A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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ON LOCAL ONE-PARAMETER GROUPS OF LOCAL TRANSFORMATIONS IN DIFFERENTIAL SPACES

In the paper there has been considered a problem of mutual correspondence between the vector fields and local one-parameter groups of local transformations in the category of differential spaces generated by a single function.

On a C^∞-manifold, every vector field X corresponds to an equivalence class of local one-parameter groups of local transformations in a 1 : 1 manner.

In this paper we consider the problem of a correspondence between the vector fields and local one-parameter groups of local transformations for a differential space (M,C), where C is a differential structure on M generated by a single function f.

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Throughout this paper, by a differential space we shall mean a couple (M,C), where C is a differential structure on a set M in the sense of S i k o r s k i (see [1]). If h is a smooth mapping from a differential space (M,C) into a differential space (N,C′), we write $h : (M,C) \to (N,C')$. The set of all C^∞-functions on the set R of all real numbers is denoted by $C^\infty(R)$. Let V be an open subset of R with the natural topological structure. By $C^\infty(V)$ we denote the set of all C^∞-functions on V. The set of all C^∞-functions on $R \times R$ is denoted by $C^\infty(R^2)$. J_ε designates the differential space $(I_\varepsilon,$

$C^{\infty}(I_{\varepsilon}))$, where $I_{\varepsilon} = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. $C(A)$ denotes the set of all functions $f|A$, where $f \in C$ and A is an open set of the topology of (M,C).

Let (M,C) be a differential space, and let U and V be two open sets of the topology of (M,C). A diffeomorphism $\varphi : (U, C(U)) \to (V, C(V))$ is called a local transformation of (M,C). U is called the domain of φ.

A local one-parameter group of local transformations of (M, C) is a set $\{U_{\alpha}, \varepsilon_{\alpha}, \varphi_t^{(\alpha)}\}_{\alpha \in A}$, where U is an open set of the topology of (M,C), ε_{α} - a positive number, and $\varphi_t^{(\alpha)}$ - a local transformation of (M,C) for each t, $|t| < \varepsilon_{\alpha}$, satisfying the following conditions:

1) $\{U_{\alpha}\}_{\alpha \in A}$ is an open cover of (M,C),

2) the domain of $\varphi_t^{(\alpha)}$, $|t| < \varepsilon_{\alpha}$, contains U_{α}, and $\varphi_0^{(\alpha)}$ is the identity transformation on U; the map $(t,p) \longrightarrow \varphi_t^{(\alpha)}(p)$ is a smooth map from $J_{\varepsilon_{\alpha}} \times (U_{\alpha}, C(U_{\alpha}))$ into (M,C),

3) if $|t|, |s|, |s+t| < \varepsilon_{\alpha}$, then $\varphi_t^{(\alpha)} \circ \varphi_s^{(\alpha)}$ is defined, its domain contains U_{α} and $(\varphi_t^{(\alpha)} \circ \varphi_s^{(\alpha)})(q) = \varphi_{t+s}^{(\alpha)}(q)$ for $q \in U_{\alpha}$,

4) if $U_{\alpha} \cap U_{\beta} \neq \emptyset$, then for each point $p \in U_{\alpha} \cap U_{\beta}$ one can choose $\varepsilon < \min(\varepsilon_{\alpha}, \varepsilon_{\beta})$ such that, for $|t| < \varepsilon$, $\varphi_t^{(\alpha)}$ and $\varphi_t^{(\beta)}$ agree on a sufficiently small neighbourhood of p.

Let $G_1 = \{U_{\alpha}, \varepsilon_{\alpha}, \varphi_t^{(\alpha)}\}_{\alpha \in A}$ and $G_1' = \{V_{\gamma}, \eta_{\gamma}, \psi_t^{(\gamma)}\}_{\gamma \in I}$ be two local one-parameter groups of local transformations. We say that G_1 and G_1' are equivalent and write $G_1 \sim G_1'$ if the following conditions is satisfied: if $U_{\alpha} \cap V_{\gamma} \neq 0$, then for each point $p \in U_{\alpha} \cap V_{\gamma}$, there is a number $\delta > 0$, $\delta < \min(\varepsilon_{\alpha}, \eta_{\gamma})$, such that, for $|t| < \delta$, $\varphi_t^{(\alpha)}$ and $\psi_t^{(\gamma)}$ agree on a sufficiently small neighbourhood of p.

To G_1 we can associate a vector field X on (M,C) as follows. For $p \in M$ and $f \in C$ we define the value $X(p)$ of the vector field X at p by

$$X(p)(f) = \frac{d}{dt}(f \circ \varphi_t^{(\alpha)})(p)\Big|_{t=0}, \quad p \in U_{\alpha}. \tag{1}$$

The vector field X is called the infinitesimal transformation of the local one-parameter groups G_1 of local transformations.

Lemma. Let C be the smallest differential structure on M containing a function f, and let X be the infinitesimal transformation of two local one-parameter groups of local transformations $G_1 = \{U_\alpha, \varepsilon_\alpha, \varphi_t^{(\alpha)}\}_{\alpha\in A}$ and $G_1' = \{V_\gamma, \eta_\gamma, \psi_t^{(\alpha)}\}_{\gamma\in I}$ of (M, C). Then, for each point $p_o \in U_\alpha \cap V_\gamma$, there exist a neighbourhood U, $U \subset U_\alpha \cap V_\gamma$, of p_o and $0 < \varepsilon < \min(\varepsilon_\alpha, \eta_\gamma)$, such that

$$(2) \qquad (f \circ \varphi_t^{(\alpha)})(p) = (f \circ \psi_t^{(\alpha)})(p) \quad \text{for } p \in U, \ t \in I_\varepsilon .$$

P r o o f. Suppose $G_1 = \{U_\alpha, \varepsilon_\alpha, \varphi_t^{(\alpha)}\}_{\alpha\in A}$ and $G_1' = \{V_\gamma, \eta_\gamma, \psi_t^{(\alpha)}\}_{\gamma\in I}$ have as their infinitesimal transformation the same vector field X. Let p_o be a point of $U_\alpha \cap V_\gamma$. $X(f)$ is a smooth function on (M,C). Thus there exist a neighbourhood U_1 of p_o contained in $U_\alpha \cap V_\gamma$ and a function $g \in C^\infty(R)$, such that

$$(3) \qquad X(p)(f) = (g \circ f)(p) \quad \text{for } p \in U_1 .$$

Denote by the symbols $\bar\varphi^{(\alpha)}$ and $\bar\psi^{(\gamma)}$ the smooth mappings $(t, p) \mapsto \varphi_t^{(\alpha)}(p)$ from $(U_\alpha, C(U_\alpha)) \times J_{\varepsilon_\alpha}$ into (M,C) and $(t,p) \mapsto \psi_t^{(\gamma)}(p)$ from $(V_\gamma, C(V_\gamma)) \times J_{\eta_\gamma}$ into (M,C), respectively. Because of the continuity of the mappings $\bar\varphi^{(\alpha)}$ and $\bar\psi^{(\alpha)}$ it follows that there are neighbourhood U_o of p_o contained in U_1 and a positive number $\varepsilon_o < \min(\varepsilon_\alpha, \eta_\gamma)$ such that $\varphi_t^{(\alpha)}(U_o) \cup \psi_t^{(\gamma)}(U_o) \subset U_1$ for $|t| < \varepsilon_o$.

From condition (3) of the definition of a local one-parameter group of local transformations, one sees that, for $p \in U_\alpha \cap V_\gamma$, $|t| < \varepsilon_o$,

$$(4) \qquad \begin{aligned} X(\varphi_t^{(\alpha)}(p))(f) &= \frac{d}{dt}\left(f \circ \varphi_t^{(\alpha)}\right)(p), \\ X(\psi_t^{(\gamma)}(p))(f) &= \frac{d}{dt}\left(f \circ \psi_t^{(\gamma)}\right)(p). \end{aligned}$$

Next from (3) we have

$$X(\varphi_t^{(\alpha)}(p))(f) = (g \circ f \circ \varphi_t^{(\alpha)})(p),$$

(5) $p \in U_o, \quad |t| < \varepsilon_o$

$$X(\psi_t^{(\alpha)}(p))(f) = (g \circ f \circ \varphi_t^{(\gamma)})(p).$$

The mapping $f \circ \bar{\varphi}^{(\alpha)}$ belongs to $C(U_\alpha) \times C^\infty(I_{\varepsilon_\alpha})$. The mapping $f \circ \bar{\psi}^{(\gamma)}$ belongs to $C(V_\gamma) \times C^\infty(I_{\eta_\gamma})$. Therefore, there exist a neighbourhood U_2 of p_o contained in U_1, a positive number $\varepsilon < \varepsilon_o$ and mappings F_φ and $F_\psi \in C^\infty(R^2)$, such that

$$(f \circ \bar{\varphi}^{(\alpha)})(p,t) = F_\varphi(f(p),t),$$

(6) for $p \in U_2, \quad |t| < \varepsilon$

$$(f \circ \bar{\psi}^{(\gamma)})(p,t) = F_\psi(f(p),t).$$

Thus, if $p \in U_2 = U, \quad |t| < \varepsilon,$ from (4), (5) and (6) we get

$$\frac{d}{dt} F_\varphi(f(p),t) = (g \circ F_\varphi)(f(p),t), \quad F_\varphi(f(p),0) = f(p),$$

$$\frac{d}{dt} F_\psi(f(p),t) = (g \circ F_\psi)(f(p),t), \quad F_\psi(f(p),0) = f(p).$$

From the uniqueness theorem for solutions of differential equations we have

$$F_\varphi(f(p),t) = F_\psi(f(p),t) \quad \text{for } p \in U, \quad |t| < \varepsilon.$$

So, by (6) we obtain (2).

Theorem. Let C be the smallest differential structure on a set M which contains a function f and let the topology (M, C) satisfy Kolmogoroff's separation axiom T_o. If two local one-parameter groups of local transformations have the same vector field as their infinitesimal transformation, then they are equivalent.

P r o o f. Since the topology of (M,C) satisfies Kolmogoroff's separation axiom T_o, then f is one-to-one. Suppose $G_1 = \{U_\alpha, \varepsilon_\alpha, \varphi_t^{(\alpha)}\}_{\alpha \in A}$ and $G_1' = \{V_\gamma, \eta_\gamma, \psi_t^{(\alpha)}\}_{\gamma \in I}$ have, as their infinitesimal transformations, the same vector field X. We shall show that $G_1 \sim G_1'$. From *Lemma*, for $p_0 \in U_\alpha \cap V_\gamma$ there exist a neighbourhood U of p_0 contained in $U_\alpha \cap V_\gamma$ and $0 < \varepsilon < \min(\varepsilon_\alpha, \eta_\gamma)$, such that

$$(f \circ \varphi_t^{(\alpha)})(p) = (f \circ \psi_t^{(\alpha)})(p)$$

for $p \in U$, $|t| < \varepsilon$. Hence, $\varphi_t^{(\alpha)}$ and $\psi_t^{(\gamma)}$ coincide on U for $|t| < \varepsilon$, and this completes the proof.

If the topology of a differential space does not satisfy the separation axiom T_0, then the 1 : 1 correspondence between the vector fields and the equivalence classes of one-parameter groups of local transformations can fail. Now we shall give suitable examples.

Example 1. Let $\pi_1: R^2 \to R$ be the canonical projection ($\pi_1(p_1, p_2) = p_1$) and let C be the smallest structure on R^2 containing π_1. Let us put

$$\varphi_t(p_1, p_2) = (p_1 + t,\ p_2 + t),$$

$$p_1, p_2, t \in R$$

$$\psi_t(p_1, p_2) = (p_1 + t,\ p_2 - t).$$

$\{R^2, \varepsilon, \varphi_t\}$ and $\{R^2, \varepsilon, \psi_t\}$, $\varepsilon > 0$, are two different, not equivalent, local one-parameter groups of local transformations which have the same vector field as their infinitesimal transformations.

Professor Z. Moszner in his paper (which will appear in Tensor) considered the function $\varphi_1: R^2 \to R$ given by the formula

$$\varphi_1(p,t) = \begin{cases} 0 & \text{for } p = 0,\ t \in R, \\ \operatorname{sgn}(\ln|p| + t)\ |p|e^t & \text{for } p \in \langle 1,\infty) \cup (-1,0),\ t \in R, \\ -\operatorname{sgn}(\ln|p| + t)\ |p|e^t & \text{for } p \in (0,1) \cup (-\infty,-1\rangle,\ t \in R. \end{cases}$$

This function will be used in the following example.

Example 2. Let r be a nonzero number and let $t \in R$. Put

$$\varphi_{rt}(p) = \begin{cases} 0 & \text{for } p = 0, \\ \operatorname{sgn}(\ln\left|\frac{p}{r}\right| + t)|p|e^{t} & \text{for } p \in \langle |r|, \infty) \cup (-|r|, 0), \\ -\operatorname{sgn}(\ln\left|\frac{p}{r}\right| + t)|p|e^{t} & \text{for } p \in (0, |r|) \cup (-\infty, -|r|\rangle, \end{cases}$$

$$\varphi_t(p) = p\,e^{t},\ f(p) = |p| \quad \text{for } p \in R.$$

Let C be the smallest differential structure on R which contains the function f. Consider the differential space (R,C). For each $r \neq 0$, $G_1^r = \{R, \varepsilon, \varphi_{rt}\}$ and $G_1 = \{R, \varepsilon, \varphi_t\}$, $\varepsilon > 0$, are two different, not equivalent, local one-parameter groups of local transformations which have the same vector field as their infinitesimal transformations.

REFERENCES

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O LOKALNYCH JEDNOPARAMETROWYCH GRUPACH PRZEKSZTAŁCEŃ NA PRZESTRZENIACH RÓŻNICZKOWYCH

W pracy rozważany jest problem wzajemnej odpowiedniości pomiędzy polami wektorowymi i lokalnymi jednoparametrowymi grupami przekształceń lokalnych w kategorii przestrzeni różniczkowych generowanych przez jedną funkcję.