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ON PROPER SUBUNIVERSES OF A BOOLEAN ALGEBRA

Let $\boldsymbol{B}$ be a boolean algebra with the universe B and let F_1, F_2 be distinct ultrafilters of $\boldsymbol{B}$. Then the set of the form $\{x \in B : F_1 \cap F_2 \cap \{x, \neg x\} \neq \emptyset\}$ is a maximal proper subuniverse of $\boldsymbol{B}$ which we shall call a *basic subuniverse*. We prove that every proper subuniverse of $\boldsymbol{B}$ is an intersection of a family of basic subuniverses. This implies that basic subuniverses are precisely maximal proper subuniverses of a boolean algebra. The same fact proved in another way can be found in [3].

For general algebraic background we refer the reader to [1] and for boolean algebras to [2]. In order to simplify notations we use the same symbols for boolean algebras and for their corresponding universes. If B is a boolean algebra, $X \subseteq B$ and $b \in B$, then by $\neg_B X$ and $(b]_B$ we denote the sets

$$\{\neg x : x \in X\}$$

and

$$\{x \in B : x \leq b\},$$

respectively. Observe that, for every filter F of B, the set

$$F \cup \neg_B F$$

is a subuniverse of B. The subuniverse of the form $F \cup \neg_B F$ will be further denoted by $B|F$. Recall that a filter B of F is an ultrafilter – i.e. a maximal proper filter – if and only if, for every $b \in B$, we have

$$\{b, \neg b\} \cap F \neq \emptyset.$$

This fact easily implies that, for every proper filter F, the equality $B = B|F$ holds precisely in the case when the filter F is an ultrafilter.

We will denote the set-theoretical operations of difference and symmetric difference by $\setminus$ and $\triangle$, respectively.

We start with the following auxiliary proposition.

Two Ultrafilters Lemma. *If B is a boolean algebra, F_1, F_2 are ultrafilters of B and $a, b \in B$, then*

$$F_1 \cap F_2 \cap \{a, \neg a, b, \neg b, a \div b, \neg(a \div b)\} \neq \emptyset.$$

Here $\div$ denotes the operation of symmetric difference, i.e.

$$a \div b = (a - b) \vee (b - a),$$

where $-$ and $\vee$ denote difference and sum, respectively.

Proof. Suppose that

$$F_1 \cap F_2 \cap \{a, \neg a, b, \neg b, a \div b, \neg(a \div b)\} = \emptyset.$$

If $a \div b \in F_1$, then

$$\{a - b, b - a\} \cap F_1 \neq \emptyset.$$

If $a - b \in F_1$, then

$$a, \neg b \in F_1$$

and, therefore,

$$\neg a, b, \neg(a \div b) \in F_2,$$

which is impossible because

$$\neg a \wedge b \wedge \neg(a \div b) = \emptyset.$$

We have shown that

$$a \div b \notin F_1$$

and this means that

$$\neg(a \div b) \in F_1.$$

In the same manner we can infer that $\neg(a \div b) \in F_2$ which gives us that

$$\neg(a \div b) \in F_1 \cap F_2,$$

thus, we get a contradiction.

By a basic subuniverse of a boolean algebra B we mean any subuniverse of the form

$$B|(F_1 \cap F_2),$$

where F_1, F_2 are distinct ultrafilters of B. Recall that

$$B|(F_1 \cap F_2) = (F_1 \cap F_2) \cup \neg_B(F_1 \cap F_2) =$$

$$\{x \in B : F_1 \cap F_2 \cap \{x, \neg x\} \neq \emptyset\} = B \setminus (F_1 \triangle F_2).$$

Let us note the following proposition.

Lemma 1. *Every basic subuniverse of a boolean algebra is a maximal proper subuniverse.*

Proof. Suppose that B is a boolean algebra and F_1, F_2 are distinct ultrafilters of B. Then $F_1 \cap F_2$ is not an ultrafilter and consequently the subuniverse of the form

$$B|(F_1 \cap F_2)$$

is proper. Now we pick an element $b \in B \setminus (B|(F_1 \cap F_2))$ and we will show that the algebra genarated by the set

$$(F_1 \cap F_2) \cup \{b\}$$

generates B. Indeed, by Two Ultrafilters Lemma it follows that, for every $a \in B \setminus (B|(F_1 \cap F_2))$,

$$\{b \div a, \neg b \div a\} \cap F_1 \cap F_2 \neq \emptyset.$$

This means that every such a can be expressed in terms of generators because

$$b \div (b \div a) = \neg b \div (\neg b \div a) = a.$$

Lemma 2. *If A is a proper subuniverse of a boolean algebra B, then for every $b \in B \setminus A$ there exists an ultrafilter F of A such that*

$$(b]_B \cap F = (\neg b]_B \cap F = \emptyset.$$

Proof. Suppose that $b \in B \setminus A$ is such that, for every ultrafilter F of A, we have

$$(b]_B \cap F \neq \emptyset$$

or

$$(\neg b]_B \cap F \neq \emptyset.$$

For every ultrafilter F of A, we pick an element $q_F \in F$ such that $q_F \leq b$ or $q_F \leq \neg b$ and we define a subset Q of A as the set of all q_F obtained in the above manner. We aim at showing that Q can be extended to a proper ideal of A and thus we have to prove that every finite subset of Q has a non-unit supremum. Suppose that, for some finite $X \subseteq Q$, $\sup(X) = \mathbf{1}$. Put

$$X_1 = (b]_B \cap X$$

and

$$X_2 = (\neg b]_B \cap X.$$

Since $X \subseteq (b]_B \cup (\neg b]_B$, then

$$X = X_1 \cup X_2$$

and thus

$$\mathbf{1} = \sup(X) = \sup(X_1) \vee \sup(X_2),$$

where $\sup(X_1) \leq b$ and $\sup(X_2) \leq \neg b$. Now, we get that

$$b - \sup(X_1) =$$
$$(b - \sup(X_1)) \wedge (\sup(X_1) \vee \sup(X_2)) =$$
$$((b - \sup(X_1)) \wedge \sup(X_1)) \vee ((b - \sup(X_1)) \wedge \sup(X_2))$$
$$\leq (b - \sup(X_1)) \wedge \neg b = \mathbf{o}.$$

This would mean that $\sup(X_1) = b$ which is not possible since $X_1 \subseteq A$ and $b \in B - A$.

We have shown that every finite subset of Q has a non-unit supremum which implies that the set $\neg_A Q$ can be extended to an ultrafilter of A. This, however is a clear contradiction because – by the definition of Q – every ultrafilter of A must contain the complement of an element of $\neg_A Q$.

Theorem 1. *Every proper subuniverse of a boolean algebra is an intersection of a family of basic subuniverses.*

Proof. Suppose that A is a proper subuniverse of a boolean algebra B and $b \in B \setminus A$. We need only to show that b does not belong to some basic subuniverse of B containing A. By Lemma 2, we get that there exists an ultrafilter F of A such that

$$(b]_B \cap F = (\neg b]_B \cap F = \emptyset.$$

Let F_1, F_2 be ultrafilters of B such that

$$b \notin F_1, \qquad \neg b \notin F_2$$

and

$$F \subseteq F_1 \cap F_2.$$

Then we have

$$b \notin B|(F_1 \cap F_2)$$

and

$$B|(F_1 \cap F_2) \supseteq A|F = A,$$

as required.

Theorem 2. *Maximal elements of the set of all proper subuniverses of a boolean algebra are precisely all its basic subuniverses.*

Proof. By Lemma 1, it follows that all basic subuniverses are maximal proper subuniverses. To prove the converse inclusion observe that every proper subuniverse must be contained in a basic subuniverse – by Theorem 1. Thus a maximal proper subuniverse must be equal to the basic subuniverse containing it.

References

[1]. G.Grätzer, Universal Algebra, 2-nd edition, Springer-Verlag, New York, 1979.

[2]. R.Sikorski, *Boolean Algebras*, Springer-Verlag, Berlin, 1964.

[3]. J.Donald Monk with the cooperation of Robert Bonnet, *Handbook of Boolean Algebras*, vol. 2, North-Holland, 1989.

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O PODALGEBRACH ALGEBRY BOOLE'A

Niech **B** będzie algebrą Boole'a z uniwersum B i niech F_1, F_2 będą różnymi ultrafiltrami **B**. Wówczas zbiór postaci $\{x \in B : F_1 \cap F_2 \cap \{x, \neg x\} \neq \emptyset\}$ jest maksymalną podalgebrą **B** która nazywać będziemy *podalgebrą bazową*. Udowodnimy, że każda właściwa podalgebrą **B** jest iloczynem rodziny podalgebr bazowych. Pozwala to stwierdzić, że podalgebry bazowe są wszystkimi podalgebrami maksymalnymi algebry Boole'a. Ten sam fakt jakkolwiek dowiedziony w inny sposób można znaleźć w [3].

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