A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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A METHOD OF SCALARIZATION OF BACOPOULOS AND SINGER IN VECTORIAL OPTIMIZATION

This work includes a generalization of Bacopoulos's and Singer's theorem refering to the scalarization of a vectorial programming for a pair of convex functions defined on a vector space.

It has been proved that Bacopoulos's and Singer's method of scalarization can also be applied in the case when the first function is linearly upper semi-continuous and the second is strictly quasi--convex.

The relation between the local and global solutions of the problem of a vectorial programming and the behaviour of the set of minimal elements under their passing to the limit of the sequence of pairs of functions have also been studied.

Definition 1. Let X be a vector space over the real field. Let f be a realvalued function defined on X and G be a subset of X. The point $g_o \in G$ is called a minimal element of the function f on G (or a solution of a scalar program $(G, f \mid G)$) if and only if $f(g_o) = \inf_{g \in G} f(g)$.

The set of all such minimal elements we shall denote by $S_G(f)$ i.e.

$$S_G(f) = \{g_o \in G : f(g_o) = \inf_{g \in G} f(g)\}.$$

Definition 2. Let X be a vector space over the real field. Let f_1, f_2 be two real-valued functions defined on X and let G be a subset of X. We shall say that $g_o \in G$ is a minimal element of the two functions (f_1, f_2) on G (or a solution of a vectorial program $(G, f_1|G, f_2|G))$ if and only if there exists no element $g \in G$ such that

$$(f_1(g), f_2(g)) < (f_1(g_o), f_2(g_o)) .$$

(the inequality $(f_1(g), f_2(g)) < (f_1(g_o), f_2(g_o))$ for a pair of numbers means that $f_1(g) \leqslant f_1(g_o)$ and $f_2(g) \leqslant f_2(g_o)$, with at least one of these inequalities being strict). The set of all minimal elements for the vectorial program is denoted by $V_G(f_1, f_2)$. In this note G most often (but not only) will be a convex set. Similarly, it is possible to define a set $V_G(f_1, \ldots, f_k)$ for every finite system of functions $f_1, \ldots, f_k$.

The examples given at the end of this note show that the method of Bacopoulos-Singer cannot be transfered in natural way on the case of three or greater number of functions (even convex).

Notice, that even if the functions f_1, f_2 and the set G are convex then the set $V_G(f_1, f_2)$ can be non-convex. In reality if

$$f_1 = \max (\varphi, \Psi),$$

where

$$\varphi(x,y) = \begin{cases} y & \text{for } (x,y) \in \{(x,y) \in R^2 : y \geqslant 0\} \\ \frac{1}{2} y & \text{for } (x,y) \in \{(x,y) \in R^2 : y < 0\} \end{cases}$$

$$\Psi(x,y) = -x \quad \text{for} \quad (x,y) \in R^2$$

and

$$f_2 = -y,$$

then

$$V_{R^2}(f_1, f_2) = \{(x,y) : y \geqslant 0 \wedge y \geqslant -x\} \cup \{(x,y) : y < 0 \wedge y \geqslant -2x\}.$$

For convex functions f_1, f_2 defined on the line $V_R(f_1, f_2)$ is convex.

Definition 3. Let X be a vector space over the real field. Let f be a realvalued function defined on X. We say that f is convex if and only if

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y)$$

for every $x, y \in X$ and every $\lambda \in [0,1]$.

We shall only consider the proper functions, i.e. finite-valued functions.

Definition 4. Let X be a vector space over the real field. Let f be a real-valued function defined on X. The function f is called quasiconvex if and only for every $x, y \in X$ and for every $\lambda \in [0,1]$ the following inequality is satisfied

$$f(\lambda x + (1 - \lambda)y) \leqslant \max (f(x), f(y)).$$

The function f is called strictly quasiconvex if and only if it is quasiconvex and for every $\lambda \in [0,1]$ equality of the above inequality holds only for $f(x) = f(y)$ (see [6]).

Definition 5. Let X be a vector space over the real field. The real-valued function $f : X \to R$ is called linearly upper semi-continuous at the point $x_o \in X$ if and only if for every $x \in X$ and for every sequence $t_n \searrow_{n\to\infty} 0$ the following inequality is satisfied

$$\limsup_{n\to\infty} (f(x_o + t_n x)) \leqslant f(x_o).$$

The real-valued function $f : X \to R$ is called linearly upper semi-continuous on X if and only if for every $x \in X$ it is linearly upper semi-continuous.

The consideration of linear upper semi-continuous functions does not require a linear space X to have any topology. Of course, if X is the topological vector space then every continuous function on X is linearly upper semi-continuous (see [7]).

Definition 6. Let X be a vector space over the real field. Let A be a subset of X. We say that the point $x \in A$ belongs to the algebraic interior of A ($x \in \operatorname{Int\,alg} A$) if and only if, for every $y \in X - \{x\}$ there exists the number $\lambda \in (0,1)$ such that

$$\{\alpha x + (1-\alpha)(\lambda x + (1-\lambda)y) : \alpha \in [0,1]\} \subset A.$$

By the definitions it is easy to prove the following theorem:

Theorem 1. Let X be a vector space over the real field. Let f be a real-valued function defined on X. The function $f : X \to R$ is quasiconvex (strictly quasiconvex) if and only if for every $x,y \in X$ the function $\varphi(\lambda) = f(\lambda x + (1-\lambda)y)$ is quasiconvex (strictly quasiconvex) on $[0,1]$.

Theorem 2. Let be a vector space over the real field. The function $f : X \to R$ is linear upper semi-continuous if and only if for every $x,y \in X$ the function $\varphi(\lambda) = f(\lambda x + (1-\lambda)y)$ is upper semi-continuous on $[0,1]$.

Theorem 3. Let X be a vector space over the real field. Let f_1, f_2 be real-valued functions on X and G be a subset of X. If $S_G(f_1) \cap S_G(f_2) \neq 0$ then $V_G(f_1,f_2) = S_G(f_1) \cap S_G(f_2)$.

Now we shall prove the theorem which is generalization of the theorem of B a c o p o u l o s and S i n g e r [2]. We need the following lemmas.

Lemma 1. Let X be a vector space over the real field, f_1, f_2 be the functions defined on X, G be a subset of X. Then if $c < \inf_{g \in G} f_1(g)$ or $c > \inf_{g \in S_G(f_2)} f_1(g)$ we have

$$V_G(f_1,f_2) \cap \{y \in X : f_1(y) = c\} = \emptyset.$$

P r o o f. We consider $c < \inf_{g \in G} f_1(g)$ then $\{y \in X : f_1(y) = c\} = \emptyset$. Therefore

$$V_G(f_1,f_2) \cap \{y \in X : f_1(y) = c\} = \emptyset.$$

We now consider such a case when $c \geq \inf_{g \in S_G(f_2)} f_1(g)$.

If $\{y \in X : f_1(y) = c\} = \emptyset$ then analogically we obtain the thesis of the theorem. If the set is nonempty then for every $y \in X$ such that $f_1(y) = c > \inf_{g \in S_G(f_2)} f_1(g)$ there exists $y' \in S_G(f_2)$ such that

$$f_1(y) > f_1(y') \quad \text{and} \quad f_2(y) \geq f_2(y') = \inf_{g \in G} f_2(g).$$

That means that $y \notin V_G(f_1, f_2)$. Thus for every $c > \inf_{g \in S_G(f_2)} f_1(g)$ the set $V_G(f_1, f_2) \cap \{y \in X : f_1(y) = c\}$ is empty. Thus the lemma is proved.

Lemma 2. Let X be a vector space over the real field, f_1, f_2 be a real-valued functions defined on X, G be a subset of X. Then if $\inf_{g \in S_G(f_2)} f_1(g) = -\infty$ then

$$V_G(f_1, f_2) = \emptyset.$$

P r o o f. Let $y \in G$ be any point. We shall show that $y \notin V_G(f_1, f_2)$. By the assumption there is $y' \in S_G(f_2)$ such that $f_1(y') < f_1(y)$. Simultaneously $\inf_{g \in G} f_2(g) = f_2(y') \leq f_2(y)$. Then $(f_1(y'), f_2(y')) < (f_1(y), f_2(y))$ and that means that $y \notin V_G(f_1, f_2)$.

Theorem 4. Let X be a vector space over the real field. Let f_1, f_2 be a real-valued function defined on X. We assume that f_1 is linearly upper semi-continuous on X and f_2 strictly quasiconvex on X. Let G be a convex subset of X. Then for every $c \in R = (-\infty, +\infty)$ satisfying the condition

$$\inf_{g \in G} f_1(g) \leq c \leq \inf_{g \in S_G(f_2)} f_1(g) \tag{1}$$

we have

$$(2)\quad V_G(f_1, f_2) \cap \{y \in X : f_1(y) = c\} = S_{G \cap \{X \in X : f_1(y) \leqslant c\}}(f_2).$$

Consequently

$$(3)\qquad V_G(f_1, f_2) =$$

$$= \bigcup_{\inf\limits_{g \in G} f_1(g) \leqslant c \leqslant \inf\limits_{g \in S_G(f_2)} f_1(g)} S_{G \cap \{y \in X : f_1(y) \leqslant c\}}(f_2).$$

P r o o f. We suppose that $c \in (-\infty, +\infty)$ satisfies (1). We assume that there exists the element

$$g_o \in S_{G \cap \{y \in X : f_1(y) \leqslant c\}}(f_2) \setminus \{y \in X : f_1(y) = c\}.$$

We shall show that

$$(4)\qquad g_o \notin S_G(f_2).$$

Indeed, if $g_o \in S_G(f_2)$ then by the assumption that $g_o \in \{y \in X : f_1(y) < c\}$ and by (1) we have

$$f_1(g) < c \leqslant \inf_{g \in S_G(f_2)} f_1(g) \leqslant f_1(g_o),$$

which contradicts (4). Because we showed that (4) is satisfied it follows by (4) that there exists $g \in G$ such that

$$(5)\qquad f_2(g) < f_2(g_o).$$

Hence for every λ, $0 \leqslant \lambda < 1$ by strict quasiconvexity of the function f_2 we have

$$(6)\qquad f_2(\lambda g_o + (1 - \lambda)g) \quad \max\,(f_2(g_o) \quad f_2(g)) = f_2(g_o).$$

The function $\varphi(\lambda) = f_1(\lambda g_o + (1 - \lambda)g)$ is an upper semi-continuous at the point $\lambda = 1$. We have $\varphi(1) = f_1(g_o) < c$. Let

$\lambda_n \nearrow 1$ thus $\limsup_{n\to\infty} \varphi(\lambda_n) \leqslant \varphi(1) < c$. Let $\varepsilon = \frac{1}{2}(c - \varphi(1)) > 0$.

Almost all numbers λ_n satisfy the following condition

$$\varphi(\lambda_n) \leqslant \varphi(1) + \varepsilon = \varphi(1) + \frac{1}{2}(c - \varphi(1)) = \frac{1}{2}(\varphi(1) + c) < c.$$

If $\lambda_o = \lambda_n$ for sufficiently large n then

$$\varphi(\lambda_o) = f_1(\lambda_o g_o + (1 - \lambda_o)g) < c, \quad 0 \leqslant \lambda_o < 1.$$

Hence and by convexity of G we obtain

$$\lambda_o g_o + (1 - \lambda_o)g \in G \cap \{y \in X : f_1(y) \leqslant c\}.$$

The former and (6) contradict our assumption that

$$g_o \in S_{G \cap \{y \in X : f_1(y) \leqslant c\}}(f_2).$$

The assumption that the element

$$g_o \in S_{G \cap \{y \in X : f_1(y) \leqslant c\}}(f_2) \setminus \{y \in X : f_1(y) = c\} \text{ exists,}$$

gives a contradiction. Therefore

$$S_{G \cap \{y \in X : f_1(y) \leqslant c\}}(f_2) \subset \{y \in X : f_1(y) = c\}. \tag{7}$$

We suppose now that there exists g_o such that

$$g_o \in S_{G \cap \{y \in X : f_1(y) \leqslant c\}}(f_2) \setminus V_G(f_1, f_2).$$

Then there exists $\bar{g} \in G$ such that

$$f_1(\bar{g}) \leqslant f_1(g_o) = c \tag{8}$$

(the equality following by (7)) and

$$f_2(\bar{g}) \leqslant f_2(g_o) \tag{9}$$

with at least one of these inequalities being strict. Then by (8) we have

$$\bar{g} \in G \cap \{y \in X : f_1(y) \leqslant c\}$$

therefore $f_2(\bar{g}) \geqslant f_2(g_o)$ (because $g_o \in S_{G \cap \{y \in X : f_1(y) \leqslant c\}}(f_2))$.

and by (9) we have

$$(10) \qquad f_2(\bar{g}) = f_2(g_o) = \inf_{g \in G \cap \{y \in X : f_1(y) \leqslant c\}} f_2(g).$$

Therefore the inequality (8) must be strict, i.e. $f_1(\bar{g}) < c$. But by (10) and by (7) we have

$$\bar{g} \in S_{G \cap \{y \in X : f_1(y) \leqslant c\}}(f_2) \subset \{y \in X : f_1(y) = c\}.$$

The assumption that there is

$$g_o \in S_{G \cap \{y \subset X : f_1(y) \leqslant c\}}(f_2) \setminus V_G(f_1, f_2)$$

gives the contradition thus

$$(11) \quad S_{G \cap \{y \in X : f_1(y) \leqslant c\}}(f_2) \subset V_G(f_1, f_2) \cap \{y \in X : f_1(y) = c\}.$$

The proof of the opposite inclusion.

Assume that there is g_o such that

$$g_o \in V_G(f_1, f_2) \cap \{y \in X : f_1(y) = c\} \setminus S_{G \cap \{y \in X : f_1(y) \leqslant c\}}(f_2).$$

Then there exists $g \in G \cap \{y \in X : f_1(y) \leqslant c\}$ satisfying the inequality

$$(12) \qquad f_2(g) \leqslant f_2(g_o).$$

But by $g \in \{y \in X : f_1(y) \leqslant c\}$ and $g_o \in \{y \in X : f_1(y) = c\}$ we have

$$(13) \qquad f_1(g) \leqslant c = f_1(g_o).$$

and by (12) we obtain that $g_o \notin V_G(f_1, f_2)$. The obtained contradition proves the opposite inclusion to (11), thus (2) is satisfied.

Because for any c satisfying the condition

$$\inf_{g\in G} f_1(g) \leqslant c \leqslant \inf_{g\in S_G(f_2)} f_1(g)$$

we have

$$S_{G\cap\{y\in X\,:\,f_1(y)\leqslant c\}}(f_2) \subset V_G(f_1,f_2)$$

thus

$$(14) \qquad V_G(f_1,f_2) \supset$$

$$\supset \bigcup_{\inf_{g\in G} f_1(g) \leqslant c \leqslant \inf_{g\in S_G(f_2)} f_1(g)} S_{G\cap\{y\in X\,:\,f_1(y)\leqslant c\}}(f_2).$$

In order to prove the opposite inclusion let

$$g_o \in V_G(f_1,f_2) \cap \{y\in X : f_1(y) = c\}.$$

Hence by (2) we have

$$g_o \in S_{G\cap\{y\in X\,:\,f_1(y)\leqslant c\}}(f_2).$$

Finally we shall show that c satisfies (1) which we have by *Lemma 1.*

In the theorem of Bacopoulos and Singer the assumptions about f_1, f_2 are symmetrical. As we shall show in the following example in which f_1, f_2 are function defined on the real line, $G = R$, f_1 is upper semi-continuous (even continuous and convex) f_2 strictly quasiconvex, the exchange of the roles of f_1, f_2 in an inequality (1) and at the sets given in the both sides of the equality (2) can cause the loss of that equality. Let the functions $f_1 : R \to R$, $f_2 : R \to R$ be defined by

$$f_1(x) = \begin{cases} -x+1 & \text{for} \quad x \leqslant 1 \\ 0 & \text{for} \quad x > 1, \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0 \\ x+1 & \text{for} \quad x > 0. \end{cases}$$

We can see at once that the first one is convex and continuous and the second one is quasiconvex. We have

$$\inf_{x\in R} f_2(x) = 0, \quad \inf_{x\in S_R(f_1)} f_2(x) = 2,$$

because $S_R(f_1) = [1,+\infty)$.

Let $c = 1$. Then c satisfies

$$\inf_{x\in R} f_2(x) \leqslant c \leqslant \inf_{x\in S_R(f_1)} f_2(x)$$

and

$$\{y : f_2(y) = 1\} = \emptyset,$$

$$S_{R\cap\{y\,:\,f_2(y)\leqslant 1\}}(f_1) = \{0\},$$

because

$$\{y : f_2(y) \leqslant 1\} = (-\infty,0].$$

Therefore

$$V_R(f_1,f_2) \cap \{y : f_2(y) = 1\} = \emptyset,$$

but

$$S_{R\cap\{y\,:\,f_2(y)\leqslant 1\}}(f_1) = \{0\}.$$

Thus for

$$c = 1,\ V_R(f_1,f_2) \cap \{y : f_2(y) = c\} \neq S_{R\cap\{y\,:\,f_2(y)\leqslant c\}}(f_1).$$

Let X be a vector space over the real field. Later we shall give examples of the group of three convex and continuous functions for which the equalities are not satisfied:

(A) $$V_G(f_1,f_2,f_3) \cap \{y\in X : f_1(y) = c\} \neq$$

$$\neq V_{G\cap\{y\in X\,:\,f_1(y)\leqslant c\}}(f_2,f_3)$$

or

(B) $V_G(f_1,f_2,f_3) \cap \{y \in X : f_1(y) = c_1\} \cap \{y \in X : f_2(y) = c_2\} \neq$

$$\neq S_{G \cap \{y \in X : f_1(y) \leqslant c_1\} \cap \{y \in X : f_2(y) \leqslant c_2\}}(f_3).$$

Hence we know that the natural equivalents of the equality of Bacopoulos and Singer are not satisfied for three functions.

Example A. Let $X = R^2$, $R = (-\infty,+\infty)$, $G = R^2$,

$$f_1(x_1,x_2) = |x_2|,$$

$$f_2(x_1,x_2) = |x_1 - 1|,$$

$$f_3(x_1,x_2) = |x_1 + 1|.$$

We shall show that $V_{R^2}(f_2,f_3) = [-1,1] \times R$. Let $(x_1, x_2) \notin$ $\notin [-1,1] \times R$ thus $x_1 \notin [-1,1]$. Therefore

$$x_1 < -1, \quad x_2 \text{ - an arbitrary number,}$$

or

$$x_1 > 1, \quad x_2 \text{ - an arbitrary number.}$$

In the first case we consider the point $(-1,x_2)$, then it can be easily proved that

$$(f_2(-1,x_2),\ f_3(-1,x_2)) < (f_2(x_1,x_2),\ f_3(x_1,x_2)).$$

Then $(x_1,x_2) \notin V_{R^2}(f_2,f_3)$.

In the second case for the point $(1,x_2)$ we have the inequality

$$(f_2(1,x_2),\ f_3(1,x_2)) < (f_2(x_1,x_2),\ f_3(x_1,x_2)),$$

thus

$$(x_1,x_2) \notin V_{R^2}(f_2,f_3).$$

Let now $(x_1,x_2) \in [-1,1] \times R$. We shall show that $(x_1,x_2) \in V_R 2(f_2,f_3)$. We assume that there exists $(x_1',x_2') \in R^2$ such that

$$(f_2(x_1',x_2'),\ f_3(x_1',x_2')) < (f_2(x_1,x_2),\ f_3(x_1,x_2)).$$

Therefore $|x_1'-1| \leqslant |x_1-1|$ and $|x_1'+1| \leqslant |x_1+1|$ with at least one of these inequalities being strict, and it is impossible. Therefore we have $V_R 2(f_2,f_3) = [-1,1] \times R$. Now we shall show that

$$V_R 2(f_1,f_2,f_3) = [-1,1] \times \{0\}.$$

Let $(x_1,x_2) \notin [-1,1] \times \{0\}$. We have two cases: $x_1 \notin [-1,1]$ or $x_2 \neq 0$. The first case, if $x_1 < -1$ then at the point $(-1,x_2)$ we have

$$f_1(-1,x_2) = |x_2| = f_1(x_1,x_2),$$

$$f_2(-1,x_2) = 2 < f_2(x_1,x_2),$$

$$f_3(-1,x_2) = 0 < f_3(x_1,x_2),$$

thus

$$(f_1(-1,x_2),f_2(-1,x_2),f_3(-1,x_2)) < (f_1(x_1,x_2),f_2(x_1,x_2),f_3(x_1,x_2)).$$

Similarly for the point (x_1,x_2) such that $x_1 > 1$ there is a point $(-1,x_2)$ for which we have inequality given above. That means that $(x_1,x_2) \notin V_R 2(f_1,f_2,f_3)$. We consider the remaining case.

Let now (x_1,x_2) be the point such that $x_2 \neq 0$. We consider a point $(x_1,0)$. Then we obtain

$$f_1(x_1,0) = 0 < |x_2| = f_1(x_1,x_2),$$

$$f_2(x_1,0) = f_2(x_1,x_2),$$

$$f_3(x_1,0) = f_3(x_1,x_2),$$

which means that

$$(f_1(x_1,0), f_2(x_1,0), f_3(x_1,0)) < (f_1(x_1,x_2), f_2(x_1,x_2), f_3(x_1,x_2)),$$

therefore

$$(x_1,x_2) \notin V_{R^2}(f_1,f_2,f_3).$$

Let $(x_1,x_2) \in [-1,1] \times \{0\}$. We assume that there exists the point $(x_1',x_2') \in R^2$ such that

$$f_1(x_1,x_2) \leqslant f_1(x_1,x_2),$$

$$f_2(x_1,x_2) \leqslant f_2(x_1,x_2),$$

$$f_3(x_1,x_2) \leqslant f_3(x_1,x_2),$$

with at least one of these inequalities being strict and it is impossible, because for f_1 we have not strict inequality and neither for f_2, f_3, because as it was shown $V_{R^2}(f_2,f_3) = [-1,1] \times R$. Thus

$$V_{R^2}(f_1,f_2,f_3) = [-1,1] \times \{0\}.$$

For $c = 1$ we have

$$V_{R^2}(f_1,f_2,f_3) \cap \{(x_1,x_2) : f_1(x_1,x_2) = 1\} =$$

$$= ([-1,1] \times \{0\}) \cap ((R \times \{-1\}) \cup (R \times \{1\})) = \emptyset$$

and

$$V_{R^2 \cap \{(x_1,x_2) \,:\, f_1(x_1,x_2) \leqslant 1\}}(f_2,f_3) =$$

$$= V_{R^2 \cap (R \times [-1,1])}(f_2,f_3) =$$

$$= [-1,1] \times [-1,1] \neq \emptyset.$$

Example B. Let $X = R$, $G = R$. Let $f_1(x) = |x|$, $f_2(x) = |x|$,

$f_3(x) = 0$ and $c_1 \neq c_2$, $c_1, c_2 > 0$. We have

$$\{y : f_1(y) = c_1\} \cap \{y : f_2(y) = c_2\} = \emptyset$$

therefore

$$V_R(f_1, f_2, f_3) \cap \{y : f_1(y) = c_1\} \cap \{y : f_2(y) = c_2\} = \emptyset$$

but

$$S_{G \cap \{y : f_1(y) \leqslant c_1\} \cap \{y : f_2(y) \leqslant c_2\}}(f_3) =$$

$$= R \cap \{x : |x| \leqslant \min(c_1, c_2)\} \neq \emptyset.$$

In the case A we can also give an easy example in which

$$V_G(f_1, f_2, f_3) = \emptyset.$$

Let $X = R$, $G = R$, $f_1(x) = x$, $f_2(x) = 0$ and $f_3(x) = 0$. Then we have

$$V_G(f_1, f_2, f_3) = \emptyset$$

but

$$V_{R \cap \{x : x \leqslant c\}}(f_2, f_3) = \{x : x \leqslant c\}.$$

At present we shall discuss in which way the set of minimal points in scalar and vectorial program changes when we consider the sequence of functions $\{f_n\}$ which is almost uniformly convergent to the function f (or the sequence of the pair of functions $\{(f_1^n, f_2^n)\}$ almost uniformly convergent to (f_1, f_2)).

See the necessary definitions ([1], [3], [5]). We shall recall that a point belonging to topological space X belongs to upper topological limit of the sequence of sets A_n ($p \in \mathrm{Ls}\, A_n$) if and only if every neighbourhood of the point p has the common points with infinite number of the sets A_n.

Theorem 5. Let X be a locally compact linear topological vector space. Let $\{f_n\}$ be a sequence of continuous functions on X. We assume that the sequence $\{f_n\}$ is almost uniformly convergent to f. Let $A = S_X(f)$. Let $A_n = S_X(f_n)$. We assume that $A \neq O$. Then $Ls\, A_n \subset A$.

P r o o f. We shall show that if $x_o \notin A$ then $x_o \notin Ls\, A_n$. Let

$$f(x_o) > \inf_{x \in A} f(x).$$

Since, as we know [6], A is closed, thus there is a neighbourhood U of the point x_o such that $U \cap A = \emptyset$. (e.g. $U = X - A$). Since X is $T_{3\frac{1}{2}}$ - space (see [3], p. 193) thus there exists a neighbourhood $\bar{U}_o \subset U$ such that $\bar{U}_o \subset U$ and $\bar{U}_o$ is compact. The function f is a continuous one, as the limit of almost uniformly convergent sequence of continuous functions. Decreasing, if necessary, the neighbourhood U_o, then by the continuity of the function f we have that for every $x \in U_o$

$$|f(x) - f(x_o)| \leq \frac{1}{3}(f(x_o) - \inf_{x \in X} f(x)).$$

Let $C = \bar{U}_o \cup \{\bar{x}\}$ where $\bar{x} \in A$. The set C is compact as the union of two compact sets. By the almost uniform convergence of the sequence $\{f_n\}$ we have that the sequence $\{f_n|C\}$ is uniformly convergent to the function $f|C$. Then there is N such that for $n > N$ and for every $x \in C$

$$|f_n(x) - f(x)| < \frac{1}{3}(f(x_o) - \inf_{x \in X} f(x)).$$

In particular for $\bar{x} \in C$

$$f_n(\bar{x}) < f(\bar{x}) + \varepsilon = \inf_{x \in X} f(x) + \varepsilon,$$

where

$$\varepsilon = \frac{1}{3}(f(x_o) - \inf_{x \in X} f(x)).$$

Then we have that for every $x \in \bar{U}_o$ and for $n > N$

$$f_n(x) > f(x) - \varepsilon > f(x_o) - 2\varepsilon = \inf_{x \in X} f(x) + \varepsilon > f_n(\bar{x}).$$

Thus for every $x \in \bar{U}_o$ and $n > N$

$$f_n(x) > f_n(\bar{x}).$$

That means that $x \notin A_n$ i.e. $U_o \cap A_n = \emptyset$ for $n > N$. Therefore $x \notin \mathrm{Ls}\, A_n$ and it is the end of the proof. We shall show that the equality not always holds.

Example. We consider a sequence of functions $\{f_n\}$ defined by $f_n(x) = \frac{1}{n}|x|$ for $x \in R$. We can see that the sequence $\{f_n\}$ is almost uniformly convergent to $f \equiv 0$. Then we have $A_n = \{0\}$, $A = R$. It is worth noticing that for the convex function of one variable we can make the sequence of broken lines $\{f_n\}$ inscribed in the graph of the function f such that $\mathrm{Ls}\, A_n = A$.

Let

$$\ldots < x_{-k}^{(n)} < x_{-k+1}^{(n)} < \ldots < x_o^{(n)} < x_1^{(n)} < \ldots < x_k^{(n)} < \ldots$$

be a sequence of divisions of the line

$$\left(x_{-k}^{(n)} \xrightarrow[k\to\infty]{} -\infty,\ n_k^{(n)} \xrightarrow[k\to\infty]{} \infty\right).$$

We assume that the diameter of division converges to zero for $n\to\infty$. Let f_n be the function which satisfies the following conditions

$$f_n(x_k^{(n)}) = f(x_k^{(n)})$$

and f_n is the continuous and linear in $\left[x_{k-1}^{(n)}, x_k^{(n)}\right]$. Then by the assumption that $A \neq \emptyset$, the sequence $\{f_n\}$ is convergent to f whereas for an arbitrary number $N > 0$ the convergence on the interval $[-N,N]$ is uniform and we have $\mathrm{Ls}\, A_n = A$. For the pair of functions (even convex ones) the analogical theorem to the above one does not hold which is shown in the following examples:

Let X be a vector space over the real field. Let H be an arbitrary Hamel's basis in X (see [4], p. 55-56). Let $x_1 \in H$. Let for $x \in X$, $a(x)$ be a coefficient of x_1 in the development of x in the relation to H. We shall construct sequences of the pairs of convex functions $\left\{(f_1^{(n)}, f_2^{(n)})\right\}$ such that $f_1^{(n)} \to f_1$ and $f_2^{(n)} \to f_2$ in every point $x \in X$. Let $G = R$. We denote

$$V_G(f_1^{(n)}, f_2^{(n)}) = V_n \quad \text{and} \quad V_G(f_1, f_2) = V.$$

We shall show examples of sequences of functions such that

(A) $V_n = \emptyset, \quad V = X,$

(B) $V_n = X, \quad V = \emptyset.$

Hence, for every topology in the example A: $\mathrm{Ls}\, V_n \subsetneqq V$, whereas in the example B: $\mathrm{Ls}\, V_n \supsetneqq V$.

Example A. Let

$$f_1^{(n)}(x) = \frac{1}{n}\, a(x), \qquad f_1(x) = 0,$$

$$f_2^{(n)}(x) = 0, \qquad f_2(x) = 0.$$

We shall show that for two sequences of functions $V_n = \emptyset$. Let $x_o \in X$. If $x' \in X$ for which $a(x') < a(x_o)$ then

$$f_1^{(n)}(x') = \frac{1}{n}\, a(x') < \frac{1}{n}\, a(x_o) = f_1^{(n)}(x_o),$$

$$f_2^{(n)}(x') = 0 = f_2^{(n)}(x_o).$$

That means that for every $x_o \in X$ there is x' such that

$$(f_1^{(n)}(x'), f_2^{(n)}(x')) < (f_1^{(n)}(x_o), f_2^{(n)}(x_o)).$$

Therefore $x_o \notin V_n$ thus $V_n = \emptyset$. Now we shall show that $V = X$. Let $x_o \in X$, as it can be easily seen $x_o \in V$, because if $x_o \notin$ $\notin V$ then there exists $x \in X$ such that $x \neq x_o$ and

$$(f_1(x), f_2(x)) < (f_1(x_o), f_2(x_o))$$

and it is impossible since $f_1(x_o) = f_1(x) = 0 = f_2(x_o) = f_2(x)$. Since $V_n = \emptyset$ thus $\mathrm{Ls}\, V_n = \emptyset$. Therefore $\mathrm{Ls}\, V_n \subsetneqq V$.

Example B. Let for $x \in X$

$$f_1^{(n)}(x) = \frac{1}{n}a(x), \quad f_1(x) = 0$$

$$f_2^{(n)}(x) = -a(x), \quad f_2(x) = -a(x).$$

We shall show that an arbitrary point $x_0 \in X$ belongs to V_n, i.e. $V_n = X$. We consider $x \in X$ and we shall show that the following case is impossible:

$$f_1^{(n)}(x) \leqslant f_1^{(n)}(x_0),$$

$$f_2^{(n)}(x) \leqslant f_2^{(n)}(x_0)$$

with at least one of these inequalities being strict. We consider three cases

1) $$a(x) = a(x_0)$$

then

$$f_1^{(n)}(x) = f_1^{(n)}(x_0) = \frac{1}{n} a(x_0),$$

$$f_2^{(n)}(x) = f_2^{(n)}(x_0) = -a(x_0).$$

2) $$a(x) > a(x_0)$$

then

$$f_1^{(n)}(x) = \frac{1}{n} a(x) > \frac{1}{n} a(x_0) = f_1^{(n)}(x_0),$$

$$f_2^{(n)}(x) = -a(x) < -a(x_0) = f_2^{(n)}(x_0).$$

3) $$a(x) < a(x_0)$$

then we have

$$f_1^{(n)}(x) = \frac{1}{n} a(x) < \frac{1}{n} a(x_0) = f_1^{(n)}(x_0),$$

$$f_2^{(n)}(x) = -a(x) > -a(x_0) = f_2^{(n)}(x_0).$$

Therefore, $V_n = X$.

Now we shall show that $V = \emptyset$. Let $x_0 \in X$ be an arbitrary point. We shall show that there always exists $x' \in X$ such that

$$f_1(x') \leqslant f_1(x_0),$$

$$f_2(x') < f_2(x_0)$$

and that means that $x_0 \notin V$ and it proves that $V = \emptyset$. It is sufficient in this case to take for x' such a point for which $a(x') > a(x_0)$. Since $V_n = X$, then $\mathrm{Ls}\, V_n = X$. Considering that $V = \emptyset$ we have that $\mathrm{Ls}\, V_n \supsetneqq V$.

The last two theorems give relation between local and global minimum for one quasiconvex function and for the pair quasiconvex functions.

Theorem 6. If the function $f : X \to R$ (X - a vector space over the real field) is strictly quasiconvex and $x_0 \in X$ is a point for which there exists a set A such that $x_0 \in \mathrm{Int\, alg}\ A$ and $f(x_0) = \inf\limits_{x \in A} f(x)$,

$$f(x_0) = \inf_{x \in X} f(x).$$

P r o o f. We assume that there exists $x' \in X$ such that $f(x') < f(x_0)$. Then for every $\lambda \in (0,1)$ we have

$$f(\lambda x' + (1 - \lambda)\, x_0) < \max(f(x'), f(x_0)) = f(x_0).$$

Since $x_0 \in \mathrm{Int\, alg}\ A$ thus there is $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0 < 1$ we have

$$\lambda x' + (1 - \lambda) x_0 \in A$$

and

$$f(x_0) > \inf_{x \in A} f(x),$$

which is impossible, therefore the point of local minimum is the point of global minimum too.

Theorem 7. Let X be a vector space over the real field. If $f_1 : X \to R$, $f_2 : X \to R$ are strictly quasiconvex functions and $x_0 \in X$ is a point for which there exists the set A such that $x_0 \in \mathrm{Int\, alg}\ A$ and $x_0 \in V_A(f_1, f_2)$ then $x_0 \in V_X(f_1, f_2)$.

P r o o f. We assume that $x_o \notin V_X(f_1,f_2)$. That means that there exists $x' \in X$ such that

$$f_1(x') \leqslant f_1(x_o),$$

$$f_2(x') \leqslant f_2(x_o)$$

with at least one of these inequalities being strict. Since $x_o \in \operatorname{Int} \operatorname{alg} A$ then there exists $\lambda_o > 0$ such that for $0 < \lambda < < \lambda_o$

$$\lambda x' + (1 - \lambda)x_o \in A$$

and both inequalities (one of them being strict) remain which is impossible.

If only f_1 is strictly quasiconvex and f_2 is upper linear semi-continuous then the last theorem is false, as we shall show in the following example.

Example. We consider two real functions defined on R

$$f_1(x) = 0 \quad \text{for} \quad x \in R,$$

$$f_2(x) = x - |x| \quad \text{for} \quad x \in R.$$

It is easy to see that the first one is convex and the second one is continuous.

Let $A = (0,1)$. We shall show that $V_A(f_1,f_2) = (0,1)$ but $V_R(f_1,f_2) = \emptyset$. It is easy to see that for every $x \in (0,1)$ there is no point $x' \in (0,1)$ such that

$$f_1(x') \leqslant f_1(x),$$

$$f_2(x') \leqslant f_2(x)$$

with at least one of these inequalities being strict, because $f_1(x) = f_2(x)$ for every $x \in (0,1)$. Therefore $x \in V_A(f_1,f_2)$ and thus $(0,1) \subset V_A(f_1,f_2)$. Since $A = (0,1)$ thus $V_A(f_1,f_2) = (0,1)$. The set $V_R(f_1,f_2)$ is empty, because for every $x \in R$ there is $x' < \min(x,0)$ for which we have

$$f_1(x) = 0 = f_1(x'),$$
$$f_2(x) > f_2(x').$$

Let X be a vector space and G be a closed convex subset X. Bacopoulos and Singer proved that for every convex and lower semi--continuous functions f_1, f_2 on X, if for every convex closed subset G_o of G

$$S_{G_o}(f_i) \neq \phi \quad (i = 1,2)$$

then the set

$$M_G(f_1,f_2) = \{(f_1(g_o), f_2(g_o)) \in R^2 : g_o \in V_G(f_1,f_2)\}$$

is either the curve or the point.

If we assume that f_1, f_2 are strictly quasiconvex, the thesis of the above theorem by Bacopoulos and Singer is false.

Example. Let $X = R$, $G = R$

$$f_1(x) = \begin{cases} x & \text{for } x \leqslant 0, \\ x+1 & \text{for } x > 0, \end{cases}$$

$$f_2(x) = \begin{cases} -x+1 & \text{for } x \leqslant 0, \\ -x & \text{for } x > 0. \end{cases}$$

It is easy to see that $V_R(f_1,f_2) = R$, because for an arbitrary $x \in R$ we cannot find the point x' such that

$$f_1(x') \leqslant f_1(x),$$

$$f_2(x') \leqslant f_2(x)$$

with at least one of these inequalities being strict, since for $x' = x$ we have both equalities, and for $x' < x$ or $x' > x$ for one function we have the strict inequality with converse direction than for the second function.

Thus we have

$$R = V_R(f_1,f_2)$$

$$M_R(f_1,f_2) = \begin{cases} x, \ -x+1 & \text{for } x \leqslant 0, \\ x+1, \ -x & \text{for } x > 0. \end{cases}$$

It is easy to see that the set $M_R(f_1,f_2)$ is not the curve (even it is not connected).

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METODA SKALARYZACJI BACOPOULOSA I SINGERA W PROGRAMOWANIU WEKTOROWYM

Praca zawiera uogólnienie twierdzenia Bacopoulosa-Singera dotyczącego skalaryzacji programowania wektorowego dla pary funkcji wypukłych określonych na przestrzeni liniowej.

Pokazano, że metoda skalaryzacji Bacopoulosa i Singera da się zastosować w przypadku, gdy pierwsza funkcja jest liniowo półciągła z góry, a druga – ściśle quasiwypukła.

Na prostych przykładach wykazano, że analogicznej metody nie można zastosować dla trójki funkcji wypukłych.

Zbadano również związek między rozwiązaniami lokalnymi i globalnymi zadania programowania wektorowego oraz zachowanie się zbioru elementów minimalnych przy przejściu do granicy ciągu par funkcji.