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A Proof of Tarski's Fixed Point Theorem by Application of Galois Connections

Abstract. Two examples of Galois connections and their dual forms are considered. One of them is applied to formulate a criterion when a given subset of a complete lattice forms a complete lattice. The second, closely related to the first, is used to prove in a short way the Knaster-Tarski's fixed point theorem.

Keywords: Closure and interior operation, Galois connection, Fixed point theorem.

1. Introduction

For given antimonotone Galois connection defined for the complete lattices, a dual form – an appropriate monotone Galois connection (a residuated pair of mappings) is considered. The pair of closure and interior operations induced on a complete lattice by such anti- and monotone Galois connections is of our interest. Two examples of Galois connections and their dual forms are introduced in the paper. First one, considered in Sect. 3, embraces a Galois connection responsible for the dual isomorphism between a complete lattice and a closure system of subsets of a meet-generating subset of the lattice. The induced closure and interior operations are of so general form that they enable to formulate a simple criterion saying when a subset B of given complete lattice $(A, \leq)$ forms a complete lattice with respect to the ordering $\leq$ (Lemma 1 and Proposition 2). This criterion is applied in Sect. 4 to prove in a simple short way the Knaster-Tarski's fixed point theorem [10] (Corollary 9). The proof is constructive in the sense that it shows the explicit form of supremum and infimum of a subset in the lattice of all fixed points of a monotone mapping (cf. [2, Theorem 5.1]). This form differs from that of [2], moreover from that of [6]. The proof is also based on some simple results (inter alia Proposition 8) concerning the second example of Galois connections introduced in the paper (Sect. 4). This example is responsible

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for well-known isomorphisms between the lattice of all closure (interior) operations defined on a complete lattice $(A, \leq)$ and the lattice of all closure (interior) systems of $(A, \leq)$. The induced closure and interior operations are here defined on the complete lattice of all monotone mappings of a complete lattice $(A, \leq)$ into itself. The closure operation C induced by the antimonotone Galois connection assigns to each monotone map α the least closure operation c defined on $(A, \leq)$ such that $\alpha \leq c$, where $\leq$ is the pointwise order on mappings from A to A induced by lattice ordering of $(A, \leq)$. In turn, the dual (monotone) Galois connection induces an interior operation Int assigning to each monotone mapping α the greatest interior operation I on $(A, \leq)$ such that $I \leq \alpha$. A crucial point of the proof of Knaster-Tarski's theorem presented here, is the fact that the set of all fixed points of a monotone map α turns out to be the intersection of the closure and interior systems of $(A, \leq)$ corresponding to closure $C(\alpha)$ and interior $Int(\alpha)$ operations, respectively.

2. Preliminaries

The paper deals mostly with the closure and interior operations defined on a complete lattice. Given a complete lattice $(A, \leq)$ any mapping $C : A \longrightarrow A$ such that for each $a \in A$, $a \leq C(a)$, $C(C(a)) \leq C(a)$ and C is monotone: $a \leq b \;\Rightarrow\; C(a) \leq C(b)$, is called a *closure operation* defined on $(A, \leq)$. Any subset $B \subseteq A$ is said to be a *closure system* or *Moore family* of the lattice $(A, \leq)$ if for each $X \subseteq B$, $\inf_A X \in B$. Given a closure operation C on $(A, \leq)$, the set of all its fixed points called *closed elements*: $\{a \in A : a = C(a)\}$, is a closure system of $(A, \leq)$. Conversely, given a closure system B of $(A, \leq)$, the map $C : A \longrightarrow A$ defined by $C(a) = \inf_A\{x \in B : a \leq x\}$, is a closure operation on $(A, \leq)$. The closure system B is just the set of all its closed elements. On the other hand, the closure system of all closed elements of a given closure operation C defines, in that way, just the operation C. Thus, there is a one to one correspondence between the class of all closure operations and of all closure systems of $(A, \leq)$ (in fact it is a dual isomorphism between respective complete lattices of all closure operations and closure systems). Any closure system B of $(A, \leq)$ forms a complete lattice with respect to the order $\leq$ such that $\inf_B X = \inf_A X$ and $\sup_B X = C(\sup_A X)$, for each $X \subseteq B$, where C is the closure operation corresponding to closure system B. Given a subset X of A, there exists the least closure system B of $(A, \leq)$ such that $X \subseteq B$, called *generated by* X. It will be denoted here by $[X]_{cl}$. It is simply the intersection of all the closure systems of $(A, \leq)$ containing X and is of the form: $[X]_{cl} = \{\inf_A Y : Y \subseteq X\}$. The closure operation C corresponding to closure system $[X]_{cl}$ is expressed by $C(a) = \inf_A\{x \in X : a \leq x\}$, any $a \in A$.

An *interior operation* and an *interior system* are the dual concepts with respect to closure ones. That is, a monotone mapping $I : A \longrightarrow A$ such that for any $a \in A,\ I(a) \leq a, I(a) \leq I(I(a))$ is said to be an *interior operation* defined on a complete lattice $(A, \leq)$. Any subset B of A is called an *interior system* of the lattice $(A, \leq)$ if for each $X \subseteq B,\ \sup_A X \in B$. Given an interior operation I on $(A, \leq)$ the set of all its fixed poits called *open elements*: $\{a \in A : a = I(a)\}$, is an interior system of $(A, \leq)$. Conversely, given an interior system B of $(A, \leq)$, the map $I : A \longrightarrow A$ defined by $I(a) = \sup_A\{x \in B : x \leq a\}$, is an interior operation on $(A, \leq)$. The interior system B is just the set of all its open elements. On the other hand, the interior system of all open elements of a given interior operation I defines, in that way, just the operation I. So, as before, a similar correspondence between the class of all interior operations and interior systems, exists (which is an isomorphism of respective complete lattices of all interior operations and all interior systems of $(A, \leq)$). Any interior system B of $(A, \leq)$ forms a complete lattice with respect to the order $\leq$ such that $\sup_B X = \sup_A X$ and $\inf_B X = I(\inf_A X)$, for each $X \subseteq B$, where I is the interior operation corresponding to interior system B. Given a subset X of A, there exists the least interior system B of $(A, \leq)$ such that $X \subseteq B$. Such an interior system is said to be *generated by* X and will be denoted as $[X]_{in}$. It is the intersection of all the interior systems of $(A, \leq)$ containing X and is of the form: $[X]_{in} = \{\sup_A Y : Y \subseteq X\}$. The interior operation I corresponding to interior system $[X]_{in}$ is defined by $I(a) = \sup_A\{x \in X : x \leq a\}$, any $a \in A$.

We shall consider the monotone and antimonotone Galois connections defined only for complete lattices. A general theory of Galois connections is to be found for example in [1,3–5,7].

Let us remind that while $(A, \leq_A),\ (B, \leq_B)$ are the complete lattices, any pair of mappings $f : A \longrightarrow B,\ g : B \longrightarrow A$ such that for each $a \in A, b \in B :\ b \leq_B f(a)$ iff $a \leq_A g(b)$, is called an *antimonotone Galois connection* for those lattices. Equivalently, such a Galois connection (f, g) fulfils the following conditions: $a \leq_A g(f(a)),\ b \leq_B f(g(b))$ for any $a \in A,\ b \in B$ and f, g are antimonotone. When the pairs $(f, g_1), (f, g_2)$ are Galois connections for the lattices $(A, \leq_A),\ (B, \leq_B)$ then $g_1 = g_2$. The first element f of an antimonotone Galois connection (f, g) for the lattices $(A, \leq_A)$, $(B, \leq_B)$ is usually called a *Galois function*. A sufficient and necessary condition for a map $f : A \longrightarrow B$ to be a Galois function is of the form: $f(\sup_A X) = \inf_B\{f(a) : a \in X\}$, for any $X \subseteq A$. Given a Galois function

f, the second unique element g of the Galois connection (f,g) is given by $g(b) = \sup_A\{a \in A : b \leq_B f(a)\}$, for each $b \in B$. This mapping g satisfies the condition: $g(\sup_B Y) = \inf_A\{g(b) : b \in Y\}$, for any $Y \subseteq B$. Given a Galois connection (f,g) for the lattices $(A, \leq_A),\ (B, \leq_B)$, the ranges $f[A],\ g[B]$ of the mappings f and g are the sets of all closed elements with respect to the closure operations $Cl_2,\ Cl_1$ respectively that are induced on B and A in the following way: for each $a \in A,\ b \in B,\ Cl_2(b) = f(g(b)),\ Cl_1(a) = g(f(a))$. Since for each $a \in g[B],\ b \in f[A] :\ g(f(a)) = a,\ f(g(b)) = b$ and moreover for any $a_1, a_2 \in g[B] :\ a_1 \leq_A a_2$ iff $f(a_2) \leq_B f(a_1)$, so the complete lattices $(g[B], \leq_A),\ (f[A], \leq_B)$ are dually isomorphic (with f being a dual isomorphism).

In turn, a pair $f : A \longrightarrow B,\ g : B \longrightarrow A$ such that for each $a \in A,\ b \in B :\ b \leq_B f(a)$ iff $g(b) \leq_A a$, is called a *monotone Galois connection* or a *residuated pair of mappings* for the lattices $(A, \leq_A),\ (B, \leq_B)$. Equivalently, a monotone Galois connection (f,g) fulfils the following conditions: $g(f(a)) \leq_A a,\ b \leq_B f(g(b))$ for any $a \in A,\ b \in B$ and f, g are monotone functions. When $(f, g_1), (f, g_2)$ are residuated pairs for the lattices $(A, \leq_A),\ (B, \leq_B)$ then $g_1 = g_2$. The first element f of a monotone Galois connection (f,g) for the lattices $(A, \leq_A),\ (B, \leq_B)$ is usually called a *residuated function* while the unique second one g–a *residual* of f. A sufficient and necessary condition for a map $f : A \longrightarrow B$ to be a residuated function is of the form: $f(\inf_A X) = \inf_B\{f(a) : a \in X\}$, for any $X \subseteq A$. Given a residuated function f, its residual g is expressed by $g(b) = \inf_A\{a \in A : b \leq_B f(a)\}$, for each $b \in B$. This mapping g satisfies the condition: $g(\sup_B Y) = \sup_A\{g(b) : b \in Y\}$. Given a residuated pair (f,g) for the lattices $(A, \leq_A),\ (B, \leq_B)$, the ranges $f[A],\ g[B]$ are, respectively, the sets of all closed and open elements with respect to the following closure and interior operations Cl, Int: for each $a \in A,\ b \in B,\ Cl(b) = f(g(b)),\ Int(a) = g(f(a))$. Since for each $a \in g[B],\ b \in f[A] :\ g(f(a)) = a,\ f(g(b)) = b$ and moreover for any $a_1, a_2 \in g[B] :\ a_1 \leq_A a_2$ iff $f(a_1) \leq_B f(a_2)$, so the complete lattices $(g[B], \leq_A),\ (f[A], \leq_B)$ are isomorphic (with f being an isomorphism).

From the very definition of Galois connections it follows that any antimonotone Galois connection (f,g) for the lattices $(A, \leq_A),\ (B, \leq_B)$ is simultaneously a residuated pair for the lattices $(A, \leq_A^{\sim}),\ (B, \leq_B)$, where $\leq_A^{\sim}$ is the converse ordering to $\leq_A$. Taking this into account, having defined a Galois function $f_{\leq_A} : A \longrightarrow B$ for the complete lattices $(A, \leq_A),\ (B, \leq_B)$ (we write down the parameter: $\leq_A$, on which the function may depend as an essential one, however in general there are the other parameters which may occur in a definition of Galois function) let us consider a mapping

$f_{\leq_A^{\sim}} : A \longrightarrow B$ which is defined exactly in the same way as the function $f_{\leq_A}$ except that instead of the parameter $\leq_A$ the converse relation is applied. Notice that when $f_{\leq_A}$ being a Galois function fulfils the condition: $f_{\leq_A}(\sup_{\leq_A} X) = \inf_{\leq_B}\{f_{\leq_A}(a) : a \in X\}$, the mapping $f_{\leq_A^{\sim}}$ has to satisfy the following one: $f_{\leq_A^{\sim}}(\inf_{\leq_A} X) = \inf_{\leq_B}\{f_{\leq_A^{\sim}}(a) : a \in X\}$, any $X \subseteq A$, that is, $f_{\leq_A^{\sim}}$ is a residuated function for the lattices $(A, \leq_A)$, $(B, \leq_B)$. Let us call such a residuated function the *dual residuated function with respect to* $f_{\leq_A}$. Moreover, when (f, g) is an antimonotone Galois connection let us call the residuated pair (f_d, g_d), where f_d is the dual residuation function with respect to f, the *dual residuated pair (or the dual Galois connection) with respect to* (f, g). Obviously, one can start not from a Galois but a residuated function (residuated pair) and define the dual Galois function (the dual antimonotone Galois connection).

Having at our disposal the Galois connections: (f, g), (f_d, g_d) for the complete lattices $(A, \leq_A)$, $(B, \leq_B)$ we are especially interested in the interior-closure pair (Int, C) of operations on $(A, \leq_A)$, where $Int = f_d \circ g_d$ and $C = f \circ g$ (the closure operation C was denoted by Cl_1 above).

In the sequel we consider two important examples of antimonotone Galois connections and their dual forms. First one enables to formulate a simple criterion saying when a given subset of a complete lattice forms a complete lattice. The second example, closely related to the first, has rather unexpected applications. It enables a very simple proving of the Knaster-Tarski's fixed point theorem.

3. A Criterion of Being a Complete Lattice

Let $(A, \leq)$ be any complete lattice and $B \subseteq A$. The following pair of mappings: $f : A \longrightarrow \wp(B)$, $g : \wp(B) \longrightarrow A$ defined by $f(a) = \{x \in B : a \leq x\}$, any $a \in A$ and $g(X) = \inf_A X$, any $X \subseteq B$, forms an antimonotone Galois connection for the lattices $(A, \leq)$, $(\wp(B), \subseteq)$. The dual residuated function with respect to f is then of the form: $f_d(a) = \{x \in B : x \leq a\}$ and its residual is defined by $g_d(X) = \inf_A\{a \in A : X \subseteq f_d(a)\} = \inf_A\{a \in A : X \subseteq \{x \in B : x \leq a\}\} = \sup_A X$, as one could expect.

These Galois connections are responsible for well-known isomorphisms of a complete lattice and a lattice of subsets of a given meet- or join-generating subset of the lattice. A subset B of a complete lattice $(A, \leq)$ is said to be *join-generating* (*meet-generating*, cf. for example [5]) or *join-dense* (*meet-dense*, e.g. [8]) iff for each $a \in A$, there is an $X \subseteq B$ such that $a = \sup_A X$ ($a =$

$\inf_A X$). For example, the set of all compact elements of an algebraic lattice is just its join-generating subset.

It is clear that the restriction of the map f to the set $\{\inf_A X : X \subseteq B\}$ (which is the closure system generated by B) is a dual isomorphism of the lattice $(\{\inf_A X : X \subseteq B\}, \leq)$ of all closed elements with respect to the closure operation $C_B = f \circ g$ to the lattice $(\{B \cap [a) : a \in A\}, \subseteq)$ (which is the closure system of $(\wp(B), \subseteq)$ corresponding to closure operation $g \circ f$; here $[a) = \{x \in A : a \leq x\}$). Similarly, the restriction of the map f_d to the set $\{\sup_A X : X \subseteq B\}$ (which is the interior system generated by B) is an isomorphism of the lattice $(\{\sup_A X : X \subseteq B\}, \leq)$ of all open elements with respect to the interior operation $I_B = f_d \circ g_d$ to the lattice $(\{B \cap (a] : a \in A\}, \subseteq)$ (being the closure system of $(\wp(B), \subseteq)$ corresponding to closure operation $g_d \circ f_d$; here $(a] = \{x \in A : x \leq a\}$).

One can easily see from their definitions that the operations I_B, C_B are of the following general form, for any $a \in A$:

(1) $I_B(a) = \sup_A\{x \in B : x \leq a\}$,

(2) $C_B(a) = \inf_A\{x \in B : a \leq x\}$.

They simply corrrespond to the interior and to closure systems of $(A, \leq)$ generated by B, respectively. The pair (I_B, C_B) is a generalization of the notion of so-called pair of interior-closure operations associated on a given subset of a complete lattice, introduced in [9] and widely applied there. In case a subset B forms a complete sublattice of the lattice $(A, \leq)$, the pair (I_B, C_B) becomes just an interior-closure pair of operations associated on B. The existence of an interior-closure pair of operations associated on B is a necessary and sufficient condition for $(B, \leq)$ to be a complete sublattice of $(A, \leq)$ (cf. [9]). This criterion will be now generalized in order to provide the sufficient and necessary conditions for the poset $(B, \leq)$ to be a complete lattice. Let us start from the crucial lemma.

LEMMA 1. *Let $D, O \subseteq A$ be any closure and interior systems of a complete lattice $(A, \leq)$, respectively. Then the following conditions are equivalent:*

(i) for each $a \in O$, $C_D(a) \in O$,

(ii) for each $a \in A$, $C_D(I_O(a)) \in O$,

(iii) for each $a \in A$, $I_O(C_D(a)) \in D$,

(iv) for each $a \in D$, $I_O(a) \in D$,

where the operations I_O, C_D are defined by (1) and (2), respectively, for the sets O, D instead of B. Moreover, any of these conditions implies that the *poset* $(D \cap O, \leq)$ *is a complete lattice in which for any* $X \subseteq D \cap O$, $\sup X = C_D(\sup_A X)$ *and* $\inf X = I_O(\inf_A X)$. *The inverse implication in general does not hold.*

Proof. Suppose that the subsets D and O of A are closure and interior systems of a complete lattice $(A, \leq)$, respectively. The equivalences $(i) \Leftrightarrow (ii)$, $(iii) \Leftrightarrow (iv)$ are obvious. In order to show the implication $(ii) \Rightarrow (iii)$ assume that for each $a \in A$, $C_D(I_O(a)) = I_O(C_D(I_O(a)))$. Then given $a \in A$ we have $C_D(I_O(C_D(a))) = I_O(C_D(I_O(C_D(a))))$. Since $I_O(C_D(a)) \leq C_D(a)$ so $C_D(I_O(C_D(a))) \leq C_D(a)$ (C_D is monotone and idempotent). Therefore, $I_O(C_D(I_O(C_D(a)))) \leq I_O(C_D(a))$ (by monotonicity of I_O) which together with the last identity implies that $C_D(I_O(C_D(a))) \leq I_O(C_D(a))$ so we obtain (iii). The proof from (iii) to (ii) goes analogously (by dual argument).

In order to prove the second part of lemma suppose (i) and consider an $X \subseteq D \cap O$. Then since O is an interior system we have $\sup_A X \in O$. So from (i) it follows that $C_D(\sup_A X) \in D \cap O$. Now, given any $a \in X$ we have $a \leq \sup_A X \leq C_D(\sup_A X)$, so $C_D(\sup_A X)$ is an upper bound of X in the poset $(D \cap O, \leq)$. When $z \in D \cap O$ is such an upper bound we obtain: $\sup_A X \leq z$, therefore $C_D(\sup_A X) \leq C_D(z) = z$. In this way, $C_D(\sup_A X)$ is the least upper bound of X in $(D \cap O, \leq)$. The form of $\inf X$ in this poset follows from the condition (iv) in a similar way.

Finally, in order to show that none of the conditions $(i) - (iv)$ needs to be true when a poset $(D \cap O, \leq)$ is a complete lattice, take for example a 4-element chain: $0 < a < b < 1$ and consider $D = \{0, b, 1\}$, $O = \{0, a, 1\}$. ■

Now let us formulate our criterion saying when a subset of given complete lattice $(A, \leq)$ forms a complete lattice with respect to the order $\leq$.

Proposition 2. *Let* $(A, \leq)$ *be a complete lattice and* $B \subseteq A$. *Consider the operations* I_B, C_B *defined by (1), (2). The following conditions are equivalent:*

(a) *for each* $a \in A$, $C_B(I_B(a)) \in B$,

(b) *for each* $a \in A$, $I_B(C_B(a)) \in B$,

(c) $(B, \leq)$ *is a complete lattice such that for any* $X \subseteq B$, $\sup X = C_B(\sup_A X)$ *and* $\inf X = I_B(\inf_A X)$.

Proof. Let $B \subseteq A$. Put $D = [B]_{cl}$, $O = [B]_{in}$. Then we have immediately $B \subseteq D \cap O$ and $C_D = C_B$, $I_O = I_B$.

$(a) \Rightarrow (b)$ & (c): Assume that (a) holds. Then the condition (ii) of Lemma 1 is satisfied. Moreover, taking any $a \in D \cap O$ we have

$C_B(I_B(a)) = a$ so from (a) it follows that $a \in B$, consequently, $B = D \cap O$. Thus, on one hand, from (ii) and Lemma 1 it follows that (iii) of Lemma 1 holds which leads to (b). On the other hand, simultaneously from (ii) and Lemma 1 it follows that (c) holds true.

$(b) \Rightarrow (a)$: By the dual argument with respect to the proof of implication $(a) \Rightarrow (b)$.

$(c) \Rightarrow (a)$: Suppose that (c) holds. Let $a \in A$. Since $I_B(a) \in [B]_{in}$ so $I_B(a) = \sup_A X$ for some $X \subseteq B$. Therefore, $C_B(I_B(a)) = C_B(\sup_A X) = \sup X$ by (c). Thus, $C_B(I_B(a)) \in B$. ■

4. The Galois Connections Involving Monotone Mappings on Complete Lattices

Let $(A, \leq)$ be a complete lattice and *Mon*–the class of all monotone mappings from A to A. Obviously, the poset $(Mon, \leq)$ is a complete sublattice of the complete lattice $(A^A, \leq)$ of all the mappings from A to A, where for any $\alpha, \beta \in A^A$, $\alpha \leq \beta$ iff for all $x \in A$, $\alpha(x) \leq \beta(x)$. For any $F \subseteq Mon$, $(\sup F)(a) = \sup_A\{\alpha(a) : \alpha \in F\}$ and $(\inf F)(a) = \inf_A\{\alpha(a) : \alpha \in F\}$, for each $a \in A$.

The main goal of this section is to prove the Knaster-Tarski's fixed point theorem using a special Galois connection. This Galois connection turns out to be significant also from the other point of view. It is responsible for well-known dual isomorphism between the complete lattice of all closure operations defined on the complete lattice $(A, \leq)$ and the complete lattice of all closure systems of $(A, \leq)$. The connection is of the form: $f : (Mon, \leq) \longrightarrow (\wp(A), \subseteq)$ is a mapping defined by $f(\alpha) = \{x \in A : \alpha(x) \leq x\}$ and $g : (\wp(A), \subseteq) \longrightarrow (Mon, \leq)$ is such that for any $B \subseteq A$, $g(B) : A \longrightarrow A$ is defined by $g(B)(a) = \inf_A\{x \in B : a \leq x\} = \inf_A(B \cap [a))$. It is obvious that $g(B)$ for each $B \subseteq A$ is monotone. Notice simply that given $B \subseteq A$, $g(B)$ is just the closure operation C_B from the previous section.

Lemma 3. *(f, g) is a Galois connection, i.e., f, g are antimonotone, for each $\alpha \in Mon$, $\alpha \leq g(f(\alpha))$ and for any $B \subseteq A$, $B \subseteq f(g(B))$.*

Proof. The proof that both f, g are antimonotone is straightforward. In order to show that given $\alpha \in Mon$, $\alpha \leq g(f(\alpha))$, notice that given $a \in A$, $g(f(\alpha))(a) = \inf_A\{x \in A : \alpha(x) \leq x \;\&\; a \leq x\}$. Consider any $x \in A$ such that $\alpha(x) \leq x$ and $a \leq x$. Then since the map α is monotone we have: $\alpha(a) \leq \alpha(x)$ which implies that $\alpha(a) \leq x$. This means that $\alpha(a)$ is a lower bound of the set $\{x \in A : \alpha(x) \leq x \ \&\ a \leq x\}$ in the lattice $(A, \leq)$. Therefore, $\alpha(a) \leq \inf_A\{x \in A : \alpha(x) \leq x \ \&\ a \leq x\}$, that is $\alpha(a) \leq g(f(\alpha))(a)$. To the end, in order to prove that for all $B \subseteq A$, $B \subseteq f(g(B))$ take any $a \in B$. Our goal is to show that $g(B)(a) \leq a$. However, in case $a \in B$ we have: $\inf_A\{x \in B : a \leq x\} = a$, so $g(B)(a) = a$. ■

Now, consider the closure operations induced by the Galois connection (f, g), $Cl_1 : Mon \longrightarrow Mon$ and $Cl_2 : \wp(A) \longrightarrow \wp(A)$ defined by $Cl_1(\alpha) = g(f(\alpha))$, for any $\alpha \in Mon$ and $Cl_2(B) = f(g(B))$, for each $B \subseteq A$. Obviously, $\{\alpha \in Mon : Cl_1(\alpha) = \alpha\} = g[\wp(A)]$ and $\{B \subseteq A : Cl_2(B) = B\} = f[Mon]$. Moreover, the mapping f restricted to the set $\{\alpha \in Mon : Cl_1(\alpha) = \alpha\}$ is a dual isomorphism between the posets $(\{\alpha \in Mon : Cl_1(\alpha) = \alpha\}, \leq)$, $(\{B \subseteq A : Cl_2(B) = B\}, \subseteq)$.

One may characterize the sets of all closed elements with respect to the first and to the second closure operations in the following way.

Proposition 4. *(1) For any $\alpha \in Mon$, $Cl_1(\alpha) = \alpha$ iff α is a closure operation on $(A, \leq)$.*

(2) For any $B \subseteq A$, $Cl_2(B) = B$ iff for any $X \subseteq B$, $\inf_A X \in B$, that is B is a closure system of the lattice $(A, \leq)$.

Proof. For (1) ($\Rightarrow$): Assume that $Cl_1(\alpha) = \alpha$. Then $\alpha = g(B)$ for some $B \subseteq A$. So α is the closure operation C_B on $(A, \leq)$ from the previous section.

($\Leftarrow$): Assume that α is a closure operation on $(A, \leq)$. Our goal is to show that $g(f(\alpha)) \leq \alpha$. For each $a \in A$ we have $g(f(\alpha))(a) = \inf_A\{x \in A : \alpha(x) \leq x \ \&\ a \leq x\}$. From the assumption it follows that given $a \in A$, $\alpha(\alpha(a)) \leq \alpha(a)$ and $a \leq \alpha(a)$, so $\alpha(a) \in \{x \in A : \alpha(x) \leq x \ \&\ a \leq x\}$, thus $\inf_A\{x \in A : \alpha(x) \leq x \ \&\ a \leq x\} \leq \alpha(a)$, that is $g(f(\alpha))(a) \leq \alpha(a)$.

For (2) ($\Rightarrow$): Assume that $Cl_2(B) = B$ and $X \subseteq B$. Then obviously, $B = f(\alpha)$ for some $\alpha \in Mon$, that is, $B = \{x \in A : \alpha(x) \leq x\}$ for some $\alpha \in Mon$. So we have furthermore $X \subseteq \{x \in A : \alpha(x) \leq x\}$. Hence, taking any $a \in X$ into account we have $\alpha(a) \leq a$ while from the monotonicity of α it follows that $\alpha(\inf_A X) \leq \alpha(a)$ (for $\inf_A X \leq a$). Thus $\alpha(\inf_A X) \leq a$, so $\alpha(\inf_A X)$ is a lower bound of X, therefore, $\alpha(\inf_A X) \leq \inf_A X$. This means that $\inf_A X \in B$.

($\Leftarrow$): Assume that for all $X \subseteq B$, $\inf_A X \in B$. It is sufficient to show that $f(g(B)) \subseteq B$. We have $f(g(B)) = \{a \in A : g(B)(a) \leq a\} = \{a \in A : \inf_A\{x \in B : a \leq x\} \leq a\} = \{a \in A : \inf_A\{x \in B : a \leq x\} = a\} = \{a \in A : \inf_A(B \cap [a)) = a\}$. So let $a \in f(g(B))$. Then $\inf_A(B \cap [a)) = a$. Since $B \cap [a) \subseteq B$ so from the assumption it follows that $\inf_A(B \cap [a)) \in B$, that is, $a \in B$. ■

As one may see, Proposition 4 yields the above-mentioned correspondence between the closure operations and closure systems of given complete lattice.

COROLLARY 5. *(1) For any monotone mapping $\alpha : A \longrightarrow A$, $Cl_1(\alpha)$ is the least closure operation $c : A \longrightarrow A$ such that $\alpha \leq c$. Explicitly, for any $a \in A:\ Cl_1(\alpha)(a) = C_{f(\alpha)}(a) = \inf_A\{x \in A : \alpha(x) \leq x \ \&\ a \leq x\}$.*

(2) For any $B \subseteq A$, $Cl_2(B)$ is the least closure system $Z \subseteq A$ such that $B \subseteq Z$ (i.e. $Cl_2(B) = [B]_{cl}$). Explicitly, $Cl_2(B) = \{\inf_A X : X \subseteq B\}$.

PROOF. It is obvious that given any poset $(Y, \leq)$ and a closure operation $Cl : Y \longrightarrow Y$, for any $y \in Y$, $Cl(y)$ is the least element $y' \in \{x \in Y : x = Cl(x)\}$ such that $y \leq y'$. So we obtain the first statements of (1) and (2) due to Proposition 4 since $\{\alpha \in Mon : Cl_1(\alpha) = \alpha\}$ is the class of all the closure operations mapping A into A, and $\{B \subseteq A : Cl_2(B) = B\}$ is the family of all the closure systems contained in A. The explicit form of the operation Cl_1 immediately follows from its definition (comp. the proof for (1) ($\Leftarrow$) of Proposition 4). In order to show the explicit form of Cl_2 we have to show, according to the proof for (2) ($\Leftarrow$) of Proposition 4, that $\{a \in A : \inf_A(B \cap [a)) = a\} = \{\inf_A X : X \subseteq B\}$. The inclusion ($\subseteq$) is obvious. In order to prove the inverse inclusion take any $X \subseteq B$. Then $X \subseteq \{x \in B : \inf_A X \leq x\} = B \cap [\inf_A X)$. Hence $\inf_A(B \cap [\inf_A X)) \leq \inf_A X$. However, on the other hand, the element $\inf_A X$ is a lower bound of the set $B \cap [\inf_A X)$. So $\inf_A X \leq \inf_A(B \cap [\inf_A X))$ and finally $\inf_A X = \inf_A(B \cap [\inf_A X))$. Thus, $\inf_A X \in \{a \in A : \inf_A(B \cap [a)) = a\}$. ■

Now let us consider the dual residuated pair of mappings with respect to Galois connection (f, g). The dual residuated function f_d should be defined by changing in the definition of f the order $\leq$ defined on Mon into its inverse order. But the order in the complete lattice of all monotone mappings from A to A is in turn defined by the order of the lattice $(A, \leq)$. So taking the inverse order on mappings means to take into consideration the inverse order of $\leq$ on A. Therefore we put $f_d(\alpha) = \{x \in A : x \leq \alpha(x)\}$. One can check that so defined map fulfils the condition for being a residuated function for the complete lattices $(Mon, \leq), (\wp(A), \subseteq)$: given $F \subseteq Mon$, $f_d(\inf F) = \{x \in A : x \leq (\inf F)(x)\} = \{x \in A : x \leq \inf_A\{\alpha(x) : \alpha \in F\}\} = \bigcap\{\{x \in A : x \leq \alpha(x)\} : \alpha \in F\} = \bigcap\{f_d(\alpha) : \alpha \in F\}$.

According to the general definition of a residual, we have for any $B \subseteq A:\ g_d(B) = \inf\{\alpha \in Mon : B \subseteq f_d(\alpha)\}$. So for each $a \in A$, $g_d(B)(a) = \inf_A\{\alpha(a) : \alpha \in Mon \ \&\ B \subseteq f_d(\alpha)\} = \inf_A\{\alpha(a) : \alpha \in Mon \ \&\ B \subseteq \{x \in A : x \leq \alpha(x)\}\}$. It is easily seen that given $a \in A$, $g_d(B)(a)$ is an upper bound of the set $\{x \in B : x \leq a\}$ in the lattice $(A, \leq)$. On the other hand, consider any upper bound z of the set $\{x \in B : x \leq a\}$, that is, $\forall x \in B\ (x \leq a \Rightarrow x \leq z)$. Then a monotone mapping α_z defined on A by $\alpha_z(x) = z$ whenever $x \leq a$ otherwise $\alpha_z(x) = 1_A$ (the unit of the complete lattice $(A, \leq)$), is such that $B \subseteq \{x \in A : x \leq \alpha_z(x)\}$. From this and the fact: $\alpha_z(a) = z$, it follows that $z \in \{\alpha(a) : \alpha \in Mon\ \&\ B \subseteq \{x \in A : x \leq \alpha(x)\}\}$ and consequently, $g_d(B)(a) \leq z$. Finally, $g_d(B)(a) = \sup_A\{x \in B : x \leq a\}$. So, given $B \subseteq A$, the mapping $g_d(B)$ is just the interior operation I_B from the previous section so it is monotone.

Now, one can consider the interior operation $Int : Mon \longrightarrow Mon$ and the closure operation $Cl : \wp(A) \longrightarrow \wp(A)$ induced by the residuated pair (f_d, g_d) that is defined by $Int(\alpha) = g_d(f_d(\alpha))$, for any $\alpha \in Mon$ and $Cl(B) = f_d(g_d(B))$, for each $B \subseteq A$. Furthermore, firstly, $\{\alpha \in Mon : Int(\alpha) = \alpha\} = g_d[\wp(A)]$ and $\{B \subseteq A :\ Cl(B) = B\} = f_d[Mon]$. Secondly, the mapping f_d restricted to the set $\{\alpha \in Mon : Int(\alpha) = \alpha\}$ is an isomorphism between the posets $(\{\alpha \in Mon : Int(\alpha) = \alpha\}, \leq)$, $(\{B \subseteq A :\ Cl(B) = B\}, \subseteq)$. Thus, the following proposition is responsible for an isomorphism between the complete lattices of all interior operations and interior systems defined on given complete lattice.

Proposition 6. *(1) For any $\alpha \in Mon$, $Int(\alpha) = \alpha$ iff α is an interior operation on $(A, \leq)$.*

(2) For any $B \subseteq A$, $Cl(B) = B$ iff for each $Y \subseteq B$, $\sup_A Y \in B$, that is B is an interior system in the lattice $(A, \leq)$.

Proof. Analogous to the proof of Proposition 4, by dual argument. ■

Corollary 7. *(1) For any monotone mapping $\alpha : A \longrightarrow A$, $Int(\alpha)$ is the greatest interior operation $I : A \longrightarrow A$ such that $I \leq \alpha$. Explicitly, for any $a \in A :\ Int(\alpha)(a) = I_{f_d(\alpha)}(a) = \sup_A\{x \in A : x \leq \alpha(x)\ \&\ x \leq a\}$.*

(2) For any $B \subseteq A$, $Cl(B)$ is the least interior system $Z \subseteq A$ such that $B \subseteq Z$ (that is $Cl(B) = [B]_{in}$). Explicitly, $Cl(B) = \{\sup_A Y : Y \subseteq B\}$.

Proof. Analogous to the proof of Corollary 5, by dual argument. ■

Example. Consider the lattice $(Mon, \leq)$ of all monotone mappings defined on 4-element lattice $(\{0, a, b, 1\}, \leq)$ with a, b – incomparable elements (Fig. 1). There are 7 closure and 7 interior operations in Mon (Fig. 2). The set Mon may be divided into 7 equivalent classes modulo the equivalence relation θ_f induced on Mon by f (that is $\alpha\theta_f\beta$ iff $f(\alpha) = f(\beta)$) as well as by f_d.

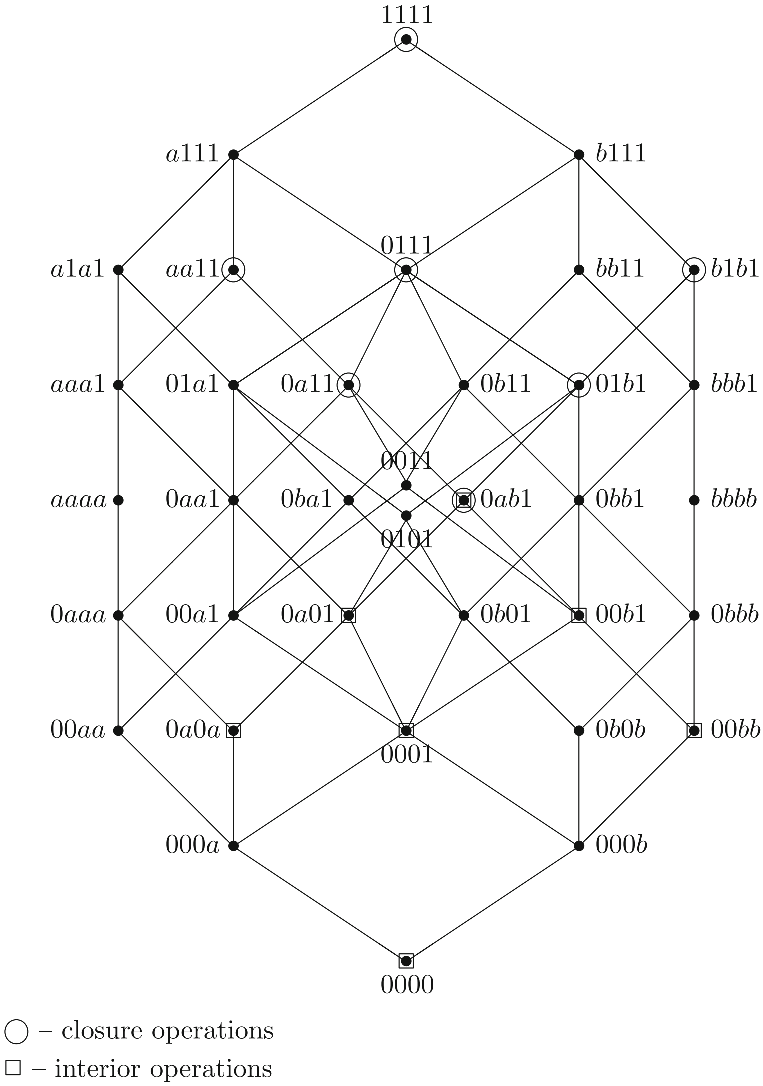

Figure 1. The lattice $(Mon, \leq)$ for the lattice $(\{0, a, b, 1\}, \leq)$

Here we write down the explicit form of each equivalence class modulo θ_f : $\{1111, a111, a1a1, b111, bb11\}$, $\{aa11, aaa1, aaaa\}$, $\{0111, 01a1, 0b11, 0ba1\}$, $\{b1b1, bbb1, bbbb\}$, $\{0a11, 0aa1, 0011, 0aaa, 00a1, 00aa\}$, $\{01b1, 0101, 0bb1, 0b01, 0bbb, 0b0b\}$, $\{0ab1, 0a01, 00b1, 0a0a, 0001, 00bb, 000a, 000b, 0000\}$. In each class at the first place a closure operation occurs. This is the unique

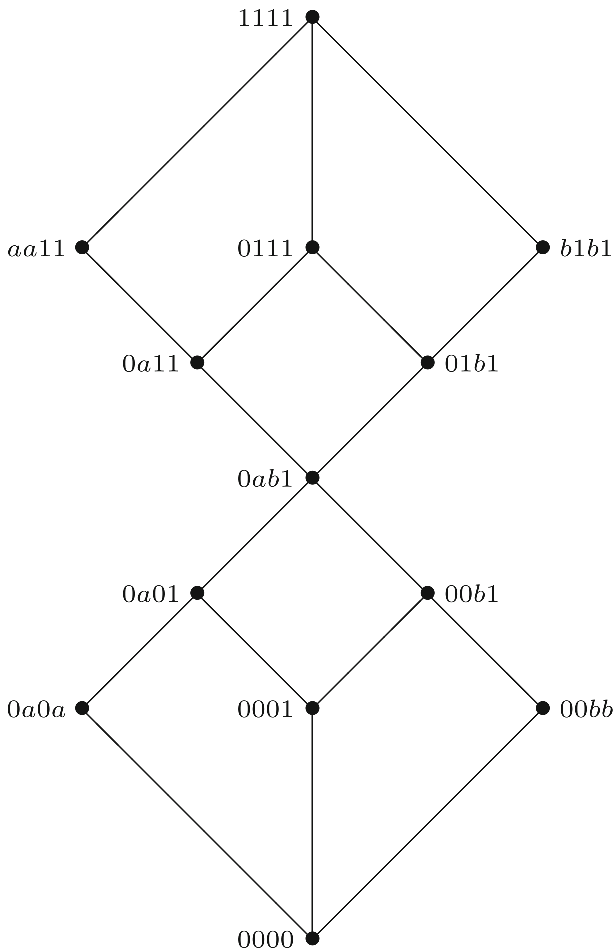

Figure 2. The lattice of all closure and interior operations on the lattice $(\{0, a, b, 1\}, \leq)$

closure operation in a given class, being the greatest element of it, denoted so far as $Cl_1(\alpha)$ or $C_{f(\alpha)}$ for any map α from the equivalence class.

Now let us proceed to a proof of Knaster-Tarski's theorem. To this aim first let us remind that given a monotone mapping $\alpha : A \longrightarrow A$ we have: $Cl_1(\alpha) = g(f(\alpha)) = C_{f(\alpha)}$. Explicitly, for each $a \in A$, $C_{f(\alpha)}(a) = \inf_A\{x \in f(\alpha) : a \leq x\} = \inf_A\{x \in A : \alpha(x) \leq x \ \& \ a \leq x\}$. The set $f(\alpha) = \{x \in A : \alpha(x) \leq x\}$, is the closure system corresponding (by dual isomorphism g) to closure operation $C_{f(\alpha)}$, so $f(C_{f(\alpha)}) = \{x \in A : C_{f(\alpha)}(x) \leq x\} = \{x \in A : C_{f(\alpha)}(x) = x\} = f(\alpha)$. Moreover, $Int(\alpha) = g_d(f_d(\alpha)) = I_{f_d(\alpha)}$, that is

$I_{f_d(\alpha)}(a) = \sup_A\{x \in f_d(\alpha) : x \le a\} = \sup_A\{x \in A : x \le \alpha(x) \;\&\; x \le a\}$. The set $f_d(\alpha) = \{x \in A : x \le \alpha(x)\}$, is the interior system corresponding (by isomorphism g_d) to interior operation $I_{f_d(\alpha)}$, so $f_d(I_{f_d(\alpha)}) = \{x \in A : x \le I_{f_d(\alpha)}(x)\} = \{x \in A : I_{f_d(\alpha)}(x) = x\} = f_d(\alpha)$. Since α is monotone, both systems: $f(\alpha),\ f_d(\alpha)$ are closed on α conceived as an unary operation on A.

PROPOSITION 8. *For all* $\alpha \in Mon$:

(1) the interior system $f_d(\alpha)$ *is closed on the operation* $C_{f(\alpha)}$*: for any* $a \in f_d(\alpha),\ C_{f(\alpha)}(a) \in f_d(\alpha)$*,*

(2) the closure system $f(\alpha)$ *is closed on the operation* $I_{f_d(\alpha)}$*: for any* $a \in f(\alpha),\ I_{f_d(\alpha)}(a) \in f(\alpha)$*.*

PROOF. Assume that $\alpha : A \longrightarrow A$ is any monotone mapping. In order to show (1) suppose that $a \in f_d(\alpha)$. Hence and from the assumption it follows that $a \le \alpha(a) \le \alpha(C_{f(\alpha)}(a))$. Moreover, $\alpha(C_{f(\alpha)}(a)) \in f(\alpha)$ for $C_{f(\alpha)}(a)$ is a closed element and the set $f(\alpha)$ of all closed elements with respect to $C_{f(\alpha)}$ is closed on α. In this way, $\alpha(C_{f(\alpha)}(a)) \in \{x \in f(\alpha) : a \le x\}$. Therefore, $\inf_A\{x \in f(\alpha) : a \le x\} \le \alpha(C_{f(\alpha)}(a))$, that is, $C_{f(\alpha)}(a) \le \alpha(C_{f(\alpha)}(a))$. This means that $C_{f(\alpha)}(a) \in f_d(\alpha)$. Analogously for (2). Obviously, the conditions $(1),\ (2)$ are equivalent due to Lemma 1 $(i) \Leftrightarrow (iv)$. ■

COROLLARY 9. (The Knaster-Tarski's fixed point theorem [10]) *Given a complete lattice* $(A, \le)$ *and a monotone function* $\alpha : A \longrightarrow A$*, the poset* $(B, \le)$*, where* $B = \{x \in A : x = \alpha(x)\}$*, is a complete lattice in which for any* $X \subseteq B,\ \sup X = C_{f(\alpha)}(\sup_A X)$ *and* $\inf X = I_{f_d(\alpha)}(\inf_A X)$*.*

PROOF. By simple application of Lemma 1 for $D = f(\alpha),\ O = f_d(\alpha)$. Any of conditions $(i) - (iv)$ of Lemma 1 is satisfied due to Proposition 8. ■

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