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ON SOME EXTREMUM PROBLEM IN THE FAMILY OF NON-DECREASING FUNCTIONS

In the paper there have been obtained, on the basis of the Ioffe-Tikhomirov extremum principle, an existential theorem and necessary conditions for the existence of extremum for the following optimization problem: minimize the functional $\int_a^b \Phi(x(t), t)dt$ under the conditions

$$x(t) = \int_a^b q(t,\tau)d\mu(\tau), \quad \int_a^b d\mu_i(\tau) = 1 \quad \text{for } i = 1, 2, \ldots, n.$$

INTRODUCTION

In the paper there have been obtained an existential theorem and necessary conditions for the existence of extremum for the following optimization problem: minimize the functional $\int_a^b \Phi(x(t), t)dt$ under the conditions $x(t) = \int_a^b q(t,\tau)d\mu(\tau)$, where $\Phi : R^n \times [a,b] \to R$, $q : [a,b] \times [a,b] \to R$, $\mu : [a,b] \to R^n$ and $x : [a,b] \to R^n$. Besides, it is assumed that $\mu(\cdot)$ is a normed and non-decreasing function, whereas $x(\cdot)$ is absolutely continuous on the interval $[a,b]$.

Necessary conditions for optimality, for the problem under consideration, have been proved on the basis of the Ioffe-Tikhomirov extremum principle.

1. FORMULATION OF THE EXTREMUM PROBLEM. AN EXISTENTIAL THEOREM

Let $\Phi(x,t)$ and $q(t,\tau)$ be functions defined on $\mathbf{R}^n \times \mathbf{R}$ and $\mathbf{R} \times \mathbf{R}$, respectively, with values in $\mathbf{R}$.

Assume that

1° $q(\cdot,\tau)$ is an absolutely continuous function for every τ,

2° $q_t(\cdot,\cdot)$ is continuous with respect to the group of variables,

3° $\Phi(\cdot,\cdot)$ and $\Phi_x(\cdot,\cdot)$ are continuous functions with respect to the group of variables.

Consider the following

Problem 1. Determine the minimal value of the functional

$$I(x) = \int_a^b \Phi x(t),t)dt, \tag{1}$$

under the conditions

$$x(t) = \int_a^b q(t,\tau)d\mu(\tau), \tag{2}$$

$$\int_a^b d\mu(\tau) = 1, \quad i = 1, \ldots, n, \tag{3}$$

$$\mu(\cdot) \in U, \tag{4}$$

where U is a set of non-decreasing vector functions defined on the interval $[a, b]$, with values in $\mathbf{R}^n$. In other words,

$$\forall(\mu(\cdot) \in U) \text{ and } \forall(t \in [a,b]),\ \mu(t) = (\mu_1(t),\ldots,\mu_n(t)) \in \mathbf{R}^n,$$

and $\mu_i(\cdot)$, for $i = 1, 2, \ldots, n$, are non-decreasing functions.

To begin with, let us notice that, under assumption 1°, $x(\cdot)$ is an absolutely continuous vector function, that is, for each $i = 1, 2, \ldots, n$, $x_i(\cdot)$ is absolutely continuous. Indeed, it follows from assumption 1^{o} that, for any $\tau \in [a,b]$,

$$q(t,\tau) = q(a,\tau) + \int_a^t q_t'(t,\tau)dt.$$

Consequently,

$$x(t) = \int_a^b q(t,\tau)d\mu(\tau) = \int_a^b (q(a,\tau) + \int_a^t q_t'(t,\tau)dt)d\mu(\tau) =$$

$$= \int_a^b q(a,\tau)d\mu(\tau) + \int_a^b (\int_a^t q_t'(t,\tau)dt)d\mu(\tau) =$$

$$= x(a) + \int_a^t (\int_a^b q_t'(t,\tau)d\mu(\tau))dt = x(a) + \int_a^t \dot{x}(t)dt,$$

which means the absolute continuity of the function $x(\cdot)$.

Let

$$U_1 := \left\{ \mu(\cdot) \in U \,\middle|\, \int_a^b d\mu_i(\tau) = 1, \ i = 1, 2, \ldots, n \right\},$$

$$U_A := \left\{ \mu(\cdot) \in U \,\middle|\, \int_a^b d\mu_i(\tau) = 1, \ i = 1, 2, \ldots, n, \ \mu(a) = A \right\},$$

where $A \in \mathbf{R}^n$ is a fixed point, and let $x(\cdot)$ be a function corresponding to $\mu(\cdot)$ through relation (2).

Of course

$$\inf_{\mu \in U_1} I(x) = \inf_{\mu \in U_A} I(x).$$

It is not hard to notice that U_A is a set of commonly bounded functions with commonly bounded variation, where by the full variation of the function $\mu(\cdot)$ we mean

$$\bigvee_a^b (\mu) = \sum_{i=1}^{n} \bigvee_a^b (\mu_i).$$

From the second theorem of Helly (cf. [3], VI, § 6) results the following

Lemma 1.1. U_A is a compact set in the topology of pointwise convergence.

Let $W^n_{11}([a,b])$ stand for a space of vector functions absolutely continuous on the interval $[a,b]$, with norm

$$\| x \| = |x(a)| + \int_a^b |\dot{x}(t)|dt.$$

Consider an operation $L : U_A \to W^n_{11}$ defined as follows

$$(L\mu)(t) := \int_a^b q(t,\tau)d\mu(\tau) = x(t). \tag{5}$$

Let us take any sequence $\{u^k\}_{k=1}^{\infty}$ of elements of the set U_A, pointwise convergent to a function μ belonging to U_A.

From the first theorem of Helly (cf. [3], VI, § 6) it follows that, for each $t \in [a,b]$, the sequences of functions $\left\{\int_a^b q(t,\tau)d\mu^k(\tau)\right\}_{k=1}^{\infty}$ and $\left\{\int_a^b q'_t(t,\tau)d\mu^k(\tau)\right\}_{k=1}^{\infty}$ converge to the functions $\int_a^b q(t,\tau)d\mu(\tau)$ and $\int_a^b q'_t(t,\tau)d\mu(\tau)$, respectively. Hence, in particular for $t = a$ and any $\varepsilon > 0$, there exists some $k_1 \in N$ such that, for each $k \geqslant k_1$, the inequality

$$\left| \int_a^b q(a,\tau)d\mu^k(\tau) - \int_a^b q(a,\tau)d\mu(\tau) \right| < \varepsilon$$

takes place.

Let

$$\varphi_k(t) := \int_a^b q'_t(t,\tau)d\mu^k(\tau) - \int_a^b q'_t(t,\tau)d\mu(\tau).$$

From this and from the above it follows that the sequence

$\{\varphi_k(\cdot)\}_{k=1}^{\infty}$ is pointwise convergent to zero in R^n. Thereby, the sequence $\{|\varphi_k(\cdot)|\}_{k=1}^{\infty}$ is pointwise convergent to zero.

By making use of assumption 2^o and the fact that $\mu^k(\cdot)$ and $\mu(\cdot)$ are non-decreasing functions, it is not difficult to show that the sequence of functions $\{|\varphi_k(\cdot)|\}_{k=1}^{\infty}$ is a sequence of commonly bounded functions. Consequently, in virtue of the Lebesgue theorem, for each $\varepsilon > 0$, there exists some $k_2 \in N$ such that, for each $k \geqslant k_2$, we have

$$\left| \int_a^b |\varphi_k(t)| \, dt \right| < \varepsilon.$$

In view of the above, for each $\varepsilon > 0$, there exists some k_o, $k_o = \max(k_1, k_2)$, such that, for each $k \geqslant k_o$, the inequality

$$0 \leqslant \| (L\mu^k) - (L\mu) \| = \| x^k - x \| =$$

$$= \left| \int_a^b q(a,\tau)d\mu^k(\tau) - \int_a^b q(a,\tau)d\mu(\tau) \right| +$$

$$\int_a^b \left| \int_a^b q_t'(t,\tau)d\mu^k(\tau) - \int_a^b q_t'(t,\tau)d\mu(\tau) \right| dt < 2\varepsilon$$

takes place. Hence, and from the arbitrariness of ε, results the following

Lemma 1.2. L is a continuous operation in the topology of pointwise convergence.

Let

$$W := \left\{ x(\cdot) \in W_{11}^n([a,b]) \,\middle|\, x(t) = \int_a^b q(t,\tau)d\mu(\tau),\ \mu(\cdot) \in U_A \right\}.$$

Since L is a continuous operation, whereas the set U_A is compact in the topology of pointwise convergence, therefore W, as the continuous image of the compact set, is a compact set in the topology of the space $W^n_{11}([a,b])$.

Lemma 1.3. $I(\cdot)$ is a functional differentiable at an arbitrary point x_0 and, for each $x \in W^n_{11}$

$$I_x(x_0)x = \int_a^b (\Phi_x(x_0(t),t),\ x(t))\,dt.$$

The proof of the above lemma runs identically as that of lemma 7.2 (cf. [1], § 7).

It follows from *Lemma 1.3* that $I(\cdot)$ is a continuous functional on the space $W^n_{11}([a,b])$.

Under the assumptions made about the functions Φ and q as well as in virtue of *Lemmas 1.1-1.3* and the Weierstrass theorem, the following one is true:

Theorem 1.1. Problem 1 possesses a solution $(x^*(\cdot),\ \mu^*(\cdot))$ where $x^*(\cdot)$ is an absolutely continuous function defined by formula (2), and $\mu^*(\cdot) \in U$.

2. THE INTEGRAL NECESSARY CONDITION

Let $X: = W^n_{11}([a,b])$, $Y: = W^n_{11}([a,b])$, while

$$f_0(x,\mu): = \int_a^b \Phi(x(t),t)\,dt, \tag{6}$$

$$F(x,\mu): = x(t) - \int_a^b q(t,\tau)\,d\mu(\tau), \tag{7}$$

$$h_i(x,\mu): = \int_a^b d\mu_i(\tau) - 1, \quad i = 1, 2, \ldots, n \tag{8}$$

and $\mu(\cdot) \in U$ where U is, as before, a set of non-decreasing vector functions.

As is well known (cf. [2], § 0.1) X and Y are Banach spaces and, besides,

$$F : X \times U \to Y,$$
$$h : X \times U \to R^n,$$

where $h = (h_1, h_2, \ldots, h_n)$.

Note that, for each fixed $\mu(\cdot) \in U$ and any $\bar{x} \in X$, we have

$$F(x + \bar{x},\mu) - F(x,\mu) = \bar{x}.$$

Hence it appears that $x \to F(x,\mu)$ is a regular mapping of class C_1. Since U is a convex set, and the Stjelties integral - a linear transformation, therefore F is a convex operator with respect to μ. The functional $f_o(\cdot,\cdot)$ does not depend explicitly on μ, so the convexity condition with respect to μ is satisfied also for the functional f_o. Making use of *Lemma 1.3*, we infer that the mapping $x \to f_o(x,\mu)$ is of class C_1 at any fixed point $x \in X$.

The operator F, the functional f_o and the vector function h satisfy the assumptions of the Ioffe-Tikhomirov extremum principle (cf. [2], I, § 1.1).

With the notations introduced above, the Lagrange function for *Problem 1* takes the form:

$$\mathcal{L}(x,\mu,\lambda_o,\lambda_1,y^*) = \lambda_o f_o(x,\mu) + (\lambda_1,h) + (y^*,F(x,\mu)), \tag{9}$$

where $\lambda_o \in R$, $\lambda_1 \in R^n$ and $\lambda_1 = (\lambda_1^1, \lambda_1^2, \ldots, \lambda_1^n)$, while $y^* \in Y^*$.

Theorem 2.1. (The integral extremum principle). If assumptions 1^o-3^o are satisfied and the pair $(x^*(\cdot),\mu^*(\cdot))$ is a solution to *Problem 1*, then there exist: an absolutely continuous function $\eta(\cdot)$ and constants $0 \leqslant \lambda_o \in \mathbf{R}$, $\lambda_1 \in \mathbf{R}^n$ and $\lambda_2 \in \mathbf{R}^n$ not vanishing simultaneously and such that

(i) $$\frac{d\eta(t)}{dt} = \lambda_o \Phi_x(x^*(t),t) \quad \text{for} \quad t \in [a,b] \quad \text{a.e.,} \quad \eta(b) = 0$$

(ii) $$\int_a^b \Big(\lambda_1 - \int_a^b \eta(t)q_t'(t,\tau)dt - \lambda_2 q(a,\tau), d[\mu(\tau) - \mu^*(\tau)]\Big) \geqslant 0$$

for each $\mu(\cdot) \in U$.

P r o o f. Let $(x^*(\cdot),\mu^*(\cdot))$ be a solution to *Problem 1*. By the Ioffe-Tikhomirov extremum principle, there exist multipliers $0 \leqslant \lambda_0 \in R$, $\lambda_1 \in R^n$ and $y^* \in Y^*$ not vanishing simultaneously, such that

(10) $$\mathcal{L}_x(x^*,\mu^*,\lambda_0,\lambda_1,y^*) = 0$$

and

(11) $$\mathcal{L}(x^*,\mu^*,\lambda_0,\lambda_1,y^*) = \min_{\mu(\cdot)\in U} \mathcal{L}(x^*,\mu,\lambda_0,\lambda_1,y^*).$$

Since $Y = W^n_{11}([a,b])$, therefore

(12) $$(y^*,F(x,\mu)) = (\lambda_2,x(a) - \int_a^b q(a,\tau)d\mu(\tau)) + \\ + \int_a^b (\eta(t),\dot{x}(t) - \frac{d}{dt}\int_a^b q(t,\tau)d\mu(\tau))dt,$$

where $\lambda_2 \in R^n$, and $\eta(\cdot) \in L^n_\infty([a,b])$.

Let us write down explicitly the Lagrange function (9) for *Problem 1* at the point $(x^*(\cdot),\mu^*(\cdot))$. Taking (6), (8) and (12) into consideration, we have

(13) $$\mathcal{L}(x^*,\mu^*,\lambda_0,\lambda_1,y^*) = \lambda_0\int_a^b \Phi(x^*(t),t)dt + \\ + \sum_{i=1}^n \lambda_1^i(\int_a^b d\mu_i^*(\tau) - 1) + \int_a^b (\eta(t),\dot{x}^*(t) - \int_a^b q'_t(t,\tau)d\mu^*(\tau))dt + \\ + (\lambda_2,x^*(a) - \int_a^b q(a,\tau)d\mu^*(\tau)).$$

Determine the differential of the function $\mathcal{L}(\cdot)$ at the point $(x^*(\cdot),\mu^*(\cdot))$. Let x be any element of X. In view of assumption 3^0, we have

$$\mathcal{L}(x^* + x,\mu^*,\lambda_o,\lambda_1,y^*) - \mathcal{L}(x^*, \mu^*, \lambda_o, \lambda_1,y^*) =$$

$$= \lambda_o \int_a^b (\Phi(x^*(t) + x(t),t) - \Phi(x^*(t),t))dt +$$

$$+ \int_a^b (\eta(t),\dot{x}(t))dt + (\lambda_2,x(a)) \doteq \lambda_o \int_a^b (\Phi_x(x^*(t),t),x(t))dt +$$

$$+ \lambda_o \int_a^b (\Phi_x(x^*(t) + \Theta(t)x(t),t) - \Phi_x(x^*(t),t),x(t))dt +$$

$$+ \int_a^b (\eta(t),\dot{x}(t))dt + (\lambda_2,x(a)),$$

where $0 \leqslant \Theta(t) \leqslant 1$ for $t \in [a,b]$.

It is easy to demonstrate that

$$\int_a^b (\Phi_x(x^*(t) + \Theta(t)x(t),t) - \Phi_x(x^*(t),t),x(t))dt =$$

$$= O(\|x\|).$$

From this and from the definition of the differential it follows that, for any $x \in X$,

$$\mathcal{L}_x(x^*,\mu^*, \lambda_o, \lambda_1,y^*)x = \tag{14}$$

$$= \lambda_o \int_a^b (\Phi_x(x^*(t),t),x(t))dt + \int_a^b (\eta(t),\dot{x}(t))dt + (\lambda_2,x(a)).$$

Integrating by parts the first addend of this last equality, we get

$$\lambda_o \int_a^b (\Phi_x(x^*(t),t)x(t))dt = \tag{15}$$

$$= \left(\lambda_0 \int_a^t \Phi_x(x^*(\tau),\tau)d\tau, x(t)\right)\Bigg|_{t=a}^{t=b} +$$

$$- \int_a^b \left(\lambda_0 \int_a^t \Phi_x(x^*(\tau),\tau)d\tau, \dot{x}(t)\right)dt =$$

$$= \left(\lambda_0 \int_a^b \Phi_x(x^*(\tau),\tau)d\tau, x(b)\right) -$$

$$- \int_a^b \left(\lambda_0 \int_a^t \Phi_x(x^*(\tau),\tau)d\tau, \dot{x}(t)\right)dt.$$

Since equality (14) holds for any $x \in X$, therefore it holds, in particular, for those x for which $x(a) = 0$. Yet then equality (10), after taking account of (14), (15) and $x(a) = 0$, will take the form

$$\text{(16)} \qquad \left(\lambda_0 \int_a^b \Phi_x(x^*(\tau),\tau)d\tau, x(b)\right) +$$

$$- \int_a^b \left(\lambda_0 \int_a^t \Phi_x(x^*(\tau),\tau)d\tau, \dot{x}(t)\right)dt +$$

$$+ \int_a^b (\eta(t), \dot{x}(t))dt = 0.$$

Since $x(\cdot) \in W^n_{11}([a,b])$ and $x(a) = 0$, therefore $x(t) = \int_a^t \dot{x}(t)dt$. From this and from (16) we obtain that

$$\left(\lambda_0 \int_a^b \Phi_x(x^*(\tau),\tau)d\tau, \int_a^b \dot{x}(t)dt\right) +$$

$$- \int_a^b (\lambda_o \int_a^t \Phi_x(x^*(\tau),\tau)d\tau, \dot{x}(t))dt +$$

$$+ \int_a^b (\eta(t), \dot{x}(t))dt = 0$$

and, next,

$$\int_a^b (\int_a^b \lambda_o\Phi_x(x^*(\tau),\tau)d\tau, \dot{x}(t))dt +$$

$$- \int_a^b (\int_a^t \lambda_o\Phi_x(x^*(\tau),\tau)d\tau, \dot{x}(t))dt +$$

$$+ \int_a^b (\eta(t), \dot{x}(t))dt = 0.$$

In virtue of the additivity of the integral, we finally get the equality

$$\int_a^b (\int_t^b \lambda_o\Phi_x(x^*(\tau),\tau)d\tau + \eta(t), \dot{x}(t))dt = 0 \tag{17}$$

for any $\dot{x}(\cdot) \in W^n_{11}([a,b])$, $x(a) = 0$ and $\eta(\cdot) \in L^n_\infty([a,b])$. The function $\int_t^b \lambda_o\Phi_x(x^*(\tau),\tau)d\tau + \eta(t)$ is an element of the space $L^n_\infty([a,b])$, whereas $\dot{x}(\cdot) \in L^n_1([a,b])$.

From this and from (17) we deduce that

$$\int_t^b \lambda_o\Phi_x(x^*(\tau),\tau)d\tau + \eta(t) = 0 \quad \text{for} \quad t \in [a,b] \quad \text{a.e.} \tag{18}$$

or, in the equivalent form,

$$\frac{d\eta(t)}{dt} = \lambda_o\Phi_x(x^*(t),t) \quad \text{for} \quad t \in [a,b] \quad \text{a.e.}, \quad \eta(b) = 0. \tag{19}$$

From (18) it also follows that $\eta(\cdot)$ is an absolutely continuous function.

Let us now make some analysis of condition (11). Making use of (13) and disregarding the addends independent of μ on the left- and right-hand sides of equality (11), we obtain the relation

$$\left(\lambda_1, \int_a^b d\mu^*(\tau)\right) - \int_a^b \left(\eta(t), \frac{d}{dt}\int_a^b q(t,\tau)d\mu^*(\tau)\right)dt +$$

$$- \left(\lambda_2, \int_a^b q(a,\tau)d\mu^*(\tau)\right) = \min_{\mu(\cdot)\in U}\left[\left(\lambda_1, \int_a^b d\mu(\tau)\right) + \right.$$

$$\left. - \int_a^b \left(\eta(t), \frac{d}{dt}\int_a^b q(t,\tau)d\mu(\tau)\right)dt - \left(\lambda_2, \int_a^b q(a,\tau)d\mu(\tau)\right)\right].$$

From the above and the assumption about the function $q(\cdot,\cdot)$ follows that

$$\left(\lambda_1, \int_a^b d[\mu(\tau) - \mu^*(\tau)]\right) +$$

$$- \int_a^b \left(\eta(t), \int_a^b q_t'(t,\tau)d[\mu(\tau) - \mu^*(\tau)]\right)dt +$$

$$- \left(\lambda_2, \int_a^b q(a,\tau)d[\mu(\tau) - \mu^*(\tau)]\right) \geqslant 0 .$$

for any $\mu(\cdot) \in U$. Hence, by changing the order of integration, we get

$$\int_a^b \left(\lambda_1, d[\mu(\tau) - \mu^*(\tau)]\right) +$$

$$- \int_a^b (\int_a^b \eta(t) q'_t(t,\tau)dt, d[\mu(\tau) - \mu^*(\tau)]) +$$

$$- \int_a^b (\lambda_2 q(a,\tau), d[\mu(\tau) - \mu^*(\tau)]) \geqslant 0.$$

By the additivity of the integral, we obtain at last that

$$\int_a^b (\lambda_1 - \int_a^b \eta(t) q'_t(t,\tau)dt - \lambda_2 q(a,\tau), d[\mu(\tau) - \mu^*(\tau)]) \geqslant 0 \tag{20}$$

for any $\mu(\cdot) \in U$, which ends the proof of the theorem.

Remark. If, in addition, it is known that $q(a,\cdot) = 0$, then the Lagrange function (13) takes the form

$$\mathcal{L}(x^*, \mu^*, \lambda_o, \lambda_1, y^*) =$$

$$= \lambda_o \int_a^b \Phi(x^*(t),t)dt + \sum_{i=1}^n \lambda_1^i (\int_a^b d\mu_i^*(\tau) - 1) +$$

$$+ \int_a^b (\eta(t), \dot{x}^*(t) - \int_a^b q'_t(t,\tau)d\mu^*(\tau))dt$$

and, in virtue of the extremum principle, we find that the multipliers $\lambda_o, \lambda_1, \eta(\cdot)$ do not vanish simultaneously.

In the sequel, by $g(\cdot)$ we shall mean a function of the form

$$g(\tau) = \lambda_1 - \int_a^b \eta(t) q'_t(t,\tau)dt - \lambda_2 q(a,\tau). \tag{21}$$

We shall write inequality (20) shortly in the form

$$\forall (\mu(\cdot) \in U), \int_a^b (g(\tau), d[\mu(\tau) - \mu^*(\tau)]) \geqslant 0. \tag{22}$$

2. THE LOCAL NECESSARY CONDITION

In conformity with the conditions of the problem, the function $\mu:[a,b]\to R^n$, and $g:[a,b]\to R^n$. Let $\mu(\cdot) = (\mu_1(\cdot), \ldots, \mu_n(\cdot))$, and $g(\cdot) = (g_1(\cdot), \ldots, g_n(\cdot))$. It is not difficult to check that from (22) follows the veracity of the inequality

$$(23)\qquad \int_a^b g_i(\tau)\,d\left[\mu_i(\tau) - \mu_i^*(\tau)\right] \geq 0$$

for any non-decreasing function $\mu_i(\cdot)$ and $i = 1, 2, \ldots, n$. Moreover, note that $g(\cdot)$ given by formula (21) is a continuous vector function.

Let

$$m_i := \min_{\tau\in[a,b]} g_i(\tau),$$

whereas

$$Z_{m_i} := \{\tau \in [a,b] \mid g_i(\tau) = m_i\} \quad \text{for } i = 1, 2, \ldots, n.$$

We shall show that

$$(24)\qquad \int_a^b g_i(\tau)\,d\mu_i^*(\tau) = m_i = 0 \quad \text{for } i = 1, 2, \ldots, n.$$

It is known that

$$\int_a^b g_i(\tau)\,d\mu_i^*(\tau) \geq m_i \int_a^b d\mu_i^*(\tau) = m_i \quad \text{for } i = 1, 2, \ldots, n.$$

Since inequality (23) is true for any non-decreasing function, therefore it also holds for a function $\bar{\mu}_i(\cdot) = \text{const}$. From this and from the above

$$m_i \leqslant \int_a^b g_i(\tau)\,d\mu_i^*(\tau) \leqslant \int_a^b g_i(\tau)\,d\tilde{\mu}_i(\tau) = 0$$

for $i = 1, 2, \ldots, n$.

Suppose that $m_i < 0$. Let $\tau_o^i \in Z_{m_i}$ $(Z_{m_i} \neq \emptyset)$, and let

$$\hat{\mu}_i(\tau) = \begin{cases} 0 & \text{for} \quad \tau \in [a, \tau_o^i], \\ 2 & \text{for} \quad \tau \in (\tau_o^i, b]. \end{cases}$$

For the function $\hat{\mu}_i(\cdot)$, in virtue of (23), we obtain

$$m_i \leqslant \int_a^b g_i(\tau)\,d\mu_i^*(\tau) \leqslant \int_a^b g_i(\tau)\,d\hat{\mu}_i(\tau) = g_i(\tau_o^i)\cdot 2 = 2m_i$$

for $i = 1, 2, \ldots, n$. Yet, the inequality obtained, $m_i \leqslant 2m_i$, is false for $m_i < 0$ and concludes the proof of equality (24).

The set Z_{m_i} is closed, therefore

$$G_i := (a,b) \setminus Z_{m_i}$$

is an open linear set for $i = 1, 2, \ldots, n$. Hence

$$G_i = \bigcup_{k=1}^{\infty} (\alpha_i^k, \beta_i^k),$$

where (α_i^k, β_i^k) for $k = 1, 2, \ldots$ are disjoint open subintervals. We shall show that, on each interval (α_i^k, β_i^k), $k = 1, 2, \ldots$ the function $\mu_i^*(\cdot)$, $i = 1, 2, \ldots, n$, is constant. Suppose that there exists an interval $(\alpha_i^{k_o}, \beta_i^{k_o})$ such that

$$\lim_{\tau \to \beta_i^{k_o}-0} \mu_i^*(\tau) > \lim_{\tau \to \alpha_i^{k_o}+0} \mu_i^*(\tau).$$

And consequently, there exists a closed interval $[c_i, d_i] \subset (\alpha_i^{k_o}, \beta_i^{k_o})$ such that

$$\mu_i^*(c_i) < \mu_i^*(d_i)$$

and $\min_{\tau\in[c_i,d_i]} g_i(\tau) = \varepsilon_i$, where $\varepsilon_i > 0$ for $i = 1, 2, \ldots, n$.

Then

$$0 = \int_a^b g_i(\tau)d\mu_i^*(\tau) = \int_a^{c_i} g_i(\tau)d\mu_i^*(\tau) + \int_{c_i}^{d_i} g_i(\tau)d\mu_i^*(\tau) +$$

$$+ \int_{d_i}^b g_i(\tau)d\mu_i^*(\tau) \geq \min_{\tau\in[c_i,d_i]} g_i(\tau)\left[\mu_i^*(d_i) - \mu_i^*(c_i)\right] =$$

$$= \varepsilon_i\left[\mu_i^*(d_i) - \mu_i^*(c_i)\right] > 0,$$

which gives a contradiction. So, $\mu_i^*(\cdot)$ is constant on each interval (α_i^k, β_i^k) for $k = 1, 2, \ldots$ and $i = 1, 2, \ldots, n$.

The non-decreasing function $\mu_i^*(\cdot)$ possesses an at most conntable number of points of discontinuity. Since $\mu_i^*(\cdot)$ is a constant function on (α_i^k, β_i^k) for $k = 1, 2, \ldots$ therefore its only points of discontinuity are those belonging to the set Z_{m_i} for $i = 1, 2, \ldots, n$. It is not hard to check, either, that in the case where $a \notin Z_{m_i}$ or $b \notin Z_{m_i}$, $\mu_i^*(a) = \lim_{\tau\to a+0} \mu_i^*(\tau)$ or, respectively, $\mu_i^*(b) = \lim_{\tau\to b-0} \mu_i^*(\tau)$ for $i = 1, 2, \ldots, n$.

Indeed, suppose that $a \notin Z_{m_i}$ and let

$$\mu_i^*(a) < \lim_{\tau\to a+0} \mu_i^*(\tau).$$

Then $g_i(a) > 0$ and

$$\int_a^b g_i(\tau)d\mu_i^*(\tau) \geq g_i(a)\left[\lim_{\tau\to a+0} \mu_i^*(\tau) - \mu_i^*(a)\right] > 0,$$

which contradicts (24).

We have thus proved the following

Theorem 3.1. (The local necessary condition). If assumptions, 1°-3° are satisfied, and

4° the function $g(\cdot) = (g_1(\cdot), g_2(\cdot), \ldots, g_n(\cdot))$, defined by formula (21), satisfies condition (22),

5° the function $\mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_n(\cdot))$, satisfies conditions (3) and (4),

6° the pair $(x^*(\cdot), \mu^*(\cdot))$, where $x^*(\cdot) = (x_1^*(\cdot), x_2^*(\cdot), \ldots, x_n^*(\cdot))$,

$$\mu^*(\cdot) = (\mu_1^*(\cdot), \mu_2^*(\cdot), \ldots, \mu_n^*(\cdot)),$$

is a solution of *Problem 1,* then, for each $i = 1, 2, \ldots, n$

1) $\displaystyle\int_a^b g_i(\tau)\, d\mu_i^*(\tau) = 0 = \min_{\tau\in[a,b]} g_i(\tau)$,

2) $\mu_i^*(\cdot)$ is a function constant on each interval on which $g_i(\cdot)$ has a constant sign,

3) points of discontinuity of the function $\mu_i^*(\cdot)$ belong to the set

$$Z_{m_i} = \{\tau\in[a,b] \mid g_i(\tau) = 0\}.$$

If $a \notin Z_{m_i}$ or $b \notin Z_{m_i}$, then $\mu_i^*(a) = \lim_{\tau\to a+0} \mu_i^*(\tau)$ or, respectively, $\mu_i^*(b) = \lim_{\tau\to b-0} \mu_i^*(\tau)$.

Example. Determine the minimal value of the functional

$$I(x) = \int_0^2 tx(t)\, dt,$$

under the conditions

$$x(t) = \int_0^2 t^2(\tau^2 - \tau)\, d\mu(\tau),$$

$$\int_0^2 d\mu(\tau) = 1,$$

where $\mu(\cdot)$ is a non-decreasing function on the interval $[0, 2]$.

Let $(x^*(\cdot), \mu^*(\cdot))$ be a solution to the problem.

Since $\Phi(x,t) = tx$ and $q(t,\tau) = t^2(\tau^2 - \tau)$, therefore $\Phi_x(x^*,t) = t$, $q_t'(t,\tau) = 2t(\tau^2 - \tau)$, $q(0,\tau) = 0$. Hence $\eta(t) = \lambda_0(t^2 - 4)/2$ and $g(\tau) = \lambda_1 + 4\lambda_0(\tau^2 - \tau)$. Note that $\lambda_0 \neq 0$, for in the contrary case, $\eta(\cdot) \equiv 0$ and $0 = \min_{\tau \in [0,2]} g(\tau) = \min_{\tau \in [0,2]} (\lambda_1) = \lambda_1$, which contradicts the extremum principle.

Hence $\lambda_0 > 0$. The function $g(\cdot)$ attains its minimum for $\tau = 0.5$. In view of the above, $\mu^*(\cdot)$ is constant on the intervals $(0, 0.5)$ and $(0.5, 2)$. Consequently,

$$\mu_i^*(\tau) = \begin{cases} \alpha & \text{for } \tau \in [0, 0.5], \\ 1 + \alpha & \text{for } \tau \in (0.5, 2], \end{cases}$$

where α is an arbitrary real number. Then

$$x^*(t) = \int_0^2 t^2(\tau^2 - \tau)d\mu^*(\tau) = t^2\left(\frac{1}{4} - \frac{1}{2}\right) \cdot 1 = -t^2/4$$

and

$$\min (I(x)) = I(x^*) = \int_0^2 t(-t^2/4)dt = -1.$$

So, the extremal function for this problem is each piecewise constant function $\mu(\cdot)$ possessing exactly one jump of value 1 for $\tau = 0.5$.

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O PEWNYM ZADANIU EKSTREMALNYM W RODZINIE FUNKCJI NIEMALEJĄCYCH

W pracy uzyskane zostało twierdzenie egzystencjalne oraz warunki konieczne istnienia ekstremum dla następującego zadania optymalizacyjnego: zminimalizować funkcjonał $\int_a^b \Phi(x(t),t)dt$, przy warunkach $x(t) = \int_a^b q(t,\tau)d\mu(\tau)$, $\int_a^b d\mu_i(\tau) = 1$ dla $i = 1, 2, \ldots, n$. Zakłada się, że $\mu(\cdot)$ jest funkcją niemalejącą, natomiast $x(\cdot)$ jest funkcją absolutnie ciągłą na przedziale $[a, b]$.

Warunki konieczne optymalności uzyskane zostały na podstawie zasady ekstremum Joffego-Tichomirowa.