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A FEW INTRODUCTORY REMARKS ON LINE ARRANGEMENTS

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Abstract. Points and lines can be regarded as the simplest geometrical objects. Incidence relations between them have been studied since ancient times. Strangely enough our knowledge of this area of mathematics is still far from being complete. In fact a number of interesting and apparently difficult conjectures has been raised just recently. Additionally a number of interesting connections to other branches of mathematics have been established. This is an attempt to record some of these recent developments.

1. Introduction

In this note we consider arrangements of lines in the projective plane $\mathbb{P}^2$ coming from a standard construction over a field $\mathbb{K}$. That means, the points in the plane represent one dimensional linear subspaces of a 3-dimesional vector space over $\mathbb{K}$. For the most of the material presented here it is irrelevant what properties the field $\mathbb{K}$ enjoys. We will clearly mark the spots where this becomes important.

Definition 1 (Arrangement)**.** An arrangement of lines is a finite set of at least two mutually distinct lines in the projective plane.

The points where lines from a given arrangement intersect are of special interest.

Definition 2 (Singular points of an arrangement)**.** We say that a point $P \in \mathbb{P}^2$ is a *singular point* of an arrangement $\mathcal{L}$, if there are at least 2 lines in $\mathcal{L}$, which pass through P.
If P is a singular point of an arrangement $\mathcal{L}$, then its *multiplicity* is the number of lines in $\mathcal{L}$ passing through P.

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Note that the notions of a singular point and multiplicity agree with corresponding notions in algebraic geometry, if we look at an arrangement $\mathcal{L}$ as the divisor $\sum_{L\in\mathcal{L}} L$.

Given an arrangement $\mathcal{L}$, for an integer $r \geq 2$, we denote by t_r the number of points of multiplicity r. Since in the projective plane any pair of lines has an intersection point, we have the following fundamental combinatorial equality

$$\binom{d}{2} = \sum_{r\geq 2} t_r \binom{r}{2}, \tag{1}$$

where d is the number of lines in $\mathcal{L}$.

The numbers $t_2, \ldots, t_d$ are basic numerical invariants of an arrangement. We consider them as coordinates of a vector $T = (t_2, t_3, \ldots, t_d)$, which we call the *$T$-vector* of an arrangement. The following natural question has motivated a lot of research, nevertheless it remains widely open.

Problem 3 (Geometrical realisability of T-vectors)**.** Decide which $(d-1)$-tuples of integers $a_2, a_3, \ldots a_d$ arise as T-vectors of line arrangements.

The equality (1) is just one of many constrains for a $(d-1)$-tuple to be a T-vector. We refer to [15] for a number of additional constrains of similar nature.

Of course in some cases, there is an easy answer to Problem 3.

Example 4 (Star configuration)**.** A $d-1$-tuple $(\binom{d}{2}, \underbrace{0,\ldots,0}_{d-2})$ is a T-vector of an arrangement of d *general* lines in $\mathbb{P}^2$. This means that there are just double points (points of multiplicity 2) in the arrangement, i.e., each pair of lines intersects in point but there are no additional incidences. Such general arrangements are also called *star configurations*, see [20] for an extensive account of this concept.

On the other extreme we the following two arrangements.

Example 5 (Pencil and near-pencil)**.** For any $d \geq 2$ the array

$$(\underbrace{0,0,\ldots,0}_{d-2},1)$$

is a T-vector of a *pencil*, i.e., an arrangement where all lines pass through the same point, i.e., there is just one singular point of multiplicity d.
Similarly, the array

$$(d-1,\underbrace{0,0,\ldots,0}_{d-4},1,0)$$

is a T-vector of a *near-pencil*. This is an arrangement where all but one line pass through the same point P, which consequently has multiplicity $d-1$. The remaining line intersects lines passing through P in altogether $d-1$ double points.

A specific $(d-1)$-tuple for which it is not known if it comes up as a T-vector of a line arrangement, has $12^2+12=156$ entries, all of which are 0, with the exception of t_{13}, which is supposed to be 157. It is expected that this *is not* a T-vector of a line arrangement. More specifically, this is the first instance, where the existence of a finite projective plane (fpp in short) of certain order (here order 12) is not known, see [31] for a survey on fpps.

A highly non-trivial constrain in Problem 3 is the following theorem due to Erdös and de Bruijn [12].

Theorem 6 (Erdös and de Bruijn)**.** *Let $\mathcal{L}$ be an arrangement of d lines, which is not a pencil. Then*

$$\sum_{r\geq 2} t_r \geq d. \tag{2}$$

Moreover, there is equality in (2) if and only if $\mathcal{L}$ is either a near-pencil or it consists of all lines in a finite projective plane.

2. Arrangements in the real projective plane

In this section we study line arrangements in the projective plane $\mathbb{P}^2(\mathbb{R})$. The central result here is that if $\mathcal{L}$ is not a pencil, then it must be

$$t_2 > 0.$$

In fact much more is known. The story begins with a question raised in 1821 by Jackson in a recreational mathematics collection [29]. We recall first its original formulation:

"Your aid I want nine trees to plant
In rows just half a score;
And let there be in each row three
Solve this: I ask no more".

Due to this formulation this problem is now known as the orchard problem. We present its two mathematical versions and then pass to generalizations.

Problem 7 (Orchard problem)**.**

a) In the original version Jackson asks if it is possible to put 9 points in the plane in such a way, that on any line through a pair of these points there is an additional point.
b) Passing to the dual version, i.e., exchanging the role of points and lines, we get a version directly related to Problem 3: Is there an arrangement of 9 real lines such that they intersect only in triple points? In other words, is the 8-tuple of integers with $t_3=12$ and all other entries zero a T-vector of a real line arrangement?

The Orchard Problem remained unsolved and went forgotten, before it was rediscovered and generalized by Sylvester in 1893 [38].

Problem 8 (Sylvester Problem)**.** Is the pencil the only arrangement of real lines which has no double points?

Sylvester interest in this question was probably motivated by a discovery of Hesse [26]. He found a non-trivial line arrangement with no double points over the complex numbers. More precisely, Hesse studied the problem in the dual version and found that 3-torsion points on an elliptic curve give rise to an interesting arrangement of 12 lines. We present his discovery in a version fitting better this notes, see [2] for an excellent account on this and related constructions.

Example 9 (Dual Hesse arrangement)**.** Let $\mathcal{L}$ be the arrangement of 9 lines defined in the complex projective plane with homogeneous coordinates $(x : y : z)$ by linear factors of the polynomial

$$(x^3 - y^3)(y^3 - z^3)(z^3 - x^3).$$

Then $t_3(\mathcal{L}) = 12$.

Remark 10. Note that the dual Hesse arrangement is a member of very interesting family of Fermat arrangement, see [40] for a recent survey on this kind of arrangements.

It took almost 50 years to answer Sylvester's question. In fact it went again forgotten and got revived by Erdös. The answer was given by another Hungarian mathematician, Tibor Gallai during the World War II and thus remained unpublished, see citation in [19].

Theorem 11 (Sylvester and Gallai)**.** *The only arrangements of lines in the real projective plane with $t_2 = 0$ are pencils.*

Over the years this theorem has been reproved in various manners. A proof attributed to Kelly by Coxeter in [8] is considered to be probably the most elegant one. For this reason it made the way to the famous "Proofs from The Book" by Aigner and Ziegler [1].

However, it was another proof which attracted a lot of attention and triggered new research directions. It is due to Melchior [34] and it provides strong numerical constrains to our central Problem 3. His method was to use in a clever way Euler's formula.

Theorem 12 (Melchior inequality)**.** *Let $\mathcal{L}$ be an arrangement of lines in the real projective plane which is not a pencil, then the following inequality holds:*

$$t_2 \geq 3 + \sum_{r \geq 3} (r-3) t_r.$$

Melchior's result shows that a non-trivial real line arrangement must not only have positive t_2 but in fact there must be at least 3 double points. Interesting generalizations of Melchior's inequality in the realms of complex line arrangements

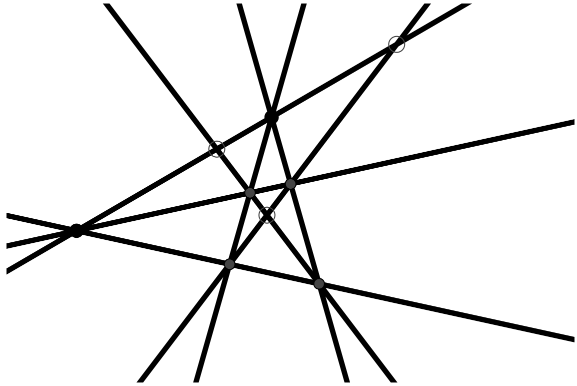

FIGURE 1. Kelly Moser arrangement

have been obtained by Hirzebruch [27], see also [39] and the article by Piotr Pokora in this volume [36].

It is then natural to wonder how many double points must there be and if this might depend on the number of lines. This problem became known as Dirac's conjecture.

Conjecture 13 (Dirac, 1951)**.** *Let $\mathcal{L}$ be an arrangement of d lines, not a pencil. If d is sufficiently large, then there must be*

$$t_2 \geq \left\lfloor \frac{d}{2} \right\rfloor .$$

The reason for the assumption that d is sufficiently large are the following two arrangements, which are in fact the only known arrangements where it is necessary to round down the term on the right in Dirac's conjecture.

Example 14 (Kelly and Moser)**.** Taking a complete quadrilateral with all its diagonals we get an arrangement of 7 lines with only 3 ordinary points. This are the points indicated by empty circles in Figure 1. All other singular points of this arrangement have multiplicity 3.

The next example is easier to explain in the dual form. In this setting we are interested in a finite set $\mathcal{P}$ of points in $\mathbb{P}^2$ and all lines determined by pairs of points in $\mathcal{P}$. The multiplicity of a line L is then the number of points in $\mathcal{P}$ which lie on it. This is in agreement with the multiplicity of the corresponding point L' in the dual projective plane since the points on L correspond to lines passing through L'. Our description is borrowed from Crowe and McKee survey [9].

Example 15 (McKee arrangement)**.** Let A, B, C, D, E be vertices of a regular pentagon. Let C', D', E' be images of C, D, E respectively under reflection along the line determined by points A and B. Let M be the midpoint of the segment AB. Finally, let I be the point at infinity on the line determined by C and D, let

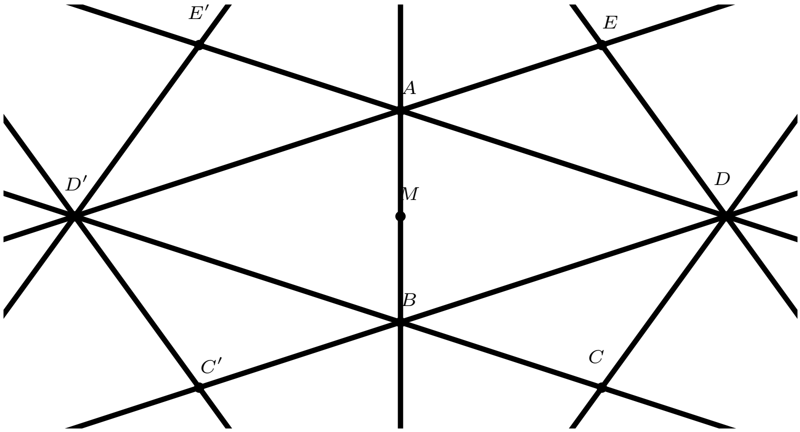

FIGURE 2. McKee configuration of points

K be the point at infinity on the line determined by D and E and let J and L be points at infinity in the direction of x and y axis respectively, see Figure 2 for points and most relevant lines in the affine part, the 4 points at the infinity are not visible. Let $\mathcal{P} = \{A, B, C, D, E, C', D', E', I, J, K, L, M\}$, then the only lines of multiplicity 2 with respect to $\mathcal{P}$ are those determined by the following pairs of points:

$$AJ,\ BJ,\ DL,\ D'L,\ MI,\ MK.$$

Dirac's conjecture has been proved in steps. In 1951 Motzkin [35] showed that for non pencil arrangements $t_2 \geq \sqrt{2d-2}$. In 1958 Kelly and Moser [30] showed that, apart of a pencil, it is always

$$t_2 \geq \frac{3}{7}d.$$

In 1993 Csima and Sawyer [10] showed that, apart of pencils and Example 14, it is always

$$t_2 \geq \frac{6}{13}d.$$

The final step so far has been made by Green and Tao [21]. They proved that Dirac's conjecture holds for d sufficiently large. Their proof allows for an effective estimate on what "sufficiently many means". The number one gets is in the magnitude of 10^5 and thus far away from 13. It is expected that the conjecture, without rounding down the number on the right, holds true for all $d \geq 14$.

We want to conclude this part with short glimpse at recent developments for complex line arrangements. We need first to define one more concept.

Definition 16 (Supersolvable arrangement)**.** We say that an arrangement $\mathcal{L}$ is *supersolvable*, if there exists a singular point of $\mathcal{L}$ which is connected by a line from

$\mathcal{L}$ to any other singular point of $\mathcal{L}$. A point with this property is called a *modular point*.

Example 17. A pencil and a near-pencil are supersolvable with all their singular points being modular.
A star configuration (see Example 4) of 4 or more lines is not supersolvable.

Whereas Example 9 shows that it might happen that $t_2 = 0$ for a non-trivial arrangement of lines in the complex projective planes, it is worth to point out that there are not too many such arrangement known. Moreover, none of known examples is supersolvable, see recent arXiv posting by Hanumanthu and Harbourne [23]. Thus it is reasonable to make the following conjecture, which parallels Sylvester's problem.

Conjecture 18 (Hanumanthu and Harbourne, 2019)**.** *Let $\mathcal{L}$ be an arrangement of complex lines which is supersolvable and which is not a pencil. Then $t_2 > 0$.*

This Conjecture is, of course, the first step towards the following bold conjecture by Anzis and Tohaneanu.

Conjecture 19 (Anzis and Tohaneanu, 2015)**.** *Let $\mathcal{L}$ be an arrangement of s complex lines which is supersolvable and which is not a pencil. Then $t_2 \geq \frac{s}{2}$.*

3. Line arrangements and the containment problem

In this section our story turns back to algebra and geometry. Building upon ideas of Swanson [37], Ein, Lazarsfeld and Smith [17] in characteristic zero and Hochster and Huneke [28] in positive characteristic (followed recently by Ma and Schwede [33] in mixed characteristic) proved that for a non-trivial homogeneous ideal I in a polynomial ring $R = \mathbb{K}[x_0, \dots, x_N]$ there is always the containment

$$I^{(m)} \subset I^r \tag{3}$$

provided $m \geq Nr$. Here $I^{(m)}$ stays for the symbolic power of I, which is defined by

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} (I^m R_{\mathfrak{p}} \cap R).$$

Of course there are ideals, an extremal example coming from ideals generated by a regular sequence, where the containment in (3) holds for m much smaller than the bound provided above (it is well-known and easy to check that m must be at least r in order that (3) holds). On the other hand examples showing that one indeed needs $m \geq Nr$ have been discovered only recently and such examples are rather rare.

Since $I^1 = I$ and $I^{(m)} \subset I$ for any $m \geq 1$, the first nontrivial containment in (3) is that of $I^{(4)}$ in I^2 for an ideal I defining points in the projective plane $\mathbb{P}^2(\mathbb{K})$. Around 2000 Huneke asked if for an ideal of points in $\mathbb{P}^2$ one has

$$I^{(3)} \subset I^2 \tag{4}$$

and he asked for a simple proof of this fact. The containment in (4) has been verified in many interesting cases, see [5], [6], [24]. However Dumnicki, Szemberg and Tutaj-Gasińska [16] showed the first example where the containment in (4) fails. This example is delivered by singular points of the dual Hesse arrangement, i.e., of our Example 9.

Theorem 20 (Dumnicki and Szemberg and Tutaj-Gasińska). *For the ideal I of points*

$$\begin{array}{lll} P_1=(1:0:0), & P_2=(0:1:0), & P_3=(0:0:1), \\ P_4=(1:1:1), & P_5=(1:1:\varepsilon), & P_6=(1:1:\varepsilon^2), \\ P_7=(1:\varepsilon:1), & P_8=(1:\varepsilon:\varepsilon), & P_9=(1:\varepsilon:\varepsilon^2), \\ P_{10}=(1:\varepsilon^2:1), & P_{11}=(1:\varepsilon^2:\varepsilon), & P_{12}=(1:\varepsilon^2:\varepsilon^2) \end{array}$$

the containment $I^{(3)} \subset I^2$ fails.

It has been quickly realized that singular points of other line arrangements also provide non-containment examples, see [25], [14], [3]. In this way also non-containment examples over the reals [11] and over the rationals [32] have been identified.

Interestingly all non-containment results identified so far in characteristic zero, show only that one needs the symbolic power 4 or higher in order to ensure the containment

$$I^{(m)} \subset I^2.$$

Passing to the third ordinary power and keeping I the ideal of points in $\mathbb{P}^2$, it is not even known if

$$I^{(5)} \subset I^3$$

might fail.

There is in fact a much more general conjecture due to Harbourne.

Conjecture 21 (Harbourne). *Let I be a radical ideal in the ring of polynomials $R = \mathbb{K}[x_0, \dots, x_N]$. Let c be the big height of I. Then for all integers $r \geq 1$,*

$$I^{(cr-c+1)} \subset I^r. \tag{5}$$

If I is an ideal of points, then its big height is equal N. Hence for $N = 2$ and $r = 2$ we are in the old containment $I^{(3)} \subset I^2$, which is false in general, as we know. However, in the case of complex numbers the conjecture can be potentially saved adding an important additional requirement that the containment in (5) holds for **all n sufficiently large**. Recent paper by Grifo, Huneke and Mukundan [22] puts Conjecture 21 in the asymptotic perspective.

4. Line arrangements and unexpected curves

In recent years we have witnessed extremely interesting developments in the theory of positivity of linear systems. In this presentation we restrict only to objects in the projective plane, which is just a piece of theory growing at awesome speed. On the other hand, it is the projective plane where the story begins with the ground-breaking article by Cook II, Harbourne, Migliore and Nagel [7]. We follow their approach.

Definition 22 (Unexpected plane curves)**.** We say that a finite set Z of reduced points in $\mathbb{P}^2$ *admits an unexpected curve* of degree $m+1$, if for a general point P, the fat point scheme mP (i.e. defined by the ideal $I(P)^m$) fails to impose independent conditions on the linear system of curves of degree m vanishing at all points of Z. In other words, Z admits an unexpected curve of degree $m+1$ if

$$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d) \otimes I(Z+mP)) \; > \; \max\left\{ h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d) \otimes I(Z)) - \binom{m+1}{2}, \; 0 \right\}.$$

It is clear that a single fat point imposes independent conditions on the *complete* linear systems of curves of fixed degree. Thus the empty set Z does not admit any unexpected curves. It came as a surprise that such non-empty sets do exist. The first example, which was the main motivation for [7], was discovered by Di Gennaro, Ilardi and Valles [13, Proposition 7.3]. This example comes from a reflexive line arrangement B_3, we refer to [40] for an extensive account on this kind of arrangements. Here we content ourselves with the definition of a root system and explicit equations of arrangement lines.

Definition 23 (Root system)**.** A *root system* is a finite collection R of vectors in an affine space V (in our case it will be $\mathbb{R}^3$) such that

- the elements in R span V;
- for each $\alpha \in R$, the vector $-\alpha$ is in R and no other multiple of α is there;
- for each α and β in R, the vector $s_\alpha(\beta)$ is also in R (here $s_\alpha(v) = v - 2\frac{\langle \alpha, v\rangle}{\langle \alpha, \alpha\rangle}\alpha$ is the reflection in the hyperplane perpendicular to α);
- for each α and β in R, the number $2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}$ is an integer.

Example 24 (The B_3 arrangement of lines)**.** The linear factors of the polynomial

$$xyz(x^2-y^2)(y^2-z^2)(z^2-x^2)$$

define an arrangement of lines in $\mathbb{P}^2$ which are reflections in the Weyl group generated by a B_3 root system. Passing to the dual setting we identify the projectivized B_3 root system consisting of points

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:0), & P_5 = (1:-1:0), & P_6 = (1:0:1), \\ P_7 = (1:0:-1), & P_8 = (0:1:1), & P_9 = (0:1:-1). \end{array}$$

Theorem 25 (Di Gennaro, Ilardi, Valles)**.** *Let Z be the set of* 9 *points defined in Example 24. Then Z admits an unexpected curve of degree* 4 *with a point of multiplicity* 3.

Explicit equations of curves whose existence is guaranteed by Theorem 25 have been found by Bauer, Malara, Szemberg and the author in [4]. Farnik, Galuppi, Sodomaco and Trok showed in [18] that this is the unique unexpected quartic curve up to projective change of coordinates.

The set Z in Theorem 25 arises as dual points of an interesting arrangement of lines. Similarly as in Section 3 sets admitting unexpected curves arise also as singular points of certain line arrangements.

Theorem 26 (Dual Hesse arrangement and an unexpected quintic)**.** *Let Z be the set of singular points of the dual Hesse arrangement defined in Example 9. Then Z admits an unexpected curve of degree 5 with a point of multiplicity 4.*

Proof. The points in Z are the following 12 points:

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:1), & P_5 = (1:1:\varepsilon), & P_6 = (1:1:\varepsilon^2), \\ P_7 = (1:\varepsilon:1), & P_8 = (1:\varepsilon:\varepsilon), & P_9 = (1:\varepsilon:\varepsilon^2), \\ P_{10} = (1:\varepsilon^2:1), & P_{11} = (1:\varepsilon^2:\varepsilon), & P_{12} = (1:\varepsilon^2:\varepsilon^2). \end{array}$$

The ideal $I(Z)$ is an almost complete intersection ideal generated by

$$x(y^3 - z^3),\ y(z^3 - x^3),\ z(x^3 - y^3),$$

see [16].

Let $P = (a : b : c)$ be a general point in $\mathbb{P}^2$. Then it is easy to check that the curve defined by the polynomial

$$\begin{aligned} Q_P(x:y:z) =& cxy((2b^3 + c^3)(z^3 - x^3) + (2a^3 + c^3)(y^3 - z^3)) \\ & bxz((2a^3 + b^3)(y^3 - z^3) + (2c^n + b^n)(x^3 - y^3)) \\ & ayz((2b^3 + a^3)(z^3 - x^3) + (2c^3 + a^3)(x^3 - y^3)) \\ & - 6a^2bcx^2(y^3 - z^3) - 6ab^2cy^2(z^3 - x^3) - 6abc^2z^2(x^3 - y^3) \end{aligned} \tag{6}$$

satisfies the assertions of the theorem. ☐

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