangements with small number of lines. Assume that we want to find a free arrangement of $d \geq 6$ lines having only triple points. Our inequality implies that $t_3 \geq 12$, and we know that the dual Hesse arrangement of $d = 9$ lines with $t_3 = 12$ is free, so our lower bound is sharp.

The next result of the section gives a lower bound on the number of double and triple points for free line arrangements.

Proposition 4.7. *Let \mathcal{L} be a free arrangement of d lines such that $t_r = 0$ for $r \geq \frac{2d}{3}$. Then*

$$2t_2 + t_3 \geq d + 3.$$

Proof. Since \mathcal{L} is free, we can use condition (\star) , namely

$$3d - 3 + \sum_{r \geq 2} (r^2 - 5r + 4)t_r \geq 0$$

since the Poincaré polynomial splits over the integers. This leads to

$$2t_2 + 2t_3 \leq 3d - 3 + \sum_{r \geq 5} (r^2 - 5r + 4)t_r \leq 3d - 3 + \sum_{r \geq 5} (r^2 - 4r)t_r.$$

Using Bojanowski's inequality

$$4t_2 + 3t_3 - 4d \geq \sum_{r \geq 5} (r^2 - 4r)t_r$$

we obtain

$$2t_2 + 2t_3 \leq 3d - 3 - 4d + 4t_2 + 3t_3,$$

so finally we get

$$d + 3 \leq 2t_2 + t_3,$$

which completes the proof. □

Observe that the above inequality is sharp for several free arrangements of lines, the simplest one is a star-configuration of $d = 3$ lines with 3 double points.

5. (n_k) -CONFIGURATIONS IN THE COMPLEX PROJECTIVE PLANE

Definition 5.1. Let $\mathcal{L} \subset \mathbb{P}_{\mathbb{C}}^2$ be an arrangement of $n \geq 4$ lines, then \mathcal{L} is called (n_k) -configuration if it consists of exactly n points of multiplicity k and we have exactly n lines in the arrangement with the property that on each line we have exactly k points of multiplicity k .

Let us observe here that usually one defines (n_k) -configurations as objects in the real projective plane, and we distinguish geometrical and topological configurations, i.e., geometrical are those which can be realized as straight lines, topological are those that can be realized with use of pseudolines. Let us recall here that a pseudoline is a simple closed curve in $\mathbb{P}_{\mathbb{R}}^2$ such that its removal does not cut $\mathbb{P}_{\mathbb{R}}^2$ in two connected components. The main open problem in this subject is to determine all those (n_k) -configurations which are geometrically realizable. Since the case of (n_3) -configurations is completely characterized, and for (n_4) -configurations the only open case is when $n = 23$ due to an interesting results by Cuntz [4], so we assume from now on that $k \geq 5$. We will follow the last section from [1].

If we assume that \mathcal{PL} is an (n_k) -configuration topologically realizable (i.e., is a pseudoline configuration) in the real projective plane, then we have the following Shmurnikov’s inequality [14]:

$$t_2 + \frac{3}{2}t_3 \geq 8 + \sum_{r \geq 4} (2r - 7.5)t_r,$$

provided that $t_n = t_{n-1} = t_{n-2} = t_{n-3} = 0$. Using a local deformation argument for \mathcal{PL} we can assume that our configuration has only k -fold and double points, so we have the following quantities:

$$t_k = n, \quad t_2 = \frac{n(n-1)}{2} - n \cdot \frac{k(k-1)}{2}.$$

Plugging this into Shmurnikov’s inequality we obtain that

$$\frac{n(n-1)}{2} - n \cdot \frac{k(k-1)}{2} - (2k - 7.5)n - 8 \geq 0,$$

and this is a necessary condition for the existence of topological (n_k) -configurations. If we restrict our attention to $k = 6$, then we can easily see that there are no (n_6) -configurations if $n \leq 40$.

Assume now that \mathcal{L} is a complex geometric (n_k) -configuration with the property that it has only double and k -fold points. Using Bojanowski’s inequality we see that the following condition is necessary:

$$n^2 - n \cdot \left(\frac{3k^2 - 6k + 6}{2} \right) \geq 0,$$

so there are no such arrangements if we have

$$n \leq \frac{3k^2 - 6k - 4}{2}.$$

If we restrict our attention to $k = 6$, then the first non-trivial case is $n = 39$, and this is an extremely important open problem. If such a configuration exists, then we will be able to construct a new example of complex compact ball-quotient surface via Hirzebruch’s construction, i.e., a minimal desingularization of the abelian cover of the complex projective plane branched along complex (39_6) -configuration. It is worth emphasizing here that ball-quotient surfaces constructed with use of abelian

covers are rather rare, and it would be very interesting to know whether we can construct a new example of such surfaces with use of line arrangements.

It seems to be quite difficult to decide whether the above (39_6) -configuration can potentially exist, and it is extreme from a viewpoint of the Bojanowski's inequality (it provides the equality). On the other hand, we can formulate the following problem.

Problem 5.2. Is it possible to construct complex (39_6) -configuration?

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