A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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ON A KIND OF METRIC DIMENSION

In the paper the properties of a dimension that can be applied to subsets of a metric space, close to the Kolmogorov metric dimension, are investigated. It is proved that dim $(X \times Y) = \dim X + \dim Y$ for any metric spaces. In th examples the dimensions of concrete spaces are examined.

1. INTRODUCTION

One of the possible ways of measuring massivness of a set in a metric space is, given an arbitrary $\varepsilon > 0$, to count the minimal number of balls of the radius ε needed to cover the set and to observe how the number changes as ε tends to zero. This method gives raise to the notion of metric dimension, as described in [4]. But it is also possible to look at this problem from the other side: given an arbitrary positive integer n, we try to find the minimal number $\varepsilon > 0$ such that the set is covered by exactly n balls of radius ε and then watch how the number changes as n tends to infinity. In this way we come to the notion of dimension which resembles that of metric dimension. In the sequel we shall call it quasi-metric dimension. The suitable definition of the dimension is due to R. Jajte.

We shall keep (with insignificant changes) the terminology and the notion of [4]. The facts from that paper which prove useful for our purposes will be reminded below.

Let (X, ρ) be a metric space and A its totally bounded nonempty subset. We say that the set $U \subset X$ is an ε - net for A.

in the space X if for any $a \in A$ there exists a point $u \in U$ such that $\rho(a,u) \le \varepsilon$.

We denote by $N_\varepsilon^X(A)$ the minimal number of points in an ε-net for A in the space X. A metric space X is called centered if for every set $U \subset X$ of the diameter $d = 2r$ there exists a point $x_0 \in X$ such that $\rho(x_0,u) \le r$ for all $u \in U$. By a theorem of Vitushkin (Theorem VII in [4]) any totally bounded space A may be embedded in a centered space X.

If $A \subset X_0$, $A \subset X$ and the space X_0 is centred, then of course $N_\varepsilon^{X_0}(A) \le N_\varepsilon^X(A)$. The number $N_\varepsilon^{X_0}(A)$, which clearly does not depend on the choice of the centred space X_0, is denoted simply by $N_\varepsilon(A)$. The number $H_\varepsilon(A) = \log_2 N_\varepsilon(A)$ is called the ε-entropy of A.

Now we can define the upper and lower metric dimension of a totally bounded space A:

$$\text{u-dm}(A) = \limsup_{\varepsilon \to 0} \frac{H_\varepsilon(A)}{\log_2 \frac{1}{\varepsilon}},$$

$$\text{l-dm}(A) = \liminf_{\varepsilon \to 0} \frac{H_\varepsilon(A)}{\log_2 \frac{1}{\varepsilon}}.$$

If $\text{l-dm}(A) = \text{u-dm}(A) = \text{dm}(A)$, the number $\text{dm}(A)$ is called the metric dimension of A and A is said to be regular with respect to the metric dimension.

We shall denote by $[x]$ the integral part of x and by $\langle x \rangle$ the smallest integer equal to or greater than x.

2. BASIC DEFINITIONS

Let (X, ρ) be a metric space, A its totally bounded nonempty subset and p an arbitrary positive number.

Definition 1. By upper (resp. lower) p-dimensional diameter of the set A in the space X we mean the number (possibly infinite)

$$\overline{D}_p^X(A) = \inf_{\xi} \limsup_{n\to\infty} n^{1/p} \Delta_n(\xi;A)$$

$$(\text{resp. } \underline{D}_p^X(A) = \inf_{\xi} \liminf_{n\to\infty} n^{1/p} \Delta_n(\xi;A)),$$

where the infimum is taken over all sequences $\xi = (x_i)$ with elements in X and where

$$\Delta_n(\xi;A) = \sup_{a\in A} \min_{1\le i\le n} \varrho(x_i,a).$$

Proposition 1. If, for some $p_o > 0$, $\overline{D}_{p_o}^X(A) < \infty$ (resp. $\underline{D}_{p_o}^X(A) < \infty$), then $\overline{D}_p^X(A) = 0$

$$(\text{resp. } \underline{D}_{p_o}^X(A) = 0) \quad \text{for every } p > p_o.$$

We omit the easy proof.

Definition 2. By upper (resp. lower) quasi-metric dimension of the set A in the space X we mean the number (possibly infinite)

$$\text{u-dqm}^X(A) = \sup\{p\in(0,\infty) : \overline{D}_p^X(A) = \infty\}$$

$$= \inf\{p\in(0,\infty) : \overline{D}_p^X(A) = 0\},$$

$$(\text{resp. } \text{l-dqm}^X(A) = \sup\{p\in(0,\infty) : \underline{D}_p^X(A) = \infty\}$$

$$= \inf\{p\in(0,\infty) : \underline{D}_p^X(A) = 0\}.$$

Definition 3. For every positive integer n and arbitrary $p > 0$ we put

$$\delta_n^X(A) = \inf_{(x_1, \ldots, x_n) \in X^n} \sup_{a \in A} \min_{1 \le i \le n} \varrho(x_i, a) = \inf_{\xi} \Delta_n(\xi; A),$$

$$\bar{d}_p^X(A) = \limsup_{n \to \infty} n^{1/p} \delta_n^X(A),$$

$$\underline{d}_p^X(A) = \liminf_{n \to \infty} n^{1/p} \delta_n^X(A).$$

Proposition 2. For every positive integer n, every sequence ξ with elements in X and arbitrary totally bounded subset A of X, we have

$$\Delta_n(\xi; A) = \Delta_n(\xi; \bar{A})$$

and consequently

$$\bar{D}_p^X(A) = \bar{D}_p^X(\bar{A}), \qquad \underline{D}_p^X(A) = \underline{D}_p^X(\bar{A}),$$

$$\bar{d}_p^X(A) = \bar{d}_p^X(A), \qquad \underline{d}_p^X(A) = \underline{d}_p^X(\bar{A}),$$

$$\text{u-dqm}^X(A) = \text{u-dqm}^X(\bar{A}), \qquad \text{l-dqm}^X(A) = \text{l-dqm}^X(\bar{A}).$$

We omit the easy proof.

Remark. Definitions 1, 2 and *3* may be introduced for any non-empty subset of X, not necessarily totally bounded. But for a set A which is not totally bounded, $\text{l-dqm}^X(A) = \text{u-dqm}^X(A) = \infty$, as can easily be verified. If A is totally bounded, its completion $\tilde{A}$ is compact and A is dense in $\tilde{A}$. Now it is clear from *Proposition 2* that is it reasonable to consider only compact sets A.

Proposition 3. The following inequalities hold for every compact A and an arbitrary positive integer n:

$$\overline{D}_p^{X_o}(A) \le \overline{D}_p^{X}(A) \le \overline{D}_p^{A}(A) \le 2\overline{D}_p^{X_o}(A),$$

$$\underline{D}_p^{X_o}(A) \le \underline{D}_p^{X}(A) \le \underline{D}_p^{A}(A) \le 2\underline{D}_p^{X_o}(A),$$

$$\delta_n^{X_o}(A) \le \delta_n^{X}(A) \le \delta_n^{A}(A) \le 2\delta_n^{X_o}(A),$$

$$\overline{d}_p^{X_o}(A) \le \overline{d}_p^{X}(A) \le \overline{d}_p^{A}(A) \le 2\overline{d}_p^{X_o}(A),$$

$$\underline{d}_p^{X_o}(A) \le \underline{d}_p^{X}(A) \le \underline{d}_p^{A}(A) \le 2\underline{d}_p^{X_o}(A),$$

where X_o denotes any centred space containing A and X an arbitrary space containing A.

Again we omit the easy proof.

Remark. It follows from *Proposition 3* that the upper and lower quasi-metric dimension of a compact set A do not depend on the choice of the space X in which the set is (isometrically) embedded, so that we can simply write u-dqm(A) and l-dqm(A) instead of u-dqm^X(A) and l-dqm^X(A). In the sequel we shall even drop the symbol X from the denotations $\delta_n^X(A)$, $\overline{d}_p^X(A)$, $\underline{d}_p^X(A)$, $\overline{D}_p^X(A)$, $\underline{D}_p^X(A)$. The exact meaning will always be clear from the context.

Theorem 1. $0 < \overline{d}_p(A) < \infty$ (resp. $0 < \underline{d}_p(A) < \infty$) if and only if $0 < \overline{D}_p(A) < \infty$ (resp. $0 < \underline{D}_p(A) < \infty$). More precisely,

(1) $$\overline{d}_p(A) \le \overline{D}_p(A) \le 8^{1/p}\, \overline{d}_p(A)$$

(1′) $$\left(\text{resp. } \underline{d}_p(A) \le \underline{D}_p(A) \le 8^{1/p}\, \underline{d}_p(A)\right).$$

P r o o f. We shall show only (1), as (1′) can be verified similarly.

For any sequence ξ (with elements in X), $\delta_n(A) \le \Delta_n\ (\xi;A)$, so that

$$\bar{d}_p(A) = \limsup_{n\to\infty} n^{1/p}\,\delta_n(A) \le \limsup_{n\to\infty} n^{1/p}\,\Delta_n\ (\xi;A).$$

Hence $\bar{d}_p(A) \le \bar{D}_p(A)$.

For any positive integer k let us choose points $x_1^{(k)}, \dots, x_k^{(k)}$ of X so that

$$\sup_{a\in A}\ \min_{1\le i\le k}\ \rho(x_i^{(k)},a) \le \delta_k(A) + 1/2^k. \tag{2}$$

Now let us define a sequence $\alpha = (a_i)$ as follows:

$$a_i = x^{(2^k)}_{i+1-2^k} \quad \text{for} \quad i = 2^k, 2^k+1, \dots, 2^{k+1}-1 \quad \text{and} \quad k = 0,1,\dots .$$

Among the first 2^n elements of the sequence there are points

$$x_1^{(2^{n-1})}, \dots, x^{(2^{n-1})}_{2^{n-1}}.$$

Hence and from (2) we get

$$\begin{aligned}\Delta_{2^k}(\alpha;A) &= \sup_{a\in A}\ \min_{1\le i\le 2^k}\ \rho(a_i,a) \\ &\le \sup_{a\in A}\ \min_{1\le i\le 2^{k-1}}\ \rho(x_i^{(2^{k-1})},a) \\ &\le \delta_{2^{k-1}}(A) + 1/2^{2^{k-1}}.\end{aligned} \tag{3}$$

For any integer $n > 1$, denote by $k(n)$ a number such that

$$2^{k(n)} \le n < 2^{k(n)+1}. \tag{4}$$

Then

(5) $$< n/4 > \leq 2^{k(n)-1}.$$

Let us notice that the sequences $(\Delta_n(\alpha;A))$ and $(\delta_n(A))$ are nonincreasing. Together with (3), (4), and (5) it gives

(6) $$n^{1/p}\,\Delta_n(\alpha;A) \leq (2^{k(n)+1,1/p}\,\Delta_{2^{k(n)}}(\alpha;A)$$

$$\leq (2^{k(n)+1,1/p}\,\delta_{2^{k(n)-1}}(A) + (2^{k(n)+1,1/p}\,/2^{2^{k(n)-1}}$$

$$\leq (2n)^{1/p}\,\delta_{<n/4>}(A) + (2n)^{1/p}\,/\,2^{n/4}.$$

Clearly for any $p > 0$

(7) $$(2n)^{1/p}\,/\,2^{n/4} \to 0 \quad \text{as} \quad n \to \infty$$

and

(8) $$(2n)^{1/p} \leq 8^{1/p} < n/4 >^{1/p}.$$

From (6), (7) and (8) we get

(9) $$\bar{D}_p(A) \leq \limsup_{n\to\infty} n^{1/p}\,\Delta_p\,(\alpha;A)$$

$$\leq 8^{1/p} \limsup_{n\to\infty} < n/4 >^{1/p}\,\delta_{< n/4 >}\,(A).$$

Obviously the sets of limit points of the sequences

$$(< n/4 >^{1/p}\,\delta_{< n/4 >}(A)) \quad \text{and} \quad (n^{1/p}\,\delta_n(A))$$

coincide. Hence

(10) $$\limsup_{n\to\infty} < n/4 >^{1/p}\,\delta_{< n/4 >}(A)$$

$$= \limsup_{n\to\infty} n^{1/p}\,\delta_n(A) = \bar{d}_p(A).$$

From (9) and (10) we conclude that

$$\bar{D}_p(A) \le 8^{1/p}\, \bar{d}_p(A).$$

This ends the proof of the theorem.

Corollary.

$$\text{u-dqm}(A) = \sup\{p \in (0,\infty) : \bar{d}_p(A) = \infty\} = \inf\{p \in (0,\infty) : \bar{d}_p(A) = 0\}$$

$$\text{l-dqm}(A) = \sup\{p \in (0,\infty) : \underline{d}_p(A) = \infty\} = \inf\{p \in (0,\infty) : \underline{d}_p(A) = 0\}.$$

Remark. The above corollary gives us a new and useful definition of quasi-metric dimension. It will allow us to find a connection between metric and quasi-metric dimension.

Definition 4. If l-dqm(A) = u-dqm(A) = dqm(A), we call the number dqm(A) the quasi-metric dimension of the set A and say that A is regular with respect to quasi-metric dimension (or simply: regular).

3. COMPARISON OF METRIC AND QUASI-METRIC DIMENSION

Theorem 2. For any compact metric space A,

$$\text{l-dm}(A) \le \text{l-dqm}(A) \le \text{u-dqm}(A) \le \text{u-dm}(A).$$

P r o o f. The inequality l-dqm(A) ≤ u-dqm(A) is obvious. We shall show that l-dm(A) ≤ l-dqm(A). The inequality u-dqm(A) ≤ ≤ u-dm(A) can be proved in a similar way.

Let us take any $p >$ l-dqm(A). Then $\underline{d}_p(A) = 0$ so that there existis a number $M < \infty$ such that

$$n^{1/p}\, \delta_n(A) < M$$

for infinitely many n. From the above inequality we easily infer that the inequality

(11)
$$\frac{\log n}{\log \frac{1}{\delta_n(A)}} < \frac{\log n}{\frac{1}{p}\log n - \log M}$$

also holds for infinitely many n (by log we mean $\log_2$). Now let us put $\varepsilon_n = \delta_n(A)$. Then $N_{\varepsilon_n}(A) = n$.
Hence and from (11) we get

$$1\text{-dm}(A) \le \liminf_{n\to\infty} \frac{\log N_{\varepsilon_n}(A)}{\log \frac{1}{\varepsilon_n}}$$

$$\le \liminf_{n\to\infty} \frac{\log n}{\frac{1}{p}\log n - \log M} = p.$$

This ends the proof of the theorem.

Remark. By virtue of the above theorem, for sets regular with respect to the metric dimension the notions of metric and quasi--metric dimension coincide. A good example is an arbitrary compact subset of n-dimensional Euclidean space with interior points. Indeed, it is not diffcult to show that such a set is regular with respect to metric dimension.

4. DIMENSION OF CARTESIAN PRODUCTS

Throughout the section X and Y denote arbitrary compact metric spaces; $X \times Y$ will always be regarded with the usual Euclidean distance. The letters ξ, η, ζ will denote sequences with elements in spaces X, Y, $X \times Y$ respectively.

We shall constantly use the following lemma, easy proof of which is omitted.

Lemma 1. Let (a_n) be a nonincreasing sequence of nonnegative real numbers and let p,q be arbitrary positive numbers.

Then

(i) $$\limsup_{n\to\infty} n^{1/p} a_n = \limsup_{n\to\infty} na_{[n^p]},$$

(ii) $$\liminf_{n\to\infty} n^{1/p} a_n = \liminf_{n\to\infty} na_{[n^p]},$$

(iii) $$\limsup_{n\to\infty} n^{1/(p+q)} a_n = \limsup_{n\to\infty} na_{[n^p][n^p]},$$

(iv) $$\liminf_{n\to\infty} n^{1/(p+q)} a_n = \liminf_{n\to\infty} na_{[n^p][n^q]}.$$

Theorem 3. u-dqm$(X\times Y) \le$ u-dqm(X) + u-dqm(Y).

P r o o f. Take arbitrary $\xi = (x_i)$ and $\eta = (y_i)$ and arbitrary positive numbers p,q. Now construct a sequence $\zeta = (z_i)$ so that for every n the set $\{z_1, \ldots, z_{[n^p][n^q]}\}$ coincides with the set

$$\{(x_i, y_j) : 1 \le i \le [n^p],\ 1 \le j \le [n^q]\}.$$

Then, for every n,

(12) $$\Delta_{[n^p][n^q]}(\zeta; X\times Y) \le \max\{\Delta_{[n^p]}(\xi; X),\ \Delta_{[n^q]}(\eta; Y)\}.$$

As any limit points of the sequence

$$(\max\{n\Delta_{[n^p]}(\xi;X),\ n\Delta_{[n^q]}(\eta;Y)\})$$

is also a limit point of at least one of the sequences

$$(n\Delta_{[n^p]}(\xi;X)),\quad (n\Delta_{[n^q]}(\eta;Y)),$$

by (12) we have

(13) $$\limsup_{n\to\infty} n\Delta_{[n^p][n^q]}(\zeta; X\times Y)$$

$$\le \max\{\limsup_{n\to\infty} n\Delta_{[n^p]}(\xi;X),\ \limsup_{n\to\infty} n\Delta_{[n^q]}(\eta;Y)\},$$

By *Lemma 1*, (13) implies

(14)
$$\overline{D}_{p+q}(X \times Y)$$

$$\leq \max \{\limsup_{n \to \infty} n^{1/p} \Delta_n(\xi;X), \ \limsup_{n \to \infty} n^{1/q} \Delta_n(\eta;Y)\}.$$

The sequences ξ, η being arbitrary, from (14) we get

(15)
$$\overline{D}_{p+q}(X \times Y) \leq \max \{\overline{D}_p(X), \overline{D}_q(Y)\}.$$

If u-dqm$(X) = \infty$ or u-dqm$(Y) = \infty$, the theorem is obvious. Suppose u-dqm$(X) < \infty$ and u-dqm$(Y) < \infty$ and take an arbitrary $r >$ u-dqm(X) + u-dqm(Y). We can find then positive numbers p and q such that $p+q = r$, $p >$ u-dqm(X), $q >$ u-dqm(Y). Now (15) implies that $\overline{D}_r(X \times Y) = \overline{D}_{p+q}(X \times Y) = 0$.

This ends the proof of the theorem.

Lemma 2. Let s,t be arbitrary positive integers. Then

(16)
$$\min \{\delta_s(X), \delta_t(Y)\} \leq 2\delta_{st}(X \times Y).$$

P r o o f. Take an arbitrary ζ and put $R = \Delta_{st}(\zeta; X \times Y)$. Suppose that

(17)
$$\delta_s(X) > 2R$$

and take an arbitrary $a_1 \in X$. By (17) there exists an $a_2 \in X$ such that $a_2 \notin B(a_1;2R)$ (where the symbol $B(a;r)$ denotes a closed ball of centre a and radius r), so that the balls $B(a_1;R)$ and $B(a_2;R)$ are disjoint. Similarly, if there exist j points $(j < s)$ $a_1, \dots, a_j \in X$ such that the balls $B(a_i,R)$ are mutually disjoint, by (17) there exists an $a_{j+1} \in X$ such that $a_{j+1} \notin B(a_i;R)$ for $i = 1, \dots, j$. By induction we conclude the existence of s points $a_1, \dots, a_s \in X$ such that the balls $B(a_i;R)$ are mutually disjant $(i = 1, \dots, s)$. In at least one of the sets $B(a_i;R) \times Y$ not more than t from the first elements of the sequence ζ will be contained.

Hence

(18)
$$\delta_t(Y) \leq R.$$

The inequalities (17) and (18) give (16).

This ends the proof of the lemma.

Theorem 4. For any positive integer m,

$$u\text{-}dqm(X^m) = m\ u\text{-}dqm(X).$$

P r o o f. By *Theorem 3*, $u\text{-}dqm(X^m) \le m\ u\text{-}dqm(X)$. Now, by an easy generalization of *Lemma 2*,

$$\delta_s(X) \le 2^m\ \delta_{s^m}(X^m). \tag{19}$$

Let us take an arbitrary $p > 0$. From (19), for every n

$$\delta_{[n^p]}(X) \le 2^m\ \delta_{[n^p]^m}(X^m). \tag{20}$$

By (20) and *Lemma 1* (iii) (after obvious generalization) we get

$$\bar{d}_p(X) \le 2^m\ \bar{d}_{pm}(X^m). \tag{21}$$

If $pm > u\text{-}dqm(X^m)$, from (21) we obtain $\bar{d}_p(X) = 0$ so that $p \ge$ $\ge u\text{-}dqm(X)$.

This ends the proof of the theorem.

Theorem 5. If X is regular, then

$$u\text{-}dqm(X \times Y) = dqm(X) + u\text{-}dqm(Y).$$

P r o o f. By *Theorem 3* it suffices to show the inequality

$$dqm(X) + u\text{-}dqm(Y) \le u\text{-}dqm(X \times Y). \tag{22}$$

If $dqm(X) = 0$ or $dqm(Y) = 0$, (22) is fairly obvious. Suppose $dqm(X) \ne 0$ and $u\text{-}dqm(Y) \ne 0$. Take an arbitrary positive number $r < dqm(X) + u\text{-}dqm(Y)$. There exist positive numbers p, q such that $p < dqm(X)$, $q < u\text{-}dqm(Y)$ and $p+q = r$. By virtue of *Lemma 1* (i), there exists an increasing sequence (k_n) of positive integers such that

(23) $$k_n \delta_{[k_n q]}(Y) \to \infty \quad \text{as} \quad n \to \infty.$$

Similarly, by *Lemma 1* (ii),

(24) $$\liminf_{n \to \infty} k_n \delta_{[k_n p]}(X) = \infty.$$

The equality (24) means that

(25) $$k_n \delta_{[k_n p]}(X) \to \infty \quad \text{as} \quad n \to \infty.$$

By *Lemma 2*, for every n

(26) $$\min \{n\delta_{[n^p]}(X),\ n\delta_{[n^q]}(Y)\}$$

$$\leq 2n\delta_{[n^p][n^q]}(X \times Y).$$

From (23), (25) and (26) we get

$$k_n \delta_{[k_n p][k_n q]}(X \times Y) \to \infty \quad \text{as} \quad n \to \infty,$$

what implies

(27) $$\limsup_{n \to \infty} n\,\delta_{[n^p][n^q]}(X \times Y) = \infty.$$

Now, by *Lemma 1* (iii) and (27) we obtain

$$\bar{d}_r(X \times Y) = \bar{d}_{p+q}(X \times Y) = \infty,$$

so that $r \leq$ u-dqm$(X \times Y)$.

This ends the proof of the theorem.

Corollary. If X and Y are regular, then $X \times Y$ is regular and

$$\text{dqm}(X \times Y) = \text{dqm}(X) + \text{dqm}(Y).$$

P r o o f. The equality will follow at once from *Theorem 5* if we show regularity of $X \times Y$. We shall make use of the proof of *Theorem 5*. We can replace (23) and (25) by

(28) $$n\,\delta_{[n^p]}(X) \to \infty \quad \text{as} \quad n \to \infty$$

and

(29) $$n\,\delta_{[n^q]}(Y) \to \infty \quad \text{as} \quad n \to \infty .$$

From (26), (28) and (29) we get

(30) $$n\,\delta_{[n^p][n^q]}(X \times Y) \to \infty \quad \text{as} \quad n \to \infty .$$

From (30) we conclude the inequality

(31) $$\mathrm{dqm}(X) + \mathrm{dqm}(Y) \leq 1\text{-}\mathrm{dqm}(X \times Y).$$

But by *Theorem 3* we have

(32) $$\text{u-dqm}(X \times Y) \leq \mathrm{dqm}(X) + \mathrm{dqm}(Y).$$

From (31) and (32) we deduce the required regularity of $X \times Y$.

This ends the proof of the corollary.

5. SOME COMMENT AND EXAMPLES

The proofs of the facts given in this section will all be omitted. Almost all of them are easy.

There is an interesting relation between Hausdorff dimension (see, for example, [1], section 14) and quasi-metric dimension. Namely, $\text{H-dim}(X) \leq 1\text{-dqm}(X)$, where H-dim denotes Hausdorff dimension. This allows us to calculate without difficulty the quasi-metric dimension of some interesting sets. For example, the Cantor set (which turns out to be regular) has the dimension equal to $\log_3 2$. Also, if M is the surface in R^3 determined by a real function defined on the unit square and satisfying there the Lipschitz condition with an arbitrary constant, then M is regular and $\mathrm{dqm}(M) = 2$ (cf. [1], section 14).

Any compact subset X of R^m with (strictly) positive m-dimensional Lebesgue measure is regular and $\mathrm{dqm}(X) = m$.

Now we shall point out a class of countable subsets of the unit interval with positive dimension. Namely, putting $X_p = \{1/n^p : n = 1,2, \ldots\}$ we get a regular set with the dimension

$dqm(X_p) = 1/(1+p)$. If we replace the natural distance on X_p by the equivalent distance

$$\varrho(x,y) = |x^3 - y^3|,$$

the new dimension will be equal to $1/(1+3p)$, which shows clearly that the dimension is not a topological invariant.

The set $\{1/2^n : n = 1,2, \dots\}$ is of dimension zero (so are all finite sets). On the other hand, the Hilbert cube has the dimension equal to infinity.

It may be difficult (even in the simplest cases) but quite interesting to find the p-dimensional diameter of a set. For example, for the unit interval I, $\overline{D}_1(I) = 1/\log 4$ (for proof, see [2], section 12.7, problem 6 and [3]).

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O PEWNYM RODZAJU WYMIARU METRYCZNEGO

W pracy bada się własności pewnego wymiaru, dającego się stosować do podzbiorów przestrzeni metrycznej, a zbliżonego do wymiaru metrycznego Kołmogorowa. Dowodzi się między innymi, że $\dim X \times Y = \dim X + \dim Y$ dla dowolnych przestrzeni metrycznych X i Y. Praca zawiera też przykłady wymiarów konkretnych przestrzeni.