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MOMENTS OF PROBABILITY DISTRIBUTIONS SEMI–ATTRACTED TO SEMI–STABLE MEASURES ON HILBERT SPACES [1]

Let H be a real separable Hilbert space, q a non–degenerate semi–stable distribution on H and $\alpha \in (0,2]$ an exponent for q.

It is proved that the probability distributions semi–attracted to the measure q have absolute moments of order β for $\beta \in (0,\alpha)$ and have no such moments for $\beta > \alpha$ and $\alpha \neq 2$.

Let H be a real separable Hilbert space with the norm $|\cdot|$. Consider the sums

$$(1) \qquad \frac{X_1 + X_2 + \cdots X_{k_n}}{a_n} + x_n$$

where X_j are independent H–valued random variables with a common distribution p, $a_n > 0$, $x_n \in H$ and $\{k_j\}$ is an increasing sequence of positive integers such that

$$(2) \qquad \lim_{n\to\infty} k_{n+1}k_n^{-1} = r < +\infty.$$

The distributions of sums (1) may be written in the form

$$(3) \qquad T_{a_n^{-1}}p^{k_n} * \delta_{x_n},$$

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where the power p^k is taken in the sense of convolution, δ_x denotes the distribution concentrated at a point $x \in H$, and the measure $T_a p$ is defined by the formula

$$T_a p(B) = p\{x \in H : ax \in B\},$$

for all Borel subsets B of H.

A probability measure on H is said to be *semi-stable* if it is a weak limit of sequence (3). W.M. Kruglov gave in [4] a characterization of semi-stable measures. Namely, a measure on H is semi-stable if and only if it is a Gaussian measure or an infinitely divisible purely Poissonian measure represented by a Lévy–Khintchine spectral measure M such that

$$T_\lambda M = \lambda^\alpha M, \tag{4}$$

for some $\alpha \in (0, 2)$ and $\lambda \in (0, +\infty) \setminus \{1\}$.

The class of semi-stable measures is a subclass of infinitely divisible measures and is a natural extension of the class of stable measures. For this reason, in the sequel, the number α in (4) will be called an exponent for a purely Poissonian semi-stable measure (the exponent for a Gaussian measures is equal to 2). Semi-stable measures have their domains of semi-attraction. Namely, by a *domain of semi-attraction* of a semi-stable measure q we mean a class of distributions p such that sequence (3) converges weakly to q for some $a_n > 0$, $x_n \in H$ and $\{k_n\}$ satisfying (2). We shall also say that *p is semi-attracted to q* if p belongs to this class.

The theorems on moments of measures attracted to stable laws can be found in [1] and [5]. We shall prove an analogous theorem for distributions semi-attracted to semi-stable measures on H. Our proof is elementary in the case $r > 1$ and, in the case $r = 1$ (the stable case), we can apply the same method. Consequently, if we reduce the problem to measures attracted to stable laws on a straight line, then we obtain a proof simpler than the classical one.

Theorem. *Let q be a non-degenerate semi-stable measure on H, and $\alpha \in (0, 2]$ an exponent for q. If a distribution p on H is semi-attracted to the measure q, then*

$$\int_H |x|^\beta p(dx) < +\infty, \qquad \textit{for} \quad \beta \in (0, \alpha)$$

and
$$\int_H |x|^\beta p(dx) = +\infty, \quad \text{for} \quad \beta > \alpha, \quad \alpha \neq 2.$$

Proof. We shall consider several cases.

CASE I. Let $\alpha \in (0,2)$. Thus the measure q is represented by a Lévy–Khintchine measure $M \not\equiv 0$. From the assumption we can find sequences $\{a_n\}$, $\{x_n\}$, $\{k_n\}$ such that

$$\lim_{n\to\infty} T_{a_n^{-1}} p^{k_n} * \delta_{x_n} = q. \tag{5}$$

Using Corollary in [2] and Lemma 7.1 in [6], we obtain the following:

for any $\varepsilon > 0$, the sequence of measures $\{k_n T_{a_n^{-1}} p\}$ restricted to the set
$$\{x \in H : |x| > \varepsilon\}$$
is weakly convergent to the measure M restricted to the same set.

In particular, we have, for some $t > 0$,

$$\lim_{n\to\infty} k_n p\{x \in H : |x| > t a_n\} = M\{x \in H : |x| > t\} > 0. \tag{6}$$

CASE IA. Let
$$\lim_{n\to\infty} k_{n+1} k_n^{-1} = r > 1.$$
Then
$$\lim_{n\to\infty} a_n a_{n+1}^{-1} = a \in (0,1)$$
and
$$a^\alpha r = 1$$
(see [4]). Of course,
$$\lim_{n\to\infty} a_n = +\infty$$
and, since $a < 1$, we can assume that $\{a_n\}$ is an increasing sequence. By (6), putting $b_n = t a_n$, we have

$$\lim_{n\to\infty} \frac{p\{x \in H : |x| > b_{n+1}\}}{p\{x \in H : |x| > b_n\}} = \lim_{n\to\infty} k_n k_{n+1}^{-1} = r^{-1}. \tag{7}$$

Let us now consider a series of the form

$$\sum_{n=1}^{\infty} b_n^{\beta} p\{x \in H : |x| > b_n\}.$$

By (7) and d'Alembert's Criterion, we obtain the convergence of the series if $a^{-\beta} r^{-1} < 1$, i.e. if $\beta \in (0, \alpha)$, and the divergence if $\beta > \alpha$. It now suffices to make use of the inequalities

$$(8) \qquad \int_{|x|>b_1} |x|^{\beta} p(dx) = \sum_{n=1}^{\infty} \int_{b_n < |x| \leq b_{n+1}} |x|^{\beta} p(dx)$$
$$\leq \sum_{n=1}^{\infty} \Big(\frac{b_{n+1}}{b_n}\Big)^{\beta} b_n^{\beta} p\{x \in H : |x| > b_n\}$$

and

$$(9) \qquad \int_{|x|>b_1} p(dx)$$
$$\geq \sum_{n=1}^{\infty} b_n^{\beta} p\{x \in H : |x| > b_n\}\Big(1 - \frac{p\{x \in H : |x| > b_{n+1}\}}{p\{x \in H : |x| > b_n\}}\Big).$$

CASE IB. Let

$$\lim_{n\to\infty} k_{n+1} k_n^{-1} = r = 1.$$

Then

$$\lim_{n\to\infty} a_n a_{n+1}^{-1} = a = 1$$

and q is a stable measure (see [4]). Consequently, the measure M has the property

$$T_{\lambda} M = \lambda^{\alpha} M,$$

for each $\lambda > 0$.

Thus, for all $t > 0$, the set

$$\{x \in H : |x| > t\}$$

is a continuity set of M and condition (6) is satisfied for each $t > 0$. This implies that

$$\lim_{n\to\infty} \frac{p\{x \in H : |x| > ta_n\}}{p\{x \in H : |x| > a_n\}} = t^{-\alpha}$$

for each $t > 0$, and, since $a = 1$, we obtain

$$\lim_{u\to\infty} \frac{p\{x \in H : |x| > tu\}}{p\{x \in H : |x| > u\}} = t^{-\alpha} \tag{10}$$

for each $t > 0$.

By putting $t = 2$ and $u = 2^n$ in (10), we have

$$\lim_{n\to\infty} \frac{2^\beta p\{x \in H : |x| > 2^{n+1}\}}{p\{x \in H : |x| > 2^n\}} = 2^{\beta-\alpha}.$$

Thus, the series of the form

$$\sum_{n=1}^{\infty} (2^n)^\beta p\{x \in H : |x| > 2^n\}$$

is convergent for $\beta \in (0, \alpha)$ and is divergent for $\beta > \alpha$. It now suffices to use inequalities (8) and (9) for $b_n = 2^n$.

CASE II. Let $\alpha = 2$. In this case we can assume that

$$\int_H |x|^2 p(dx) = +\infty.$$

Now, q is a Gaussian measure on H represented by a non-negative, self-adjoint operator S with a positive finite trace. Let $\{a_n\}$, $\{x_n\}$, $\{k_n\}$ be sequences such that the condition of form (5) is satisfied. In the same way as in the proof of Theorem 3.2 in [3], condition (5) implies now

$$\lim_{n\to\infty} k_n a_n^{-2} \int_{|x|<ta_n} |x|^2 p(dx) = trS > 0 \tag{11}$$

for each $t > 0$.

CASE IIA. Let $r > 1$. Then $a \in (0, 1)$ and $a^2 r = 1$ (see [4]). We have

$$\lim_{n\to\infty} a_n = +\infty$$

and, since $a < 1$, we can assume that $\{a_n\}$ is an increasing sequence. For $\beta \in (0, 2)$, we have the following inequality:

$$\int_{|x|\geq a_1} |x|^\beta p(dx) \leq \sum_{n=1}^{\infty} a_n^{\beta-2} \int_{|x|<a_{n+1}} |x|^2 p(dx). \tag{12}$$

By using (11) for $t = 1$, we obtain

$$\lim_{n\to\infty} \frac{a_{n+1}^{\beta-2} \int_{|x|<a_{n+1}} |x|^2 p(dx)}{a_n^{\beta-2} \int_{|x|<a_n} |x|^2 p(dx)} = a^{-\beta} \lim_{n\to\infty} k_n k_{n+1}^{-1}$$
$$= a^{-\beta} r^{-1} < a^{-2} r^{-1} = 1.$$

Thus the series in (12) is convergent and

$$\int_H |x|^2 p(dx) < +\infty$$

for $\beta \in (0, 2)$.

CASE IIB. Let $r = 1$. Condition (11) implies that

$$\lim_{n\to\infty} \frac{\int_{|x|<ta_n} |x|^2 p(dx)}{\int_{|x|<a_n} |x|^2 p(dx)} = 1$$

for each $t > 0$ and, since $a = 1$, we further have

$$\lim_{u\to\infty} \frac{\int_{|x|<tu} |x|^2 p(dx)}{\int_{|x|<u} |x|^2 p(dx)} = 1 \tag{13}$$

for each $t > 0$.

Putting $t = 2$ and $u = 2^n$ in (13), we obtain, for $\beta \in (0, 2)$,

$$\lim_{n\to\infty} \frac{2^{\beta-2} \int_{|x|<2^{n+1}} |x|^2 p(dx)}{\int_{|x|<2^n} |x|^2 p(dx)} = 2^{\beta-2} < 1.$$

The above inequality means that the series

$$\sum_{n=1}^{\infty} (2^n)^{\beta-2} \int_{|x|<2^n} |x|^2 p(dx)$$

is convergent. It now suffices to make use of inequality (12) by putting $a_n = 2^n$.

Our theorem implies the following

Corollary. *Every semi-stable measure on H has exactly one exponent.*

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MOMENTY ROZKŁADÓW PÓŁPRZYCIĄGANYCH PRZEZ MIARY PÓŁSTABILNE W PRZESTRZENI HILBERTA

Niech H będzie rzeczywistą, ośrodkową przestrzenią Hilberta, q – niezdegenerowanym półstabilnym rozkładem prawdopodobieństwa na H, a $\alpha \in (0,2]$ – wykładnikiem rozkładu q. W pracy udowodniono, że rozkład prawdopodobieństwa na H półprzyciągany przez q ma momenty absolutne rzędu β dla $\beta \in (0,\alpha)$ i nie ma takich momentów dla $\beta > \alpha$ i $\alpha \neq 2$.

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