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# ON SOME CONSERVATIVE NONLINEAR INTEGRAL OPERATORS

*To Professor Lech Włodarski on His 80th birthday*

**There are given sufficient conditions in order that a nonlinear integral operator defines a conservative method of summability.**

**1.** There was done a lot of work concerning linear methods in summability, as well matrix methods as continuous methods. Among else, I should like to mention the work done by Professor L. Włodarski ([7], [8]), concerning continuous methods. In this paper we shall derive sufficient conditions in order that a nonlinear integral method of summability for functions, defined by means of a nonlinear integral operator, be conservative (convergence preserving) in some modular function spaces. In place of considering matrix methods and continuous methods separately, we shall deal with convergence in the sense of a filter. This kind of approach was started in case of linear methods in [2] and developed later for nonlinear methods (see e.g. [6], [1]) in case of problems of approximation. The case of a semigroup was started in [5].

Let $G$ be a semigroup. Let $\mu$ be a measure on a $\sigma$-algebra $\Sigma$ of subsets of $G$ and let $L^0(G)$ denote the space of all extended real-valued, $\Sigma$-measurable and finite $\mu$-almost everywhere functions on $G$, with equality $\mu$-almost everywhere. The measure $\mu$ is called *compatible with the structure of the semigroup* $G$, if $A \in \Sigma$ implies $sA \in \Sigma$ and $At \in \Sigma$, and $f \in L^0(G)$ implies $f(s\,\cdot) \in L^0(G)$ and $f(\cdot\, t) \in L^0(G)$, for every $s \in G$, $t \in G$ (see [5]). A function $K : G \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the real line, is called a *kernel function*, if $K(\cdot, u) \in L^1(G)$ for all $u \in \mathbb{R}$ and $K(t, 0) = 0$ for all $t \in G$. Let $L : G \to \mathbb{R}_0^+ = [0, \infty)$, $L \in L^1(G)$ and let $D = \int_G L(t)d\mu(t)$, $p(t) = L(t)/D$. Next, let $\psi : G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be $\Sigma$-measurable on $G$ for every value of the second variable and $\psi(t, 0) = 0$, $\psi(t, u) > 0$ for $u > 0$, $\psi(t, u)$ is a continuous, nondecreasing function of $u$ tending to $+\infty$ as $u \to \infty$ for every $t \in G$. We say that a kernel function $K$ is $(L, \psi)$-*Lipschitz*, if the inequality

$$|K(t, u + h) - K(t, u)| \leq L(t)\,\psi(t, |h|)$$

is satisfied for all $u, h \in \mathbb{R}$, $t \in G$ (see [6], [5]). It is easily observed that if $\mu$ is compatible with the structure of $G$, then the function $K(\cdot, f(s\,\cdot))$ is $\Sigma$-measurable for every $s \in G$, if $K$ is $(L, \psi)$-Lipschitz, because then $K$ is continuous with respect to the second variable.

**2.** We shall investigate the *nonlinear integral operator* $T$ defined by

$$(Tf)(s) = \int_G K(t, f(st))d\mu(t), \tag{1}$$

where $f \in \operatorname{Dom} T$ and $\operatorname{Dom} T$ is the set of all functions $f \in L^0(G)$ such that $(Tf)(s)$ exists for $\mu$-almost every $s \in G$ and is a $\Sigma$-measurable function of $s$ in $G$.

We shall need the notion of the modular in the space $L^0(G)$. A *modular* in a real vector space $X$ is defined as a functional $\rho : X \to \overline{\mathbb{R}_0^+} = [0, \infty]$ such that

$1^0$ $\rho(0) = 0$, $\rho(f) > 0$ for $f \neq 0$, $f \in X$,

$2^0$ $\rho(-f) = \rho(f)$ for $f \in X$,

$3^0$ $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for $f, g \in X$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

A modular $\rho$ in $X$ generates a modular space $X_\rho$ defined as

$$X_\rho = \{f : f \in X,\ \rho(\lambda f) \to 0 \ \text{ as } \lambda \to 0^+\}.$$

(see [3]). A modular $\rho$ in $L^0(G)$ is called *monotone*, if $f, g \in L^0(G)$ and $|f| \leq |g|$ imply $\rho(f) \leq \rho(g)$. A modular $\rho$ is called *J-convex*, if for every $\Sigma$-measurable function $p : G \to \mathbb{R}_0^+$ such that

$$\int_G p(t)\, d\mu(t) = 1$$

and for every $\Sigma$-measurable function $F : G \times G \to \mathbb{R}$ there holds the inequality

$$\rho\left(\int_G p(t)|F(t, \cdot)|d\mu(t)\right) \leq \int_G p(t)\, \rho(F(t, \cdot))d\mu(t).$$

Let $\mathfrak{U}$ be a filter of subsets of $G$, possesing a basis $\mathfrak{U}_0$ such that $\mathfrak{U}_0 \subset \Sigma$. A modular $\eta$ in $L^0(G)$ is called $(\tau, \mathfrak{U})$*-bounded* if there are a number $c \geq 1$ and a $\Sigma$-measurable, bounded function $h : G \to \mathbb{R}_0^+$ such that $h(t) \to 0$ in the sense of the filter $\mathfrak{U}$ and

$$\eta[f(t \cdot)] \leq \eta(cf) + h(t), \quad t \in G$$

for all $f \in L^0(G)$ such that $\eta(f) < \infty$; we shall write $h_U = \sup\{h(t) : t \in U\}$ for $U \in \mathfrak{U}$ (see [1], [4]).

Finally, we shall say that $(\rho, \psi, \eta)$ where $\rho, \eta$ are modulars in $L^0(G)$ and $\psi : G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a *properly directed triple*, if there exists a set $G_0 \subset G$ with $\mu(G \setminus G_0) = 0$ such that for every $\lambda \in (0, 1)$ there is a number $C_\lambda \in (0, 1)$ satisfying the inequality

$$\rho[C_\lambda\, \psi(t, |F(\cdot)|)] \leq \eta(\lambda F(\cdot))$$

for all $t \in G_0$ and all $F \in L^0(G)$ (see [1], condition (1)). This condition implies the inequality

$$\rho[C_\lambda\, \psi(t, F_t(\cdot))] \leq \eta(\lambda F_t(\cdot))$$

for every $t \in G_0$ and for any family $(F_t(\cdot))_{t \in G}$ of functions $F_t \in L^0(G)$.

**3.** We shall prove now the following

**Theorem 1.** *Let $\mu$ be a measure in the semigroup $G$, compatible with the structure of $G$, defined in the $\sigma$-algebra $\Sigma$ of subsets of $G$. Let $\mathfrak{U}$ be a filter of subsets of $G$ with a basis $\mathfrak{U}_0 \subset \Sigma$. Let $\rho$ and $\eta$ be monotone modulars in $L^0(G)$, where $\rho$ is $J$-convex and $\eta$ is $(\tau, \mathfrak{U})$-bounded. Let $K$ be an $(L, \psi)$-Lipschitz kernel function and let $(\rho, \psi, \eta)$ be a properly directed triple. Let the nonlinear integral operator $T$ be defined by (1) and let $f, g \in (L^0(G))_\eta \cap \mathrm{Dom}\, T$, $0 < \lambda < 1$, $0 < a \leq C_\lambda D^{-1}$. Then for an arbitrary $U \in \mathfrak{U}_0$ there holds the inequality*

$$\begin{aligned}\rho[a(Tf - Tg)] &\leq \eta(c\lambda(f-g)) + h_U \\ &\quad + [\eta(2c\lambda f) + \eta(2c\lambda g) + 2h_G] \int_{G\setminus U} p(t)\, d\mu(t),\end{aligned} \tag{2}$$

*where $c$ is the constant from the definition of $(\tau, \mathfrak{U})$-boundedness of $\eta$.*

*Proof.* Let $f, g \in \mathrm{Dom}\, T$. By the $(L, \psi)$-Lipschitz condition for the kernel function $K$, we obtain

$$|(Tf)(s) - (Tg)(s)| \leq \int_G p(t)\, D\psi(t, |f(\cdot t) - g(\cdot t)|)\, d\mu(t)$$

for $\mu$-almost every $s \in G$. Hence, by monotony and $J$-convexity of $\rho$ and by the assumption that the triple $(\rho, \psi, \eta)$ is properly directed, we obtain for $0 < a \leq C_\lambda D^{-1}$, $0 < \lambda < 1$

$$\begin{aligned}\rho[a(Tf - Tg)] &\leq \int_G p(t)\, \rho\{a D\psi[t, |f(\cdot t) - g(\cdot t)|]\} d\mu(t) \\ &\leq \int_G p(t)\, \eta\{\lambda[f(\cdot t) - g(\cdot t)]\} d\mu(t).\end{aligned}$$

Let us denote for $A \in \Sigma$

$$\nu(A) = \int_A p(t)\, \eta\{\lambda[f(\cdot t) - g(\cdot t)]\} d\mu(t).$$

Then we have for arbitrary $U \in \mathfrak{U}_0$

$$\rho[a(Tf - Tg)] \leq \nu(U) + \nu(G \setminus U). \tag{3}$$

By $(\tau, \mathfrak{U})$-boundedness of $\eta$,

$$\begin{aligned} \nu(U) &\leq \int_U p(t)\, \eta[\lambda(f-g)]d\mu(t) + \int_U p(t)\, h(t) d\mu(t) \\ &\leq \eta[c\,\lambda(f-g)] + h_U. \end{aligned}$$

Again, by monotonity and $(\tau, \mathfrak{U})$-boundedness of $\eta$,

$$\begin{aligned} \nu(G \setminus U) &\leq \int_{G\setminus U} p(t)\, \eta[2\,\lambda\, f(\cdot\, t)]d\mu(t) + \int_{G\setminus U} p(t)\, \eta[2\,\lambda\, g(\cdot\, t)]d\mu(t) \\ &\leq \int_{G\setminus U} p(t)\, [\eta(2\,c\,\lambda\, f) + \eta(2\,c\,\lambda\, g)]d\mu(t) + 2\int_{G\setminus U} p(t)\, h(t) d\mu(t) \\ &\leq [\eta(c\,\lambda\, f) + \eta(2\,c\,\lambda\, g) + 2\,h_U] \int_{G\setminus U} p(t) d\mu(t). \end{aligned}$$

Hence, by inequality (3), we get (2).

**4.** Let $\mathcal{W}$ be an abstract, nonempty set of indices and let $\mathfrak{W}$ be a filter of subsets of the set $\mathcal{W}$. Let $(f)_{w\in\mathcal{W}}$ be a filtered family of functions $f_w \in (L^0(G))_\eta \cap \operatorname{Dom} T$, and let $(F_w)_{w\in\mathcal{W}}$ be a filtered family of functions $F_w \in (L^0(G))_\rho$. We say that $(f_w)_{w\in\mathcal{W}}$ is *$\eta$-convergent* to a function $f$ [resp. $(F_w)_{w\in\mathcal{W}}$ is *$\rho$-convergent* to a function $F$], if there is a $\lambda > 0$ such that for every $\varepsilon > 0$ there exists a set $W \in \mathfrak{W}$ such that for all $w \in W$ there holds the inequality $\eta[\lambda(f_w - f)] < \varepsilon$ [resp. $\rho[\lambda(F_w - F)] < \varepsilon$]. We say that $(f_w)_{w\in\mathcal{W}}$ is *$\eta$-Cauchy* [resp. $(F_w)_{w\in\mathcal{W}}$ is *$\rho$-Cauchy*], if there is a $\lambda > 0$ such that for every $\varepsilon > 0$ there exists a set $W \in \mathfrak{W}$ such that for all $v, w \in W$ there holds the inequality $\eta[\lambda(f_v - f_w)] < \varepsilon$ [resp. $\rho[\lambda(F_v - F_w)] < \varepsilon$]. Obviously, $\eta$-convergence of $(f_w)_{w\in\mathcal{W}}$ implies $(f_w)_{w\in\mathcal{W}}$ to be $\eta$-Cauchy [resp. $\rho$-convergence of $(F_w)_{w\in\mathcal{W}}$ implies $(F_w)_{w\in\mathcal{W}}$ to be $\rho$-Cauchy ]. If the converse implication holds we say, that the space $(L^0(G))_\eta$ is *$\eta$-complete* [resp. $(L^0(G))_\rho$ is *$\rho$-complete*] with respect to the filter $\mathfrak{W}$.

Let

$$(T_w f)(s) = \int_G K_w(t, f(st))d\mu(t).$$

We shall say that the family of nonlinear operators $T_w$ given by (1) is *conservative* [resp. *Cauchy-conservative*] from $(L^0(G))_\eta \cap \operatorname{Dom} T_w$ to $(L^0(G))_\rho$ with respect to the filter $\mathfrak{W}$, if for every $\eta$-convergent [resp. $\eta$-Cauchy] filtered family $(f_w)_{w\in\mathcal{W}}$ of functions $f_w \in (L^0(G))_\eta$ $\cap \operatorname{Dom} T_w$, the filtered family $(T_w f_w)_{w\in\mathcal{W}}$ is $\rho$-convergent [resp. $\rho$-Cauchy]. We shall still need the notion of weak singularity of an $(L_w, \psi)$-Lipschitz kernel function $K$. We say that the $(L_w, \psi)$-Lipschitz kernel function $K$ is *weakly singular*, if for every $\varepsilon > 0$, $U \in \mathfrak{U}_0$ there exists a set $W \in \mathfrak{W}$ such that

$$\int_{G\setminus U} p_w(t)\, d\mu(t) < \varepsilon \quad \text{for all} \quad w \in W.$$

There holds the following

**Theorem 2.** *Let all assumptions of Theorem 1 be satisfied and let, moreover, the $(L, \psi)$-Lipschitz kernel $K$ be weakly singular. Then the family of nonlinear integral operators $T_w$ given by (1) is Cauchy conservative from $(L^0(G))_\eta \cap \operatorname{Dom} T_w$ to $(L^0(G))_\rho$ with respect to the filter $\mathfrak{W}$.*

*Proof.* By inequality (2), we have

$$\rho\,[a\,(T_v f_v - T_w f_w)] \le \eta\,(c\,\lambda(f_v - f_w)) + h_U + [\eta\,(2\,c\,\lambda\, f_v) + \eta\,(2\,c\,\lambda\, f_w) + 2\,h_G] \int_{G\setminus U} p_w(t)\, d\mu(t)$$

for $U \in \mathfrak{U}_0$, $v, w \in \mathcal{W}$, $0 < \lambda < 1$, $0 < a \le C_\lambda D^{-1}$. Let us remark that if $(f_w)_{w\in\mathcal{W}}$ is $\eta$-Cauchy and $f_w \in (L^0(G))_\eta \cap \operatorname{Dom} T_w$ for $w \in \mathcal{W}$, then there exists a set $W_1 \in \mathfrak{W}$ such that the family

$(\eta(2c\lambda f_w))_{w\in W_1}$ is bounded for sufficiently small $\lambda > 0$. This follows from the inequality

$$\eta(2c\lambda f_w) \le \eta[4c\lambda(f_w - f_{w_0})] + \eta(4c\lambda f_{w_0})$$

for $w$, $w_0 \in \mathcal{W}$, since we may take a set $W_1 \subset \mathfrak{W}$ such that $\eta[4c\lambda(f_w - f_{w_0})] < 1$ for $w$, $w_0 \in W_1$ and because $f_{w_0} \in (L^0(G))_\eta$, we have $\eta(4c\lambda f_{w_0}) < \infty$, if $\lambda > 0$ is sufficiently small. Thus, taking $w_0 \in W_1$ fixed, we have

$$\eta(2c\lambda f_w) < 1 + \eta(4c\lambda f_{w_0}) = M < \infty$$

for $w \in W_1$. Let us choose an arbitrary $\varepsilon > 0$. Since $K$ is weakly singular and $h(t) \to 0$ in the sense of the filter $\mathfrak{U}$, we may find a set $U \in \mathfrak{U}_0$ such that

$$h_U < \frac{1}{3}\varepsilon \quad \text{and} \quad \int\limits_{G\setminus U} p_w(t)\, d\mu(t) < \frac{\varepsilon}{6(M + h_G)}$$

for $w \in W_1$. Then

$$\rho[a(T_v f_v - T_w f_w)] \le \eta[c\lambda(f_v - f_w)] + \frac{2}{3}\varepsilon$$

for $v$, $w \in W_1$. Since $(f_w)_{w\in\mathcal{W}}$ is $\eta$-Cauchy, we may find $W_2 \in \mathfrak{W}$ such that

$$\eta[c\lambda(f_v - f_w)] < \frac{1}{3}\varepsilon$$

for $\lambda > 0$ sufficiently small, if only $v$, $w \in W_2$. Hence, taking $v$, $w \in W = W_1 \cap W_2$ and $a > 0$ sufficiently small, we have

$$\rho[a(T_v f_v - T_w f_w)] < \varepsilon.$$

Thus, $(T_w f_w)_{w\in\mathcal{W}}$ is $\rho$-Cauchy.

From Theorem 2 it follows immediately

**Theorem 3.** *Let us suppose the assumptions of Theorem 2 to be satisfied. Let the space $(L^0(G))_\rho$ be $\rho$-complete. Then the family of nonlinear integral operators $T_w$ given by (1) is conservative from $(L^0(G))_\eta \cap \operatorname{Dom} T_w$ to $(L^0(G))_\rho$ with respect to the filter $\mathfrak{W}$.*

## References

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# O PEWNYCH ZACHOWAWCZYCH NIELINIOWYCH OPERATORACH CAŁKOWYCH

Niech

$$(Tf)(s) = \int GK(t, f(ts))d\mu(t)$$

będzie nieliniowym operatorem całkowym, przyczym $\mu$ jest miarą w półgrupie $G$, zgodną z działaniem w tej półgrupie. Oszacowano wartość $\rho[a(Tf - Tg)]$, gdzie $\rho$ jest modularem nad przestrzenią $L^0(G)$. Wynik zastosowano do uzyskania warunków dostatecznych na to, by sfiltrowana rodzina $(T_w)$ takich przekształceń była zacho-

wawcza, tj. by zbieżność rodziny funkcji $(f_w)$ pociągała zbieżność rodziny funkcji $(T_w f_w)$ w przestrzeni modularnej $(L^0(G))_\rho$.

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