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ON ALMOST INVARIANT SUBSETS OF THE REAL LINE

Some partitions of the real line, consisting of almost invariant sets, are considered and one theorem of Sierpiński concerning such partitions is generalized.

Let E be an infinite basic set. We denote by $Sym(E)$ the family of all bijective mappings acting from E onto E. Obviously, $Sym(E)$ is a group with respect to the operation of composition of mappings. Let us fix a subgroup G of $Sym(E)$. The pair (E, G) is usually called a space equipped with a transformation group. If the group G acts transitively in E, then the pair (E, G) is called a homogeneous space (with respect to G).

Let X be a subset of E. We say that X is almost G-invariant (or X is almost invariant with respect to G) if, for each transformation $g \in G$, we have the inequality

$$card(g(X) \triangle X) < card(E),$$

where the symbol $\triangle$ denotes, as usual, the operation of symmetric difference of two sets.

Evidently, the following three relations hold:

1) if a set X is almost G-invariant, then the set $E \setminus X$ is almost G-invariant, too;

2) if sets X and Y are almost G–invariant, then the set $X \cup Y$ is almost G–invariant, too;

3) if we have the inequality

$$cf(card(E)) > \omega$$

and $\{X_n : n \in \omega\}$ is an arbitrary countable family of almost G–invariant subsets of E, then $\cup\{X_n : n \in \omega\}$ also is an almost G–invariant subset of E.

In particular, relations 1) and 2) show us that the family of all almost G–invariant sets forms an algebra of subsets of E. Relation 3) shows us that if $cf(card(E)) > \omega$, then the same family forms a σ–algebra of subsets of E.

We want to remark that almost invariant sets play an important role in the theory of invariant (or, more generally, quasiinvariant) measures. Some applications of such sets to the theory of invariant extensions of the classical Lebesgue measure are considered in [1] and [2].

There are many interesting examples of almost invariant subsets of the real line $\mathbf{R}$ (see, for instance, [2], [3] and [4]). One of the earliest examples is due to Sierpiński (see [5]). Namely, Sierpiński constructed, using the method of transfinite recursion, a partition $\{X, Y\}$ of $\mathbf{R}$ such that

a) $card(X) = card(Y) = card(\mathbf{R})$;

b) for each $g \in \mathbf{R}$, the inequalities

$$card((g + X)\triangle X) < card(\mathbf{R}), \quad card((g + Y)\triangle Y) < card(\mathbf{R})$$

are fulfilled.

In particular, both the sets X and Y are almost $\mathbf{R}$–invariant subsets of $\mathbf{R}$. Moreover, it is possible to show, by the same method, that the partition $\{X, Y\}$ mentioned above can have some additional properties. For instance, the sets X and Y can be Bernstein subsets of the real line $\mathbf{R}$ (for the definition of a Bernstein subset of $\mathbf{R}$, see e.g. [3] or [4]). Notice also that if Martin's Axiom holds, then one of the sets X and Y can be a Lebesgue measure zero subset of $\mathbf{R}$ (or a first category subset of $\mathbf{R}$). But it is reasonable to remark here that X and Y cannot be Borel subsets of the real line.

If the Continuum Hypothesis holds, then the Sierpiński partition $\{X, Y\}$ satisfies the following two conditions:

(a) $card(X) = card(Y) = card(\mathbf{R})$;

(b) for each $g \in \mathbf{R}$, we have

$$card((g + X)\triangle X) \leq \omega, \quad card((g + Y)\triangle Y) \leq \omega.$$

Conversely, it is not difficult to prove that the existence of a partition $\{X, Y\}$ of $\mathbf{R}$ satisfying conditions (a) and (b) implies the Continuum Hypothesis.

Now, let G be an uncountable subgroup of the additive group of $\mathbf{R}$. Suppose that $\{X, Y\}$ is a partition of the real line such that

(1) $card(X) = card(Y) = card(\mathbf{R})$;

(2) for any element $g \in G$, the inequalities

$$card((g + X)\triangle X) \leq \omega, \quad card((g + Y)\triangle Y) \leq \omega$$

are fulfilled.

Then the following question naturally arises: does the Continuum Hypothesis hold in such a situation? Clearly, the answer to this question is negative. Indeed, it is easy to see that if G is a proper subgroup of $\mathbf{R}$ with

$$card(G) = card(\mathbf{R}),$$

then the partition

$$\{X, Y\} = \{G, \mathbf{R} \setminus G\}$$

consists of two G–invariant subsets of the real line and, hence, satisfies conditions (1) and (2), but the cardinality of the continuum (denoted by $\mathbf{c}$) can be strictly greater than the first uncountable cardinal number ω_1. Thus, if we want to deduce the Continuum Hypothesis from the existence of a partition $\{X, Y\}$ of $\mathbf{R}$ satisfying conditions (1) and (2), we must have some additional information about $\{X, Y\}$. In our further considerations we shall discuss some properties of $\{X, Y\}$ which enable us to obtain the corresponding result. Notice that those properties will be formulated in terms of G–orbits of points of $\mathbf{R}$.

Suppose that G is an arbitrary subgroup of the additive group of the real line. Let us recall that the G–orbit of a point $z \in \mathbf{R}$ is the following set:

$$G(z) = \{g + z \ : \ g \in G\}.$$

Obviously, the family of all G–orbits forms a partition of $\mathbf{R}$ (which is usually called the partition of $\mathbf{R}$ canonically associated with the given group G).

Let $\{X, Y\}$ be a partition of $\mathbf{R}$ into two subsets of $\mathbf{R}$.

We say that a G–orbit $G(z)$ is X–admissible (respectively, Y–admissible) if $G(z) \subseteq X$ (respectively, $G(z) \subseteq Y$).

We say that a G–orbit $G(z)$ is in general position with respect to the partition $\{X, Y\}$ if

$$G(z) \cap X \neq \emptyset, \quad G(z) \cap Y \neq \emptyset.$$

Using these notions we can consider the following example.

Example 1. Let G be a countable subgroup of $\mathbf{R}$ and let $\{X, Y\}$ be a partition of $\mathbf{R}$ such that

1) $card(X) > \omega, \quad card(Y) > \omega$;

2) for all elements $g \in G$, we have the inequalities

$$card((g + X)\Delta X) \leq \omega, \quad card((g + Y)\Delta Y) \leq \omega.$$

Then it can be shown that the subsequent three relations hold:

a) there exists an uncountable family of X–admissible G–orbits;

b) there exists an uncountable family of Y–admissible G–orbits;

c) the family of all those G–orbits which are in general position with respect to the partition $\{X, Y\}$ is at most countable.

Indeed, suppose that relation c) is not true. Then there exists an uncountable subset $\{z_i \ : \ i \in I\}$ of $\mathbf{R}$ such that the corresponding family of G–orbits

$$\{G(z_i) \ : \ i \in I\}$$

is disjoint and all G–orbits from this family are in general position with respect to $\{X, Y\}$. Consequently, for each index $i \in I$, there are elements

$$x_i \in G(z_i) \cap X, \quad y_i \in G(z_i) \cap Y.$$

Let us put

$$\{g_i \ : \ i \in I\} = \{y_i - x_i \ : \ i \in I\}.$$

Evidently, we have $g_i \in G$ for every $i \in I$. Since G is a countable group and I is an uncountable set, there exists an element $g \in G$ such that

$$card(\{i \in I : g_i = g\}) > \omega.$$

Hence, we obtain the inequality

$$card((g + X) \cap Y) > \omega,$$

which yields a contradiction with condition 2). This contradiction shows us that relation c) is true. Now, taking into account condition 1), it is easy to show that relations a) and b) are fulfilled, too.

Conversely, suppose that a partition $\{X, Y\}$ of the real line satisfies relations a), b) and c). Then it is not difficult to check that conditions 1) and 2) hold for $\{X, Y\}$.

Thus, for an arbitrary countable subgroup G of $\mathbf{R}$, we have a simple geometrical (or, if one prefers, algebraic) characterization of all partitions $\{X, Y\}$ of $\mathbf{R}$ satisfying conditions 1) and 2). We shall see below that for uncountable subgroups G of $\mathbf{R}$ we have an essentially different situation.

Let G be an uncountable subgroup of the real line $\mathbf{R}$ and let $\{X, Y\}$ be a partition of this line into two uncountable subsets. Further, let $G(z)$ be the G-orbit of a point $z \in \mathbf{R}$. We say that $G(z)$ is X-singular if

$$0 < card(G(z) \cap X) \leq \omega.$$

In the analogous way, we say that $G(z)$ is Y-singular if

$$0 < card(G(z) \cap Y) \leq \omega.$$

It immediately follows from this definition that every X-singular (Y-singular) G-orbit is in general position with respect to the given partition $\{X, Y\}$.

Let G be again an uncountable subgroup of $\mathbf{R}$ and $\{X, Y\}$ be a partition of $\mathbf{R}$ into two uncountable sets. We say that the partition $\{X, Y\}$ is admissible for the group G if the following three conditions are fulfilled:

(1) the family of all X-singular G-orbits is at most countable;

(2) the family of all Y–singular G–orbits is at most countable;

(3) if the G–orbit $G(z)$ of an arbitrary point $z \in \mathbf{R}$ is not X–singular and is not Y–singular, then $G(z)$ is X–admissible or Y–admissible.

It is easy to see that the following statement is true.

Proposition 1. *If a partition $\{X, Y\}$ of the real line $\mathbf{R}$ is admissible for a subgroup G of $\mathbf{R}$, then*

$$card((g + X)\triangle X) \leq \omega, \quad card((g + Y)\triangle Y) \leq \omega,$$

for all elements $g \in G$. In particular, X and Y are almost G–invariant subsets of $\mathbf{R}$.

We want to make some remarks in connection with Proposition 1. Let G and $\{X, Y\}$ satisfy the assumptions of this proposition and let κ be an uncountable cardinal number strictly less than the cardinality of the continuum $\mathbf{c}$. In general, we cannot deduce, for the group G, the inequality

$$card(G) \leq \kappa.$$

Indeed, let us consider a subgroup Γ of $\mathbf{R}$ such that

a) $card(\Gamma) = \mathbf{c}$;

b) $card(\mathbf{R}/\Gamma) > \omega$.

Then it is not difficult to construct a partition $\{A, B\}$ of the real line $\mathbf{R}$ into two uncountable subsets of $\mathbf{R}$ such that

(1) the family of all A–singular Γ–orbits is infinite and countable;

(2) the family of all B–singular Γ–orbits is infinite and countable;

(3) the family of all A–admissible Γ–orbits is uncountable;

(4) the family of all B–admissible Γ–orbits is uncountable;

(5) if a Γ–orbit is not A–singular and is not B–singular, then it is A–admissible or B–admissible.

We see, in particular, that

$$card(A) = card(B) = \mathbf{c}$$

and the partition $\{A, B\}$ is admissible for the group Γ.

Thus, we can conclude that the existence of an admissible partition $\{X, Y\}$ of $\mathbf{R}$, for a given uncountable subgroup G of $\mathbf{R}$, does not imply, in general, any upper estimation of the cardinality of G.

On the other hand, we shall show in our further considerations that if G is an uncountable subgroup of $\mathbf{R}$ and $\{X, Y\}$ is a partition of $\mathbf{R}$ satisfying the relations

$$card(X) > \omega, \quad card(Y) > \omega,$$

$$card((g+X)\triangle X) \leq \omega, \quad card((g+Y)\triangle Y) \leq \omega,$$

for all elements $g \in G$, and, in addition, $\{X, Y\}$ is not admissible for the group G, then the equality

$$card(G) = \omega_1$$

is fulfilled.

In order to establish this result, we need the following

Proposition 2. *Let $(G, +)$ be an arbitrary uncountable commutative group and let $\{X, Y\}$ be a partition of G into two uncountable subsets such that*

$$card((g+X)\triangle X) \leq \omega, \quad card((g+Y)\triangle Y) \leq \omega,$$

for every element $g \in G$. Then we have the equality $card(G) = \omega_1$.

Proof. The argument is essentially due to Sierpiński (cf. [5]). First of all we deduce from the assumptions of Proposition 2 that

$$card(X) = card(Y) = card(G).$$

Further, since $card(X) > \omega$, we can fix a subset Z of X such that

$$card(Z) = \omega_1.$$

Then it is not difficult to check that the inclusion

$$Y \subseteq \cup\{(X - z) \setminus X \ : \ z \in Z\}$$

holds. But, for each $z \in Z$, we have the inequality

$$card((X - z) \setminus X) \leq \omega.$$

Consequently, we get the inequality

$$card(Y) \leq \omega \cdot \omega_1 = \omega_1.$$

Taking into account the fact that $card(Y) = card(G)$ and that G is an uncountable group, we obtain the desired equality $card(G) = \omega_1$.

Now, we can formulate and prove the next statement.

Proposition 3. *Let G be an uncountable subgroup of the additive group of the real line $\mathbf{R}$ and let $\{X, Y\}$ be a partition of this line such that*

1) $card(X) > \omega$, $card(Y) > \omega$;

2) for all elements $g \in G$, we have

$$card((g + X)\Delta X) \leq \omega, \quad card((g + Y)\Delta Y) \leq \omega;$$

3) the partition $\{X, Y\}$ is not admissible for the group G.

Then the equality $card(G) = \omega_1$ is true.

Proof. Since the partition $\{X, Y\}$ is not admissible for G, at least one of the following three assertions holds:

a) there exists a G-orbit $G(z)$ such that

$$card(G(z) \cap X) > \omega, \quad card(G(z) \cap Y) > \omega;$$

b) the family of all X-singular G-orbits is uncountable;

c) the family of all Y-singular G-orbits is uncountable.

First let us consider the case when assertion a) is true. Let $G(z)$ be an arbitrary G-orbit satisfying a). It is easy to see that the set $G(z)$ can canonically be equipped with the structure of a commutative group isomorphic to the original group G. We denote this new group by the symbol $(G^*, +)$ (notice that the zero of G^* coincides with the point z). Also it is not difficult to verify that the sets

$$X^* = G(z) \cap X, \quad Y^* = G(z) \cap Y$$

form a partition of the group G^* such that

$$card(X^*) > \omega, \quad card(Y^*) > \omega$$

and, for each element $g \in G^*$, the inequalities

$$card((g + X^*)\triangle X^*) \leq \omega, \quad card((g + Y^*)\triangle Y^*) \leq \omega$$

are fulfilled. Hence, we can directly apply Proposition 2 to the group $(G^*, +)$ and to the partition $\{X^*, Y^*\}$. Applying the above–mentioned proposition, we obtain

$$card(G) = card(G^*) = \omega_1.$$

Now, let us consider the case when assertion b) is true. We take an uncountable family $\{z_i \ : \ i \in I\}$ of points of $\mathbf{R}$ such that the corresponding family of G–orbits

$$\{G(z_i) \ : \ i \in I\}$$

is disjoint and, for each index $i \in I$, we have the inequalities

$$0 < card(G(z_i) \cap X) \leq \omega.$$

Of course, without loss of generality, we may assume that

$$card(I) = \omega_1.$$

We want to show that $card(G) = \omega_1$, too. Suppose otherwise, i.e. $card(G) > \omega_1$. For any $i \in I$, let us denote

$$X_i = G(z_i) \cap X$$

and then let us put

$$Z = \cup\{X_i - z_i \ : \ i \in I\}.$$

Obviously, we have the relations

$$Z \subseteq G, \quad card(Z) \leq \omega_1.$$

Consequently, there exists an element $h \in G$ such that

$$(h + Z) \cap Z = \emptyset.$$

Now, it is easy to check that, for every $i \in I$, the inclusion

$$h + X_i \subseteq G(z_i) \cap Y$$

is fulfilled. From this fact we immediately obtain the inequality

$$card((h + X) \cap Y) \geq \omega_1,$$

which gives us a contradiction with the relation

$$card((h + X) \Delta X) \leq \omega.$$

Taking into account the obtained contradiction, we conclude that the desired equality $card(G) = \omega_1$ must be true.

The case when assertion c) holds is analogous to the previous case.

Thus, the proof of Proposition 3 is complete.

The next statement is an easy consequence of Proposition 3.

Proposition 4. *Let G be a subgroup of the real line with $card(G) = \mathbf{c}$ and let $\{X, Y\}$ be a partition of this line into two uncountable subsets such that*

$$card((g + X) \Delta X) \leq \omega, \quad card((g + Y) \Delta Y) \leq \omega,$$

for all elements $g \in G$. Then at least one of the following two assertions is true:

1) the Continuum Hypothesis ($\mathbf{c} = \omega_1$);

2) the partition $\{X, Y\}$ is admissible for the group G.

Notice, in connection with Proposition 4, that if $G = \mathbf{R}$, then assertion 2) is false and, therefore, the Continuum Hypothesis is fulfilled.

The next example describes a situation where $\{X, Y\}$ is a partition of $\mathbf{R}$ into two uncountable almost G-invariant subsets and all G-orbits are X-singular (cf. [6] and [7]).

Example 2. Suppose that the Continuum Hypothesis holds. Let $\mathbf{R}^2$ be the Euclidean plane. Consider the straight line $\{0\} \times \mathbf{R}$ lying in this plane. Clearly, $\{0\} \times \mathbf{R}$ is an uncountable subgroup of the additive group of $\mathbf{R}^2$. Since we have the equalities

$$card(\{0\} \times \mathbf{R}) = \mathbf{c} = \omega_1,$$

there exists an ω_1–sequence $\{\Gamma_\xi \ : \ \xi < \omega_1\}$ of subsets of $\{0\} \times \mathbf{R}$ such that

a) $card(\Gamma_\xi) = \omega$, for each ordinal $\xi < \omega_1$;

b) Γ_ξ is a subgroup of $\{0\} \times \mathbf{R}$, for each ordinal $\xi < \omega_1$;

c) the family $\{\Gamma_\xi \ : \ \xi < \omega_1\}$ is strictly increasing with respect to inclusion;

d) the union of this family is equal to $\{0\} \times \mathbf{R}$.

Further, let $\{x_\xi \ : \ \xi < \omega_1\}$ be an injective ω_1–sequence consisting of all points of the straight line $\mathbf{R} \times \{0\}$. Then we put

$$\Gamma = \{0\} \times \mathbf{R},$$

$$A = \cup\{\Gamma_\xi + x_\xi \ : \ \xi < \omega_1\},$$

$$B = \mathbf{R}^2 \setminus A.$$

One can easily verify that

1) $\{A, B\}$ is a partition of $\mathbf{R}^2$ into two uncountable sets;

2) for any element $g \in \Gamma$, we have

$$card((g + A)\triangle A) \leq \omega, \quad card((g + B)\triangle B) \leq \omega;$$

3) all Γ–orbits are A–singular; more precisely, the intersection of every straight line lying in $\mathbf{R}^2$ and parallel to the line $\{0\} \times \mathbf{R}$ with the set A is infinite and countable.

Now, let us consider the real line $\mathbf{R}$ and the plane $\mathbf{R}^2$ as abstract groups. Then it is not difficult to see that these two groups are isomorphic. Let us take an arbitrary isomorphism

$$f \ : \ \mathbf{R}^2 \rightarrow \mathbf{R}$$

between these groups (notice that the existence of such an isomorphism cannot be proved without uncountable forms of the Axiom of Choice, since f is a real-valued function nonmeasurable with respect to the standard two-dimensional Lebesgue measure). Finally, let us put

$$G = f(\Gamma), \quad X = f(A), \quad Y = f(B).$$

Evidently, G is an uncountable subgroup of $\mathbf{R}$, $\{X, Y\}$ is a partition of $\mathbf{R}$ into two uncountable sets, all G-orbits are X-singular and

$$card((g + X)\triangle X) \leq \omega, \quad card((g + Y)\triangle Y) \leq \omega,$$

for each element $g \in G$.

Actually, the argument presented above shows us that a more general fact is true. Namely, suppose again that the Continuum Hypothesis holds and let G be an arbitrary uncountable subgroup of $\mathbf{R}$ satisfying the inequality

$$card(\mathbf{R}/G) > \omega.$$

Then there exist X and Y such that

(1) $\{X, Y\}$ is a partition of $\mathbf{R}$ into two uncountable sets;

(2) for any $g \in G$ we have the inequalities

$$card((g + X)\triangle X) \leq \omega, \quad card((g + Y)\triangle Y) \leq \omega;$$

(3) all G-orbits are X-singular.

Remark. Since our considerations were based only on algebraic properties of the real line $\mathbf{R}$, we can establish the corresponding analogues of the preceding results for various uncountable commutative groups.

References

[1]. S.Kakutani, J.Oxtoby, *Construction of a nonseparable invariant extension of the Lebesgue measure space*, Ann. Math. **52** (1950), 580–590.

[2]. A.B.Kharazishvili, *Invariant Extensions of the Lebesgue Measure*, Izd. Tbil. Gos. Univ., Tbilisi, 1983.

[3]. J.C.Oxtoby, *Measure and Category*, Springer-Verlag, Berlin, 1971.

[4]. J.C.Morgan II, *Point Set Theory*, Marcel Dekker, Inc., New York and Basel, 1990.

[5]. W.Sierpiński, *Sur les translations des ensembles lineares*, Fund. Math. **19** (1932), 22–28.

[6]. A.B.Kharazishvili, *On translations of sets and functions*, Journal of Applied Analysis **1** no. 2 (1995), 145–158.

[7]. A.B.Kharazishvili, *Some Questions of Set Theory and Measure Theory*, Izd. Tbil. Gos. Univ., Tbilisi, 1978.

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O PRAWIE NIEZMIENNICZYCH PODZBIORACH PROSTEJ

W pracy rozważa się podziały prostej składające sie ze zbiorów prawie niezmienniczych. Uogólnione zostało pewne twierdzenie Sierpińskiego dotyczące takich podziałów.

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