A C T A U N I V E R S I T A T I S L O D Z I E N S I S
FOLIA MATHEMATICA 1, 1984

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ON SOME CONTROL PROBLEM

In the paper there has been considered an optimal control problem in a arbitrary convex class of controls. The integral maximum principle for optimal controls as well as a local necessary condition for monotone controls have been proved.

INTRODUCTION

Let us consider the following optimization problem:

$$I(x,u) = \int_0^1 f^o(x,u,t)\,dt \to \min, \quad \dot{x} = f(x,u,t), \quad x(o) = x_o,$$

$u \in \mathcal{U}$. (The exact assumptions on the functions f^o, f, x, u and the set $\mathcal{U}$ are given at the beginning of § 1).

In the case when $\mathcal{U}$ is the class of Pontryagin admissible controls, the extremal problem formulated above is a classical problem of optimal control and was investigated in many papers and monographs (cf. e.g. [1-5]).

In the present paper we assume that $\mathcal{U}$ is an arbitrary convex class of measurable controls. $\mathcal{U}$ may be, for instance a family of monotone controls with values belonging to a given set M, a family of controls with bounded variation, and the like.

In the paper we have proved the integral maximum principle for optimal controls in the class $\mathcal{U}$ as well as some local condition in the case when $\mathcal{U}$ is the family of monotone controls.

The proof is based on the Euler equation which was derived in paper [6].

AN OPTIMAL CONTROL PROBLEM

Let $f^o : R^n \times R^m \times R \to R$, $f : R^n \times R^m \times R \to R^n$, f^o_x, f^o_u, f_x, f_u be functions continuous with respect to (x,u) and measurable with to t. Besides, let f^o_x, f^o_u, f_x, f_u be bounded in any bounded set of the space $R^n \times R^m \times R$.

Let $\mathcal{U}$ be an arbitrary convex class of measurable controls $u : R \to R^m$.

Consider an optimal control problem of the form

$$I(x,u) = \int_0^1 f^o(x(t),\ u(t),t)dt \to \min \tag{1}$$

under the conditions

$$\dot{x}(t) = f(x(t),\ u(t),t), \tag{2}$$

$$x(0) = x_o, \tag{3}$$

$$u(\cdot) \in \mathcal{U}, \tag{4}$$

where x_o is a fixed point of the space R^n, while $x(\cdot)$ is an absolutely continuous function.

We are going to prove

Theorem 1 (integral maximum principle). If (x^o,u^o) is a solution to problem (1-4), then there exist some $\lambda_o \geq 0$ and absolutely continuous function ψ, such that

$$\int_0^1 \left[-\lambda_o\ f^o_u(x^o,u^o,t) + f^*_u(x^o,u^o,t)\ \psi(t)\right] u_o(t)dt \tag{5}$$

$$= \max_{u(\cdot)\in\mathcal{U}} \int_0^1 \left[-\lambda_o f^o_u(x^o,u^o,t) + f^*_u(x^o,u^o,t)\ \psi(t)\right] u(t)dt,$$

$$\dot{\psi}(t) \; -f_x^*(x^o(t),u^o(t),t) \;\; \psi(t) + \lambda_o f_x^o(x^o(t),u^o(t),t) \qquad \text{a.e.},$$

$$\psi(1) = 0,$$

$$\lambda_o \neq 0 \quad \text{or} \quad \psi(t) \neq 0 \quad \text{for} \quad t \in [0,1].$$

P r o o f. Let us adopt $X = C^n(0,1) \times L_\infty^m(0,1)$, where $C^n(0,1)$ is a space of functions continuous on the interval $[0,1]$ with norm $\|x\| = \max_{t \in [0,1]} |x(t)|$, while $L_\infty^m(0,1)$ is a space of essentially bounded functions with norm vrai sup $|u(t)|$.

Denote by Z_1, Z_2 the sets

$$Z_1 = \{(x,u) \in X, \; x(t) = x_o + \int_0^t f(x(t),u(t),t)dt\},$$

$$Z_2 = \{(x,u) \in X, \;\; u \in \mathcal{U}\},$$

So, problem (1-4) may be formulated in the form

$$I(x,u) \to \min, \quad (x,u) \in Z_1 \cap Z_2.$$

The cone of directions of decrease of the functional I is of the form

$$C_o = \{(\bar{x},\bar{u}) \in X, \; \int_0^1 [f_x^o(x^o,u^o,t)\bar{x} + f_u^o(x^o,u^o,t)\bar{u}]dt < 0\},$$

whereas the dual cone

$$C_o^* = \{f_o \in X^*, \; f_o(\bar{x},\bar{u}) =$$

$$= -\lambda_o \int_0^1 [f_x^o(x^o,u^o,t)\bar{x} + f_u^o(x^o,u^o,t)\bar{u}]dt, \; \lambda_o \geq 0\}$$

(cf. [2]).

Assume momentarily that

(6) $$C_o \neq \emptyset.$$

The cone tangent to the set Z_1 at (x^0,u^0) is defined by the formula

$$C_1 = \{(\bar{x},\bar{u}) \in X,\ \dot{\bar{z}} = f_x(x^0,u^0,t)\bar{x} + f_u(x^0,u^0,t)\bar{u},\ \bar{x}(0) = 0\}.$$

(C_1 is a space tangent to Z_1 at (x^0,u^0)).

Denote by C_2 cone tangent to the set Z_2 at the point (x^0, u^0). Since $Z_2 = X \times \mathcal{U}$, therefore C_2 is of the form

$$C_2 = X \times \tilde{C}_2, \tag{7}$$

where $\tilde{C}_2 \subset L^m_\infty$ is a cone tangent to the set $\mathcal{U}$ at the point u^0. We shall further show that the cones C_1 and C_2 satisfy assumption (3) of theorem 4.1 (cf [6]) i.e. that $C_1 \cap C_2$ is contained in a cone tangent to $Z_1 \cap Z_2$. Denote by P an operator $P : C^n \times L^m \to C^n$ defined by the formula .

$$P(x,u) = x(t) - x_0 - \int_0^1 f(x(t),u(t),t)dt.$$

The set Z_1 can be represented in the form

$$Z_1 = \{(x,u) \in X,\ P(x,u) = 0\}.$$

It is easily checked that, in same neighbourhood V_0 of the point (x^0,u^0), the operator P satisfies the assumptions of the implicit function theorem (see [2] example 9.3 and [3]). Consequently, the set Z_1 can be represented in the neighbourhood V_0 in the form

$$Z_1 = \{(x,u) \in X,\ x = \varphi(u)\}, \tag{8}$$

where $\varphi : L^m_\infty \to C^n$ is an operator of class C^1, satisfying the condition $P(\varphi(u),u) = 0$ for u such that $(\varphi(u),u) \in V_0$. From this we infer that the cone C_1 can be represented in the form

$$C_1 = \{(\bar{x},\bar{u}) \in X,\ \bar{x} = \varphi_u(u^0)\bar{u}\}. \tag{9}$$

Let $(\bar{x},\bar{u})$ be any element of the set $C_1 \cap C_2$. So, there exists an operator $v^2_u : R \to \mathcal{U}$ such that

$\frac{v_u^2(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0^+$ and

$$(10) \qquad (x^0,u^0) + \varepsilon(\bar{x},\bar{u}) + (v_x^2(\varepsilon),v_u^2(\varepsilon)) \in Z_2$$

for a sufficiently small ε and with any $v_x^2(\varepsilon)$.

It follows from (8) that, with a sufficiently small ε, we have

$$(\varphi(u^0 + \varepsilon\bar{u} + v_u^2(\varepsilon)), \quad u^0 + \varepsilon\bar{u} + v_u^2(\varepsilon)) \in Z_1.$$

Since φ is a differentiable operator, therefore

$$\varphi(u^0 + \varepsilon\bar{u} + v_u^2(\varepsilon)) = \varphi(u^0) + \varepsilon\varphi_u(u^0)\bar{u} + v_x^1(\varepsilon)$$

for some v_x^1 such that $1/\varepsilon \;\; v_x^2(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Taking account of (8) and (9), we get

$$(11) \qquad (x^0,u^0) + \varepsilon(\bar{x},\bar{u}) + (v_x^1(\varepsilon), \; v_u^2(\varepsilon)) \in Z_1.$$

If in formula (10) we take $v_x^2(\varepsilon) = v_x^1(\varepsilon)$, then it follows from (10) and (11) that $(\bar{x},\bar{u})$ is a vector tangent to the set $Z_1 \cap Z_2$. Consequently, $C_1 \cap C_2$ is contained in the cone tangent to the set $Z_1 \cap Z_2$.

From theorem 3.3 (cf. [6]) (7) and (9) it follows that the cones C_1^* and C_2^* are of the same sense. Making use of theorem 4.1 ([6]) we obtain the Euler equation of the form

$$f_0(\bar{x},\bar{u}) + f_1(\bar{x},\bar{u}) + f_2(\bar{x},\bar{u}) = 0$$

for any $(\bar{x},\bar{u}) \in X$, where $f_i \in C_i^*$, $i = 0,1,2$ (see [6]).

Further, proceeding analogously as in ([2], § 12), we get the proposition of *Theorem 1*. In the singular case, i.e. when condition (6) is not satisfied, we also obtain the proposition.

It $\mathcal{U}$ is, for example, the family of all measurable functions with values belonging to a convex set $M \subset R^m$, then from *Theorem 1* one can obtain a generalization of the local maximum principle (cf. [2] § 12).

In *Theorem 1* we do not assume that the set of controls possesses interior points. This enables us to examine various non-standard classes of controls and to obtain for them necessary conditions for optimality. For instance, let us consider a set of controls $u = (u^1, \dots, u^m)$ such that

$$(12) \quad u^i(t) \in [0, M^i] \quad \text{for } i = 1,2, \dots, m; \quad t \in [0,1]; \; u(o) = 0,$$

and u^i are non-decreasing functions on the interval $[0,1]$, where $M^i > 0$ are fixed for $i = 1,2, \dots, m$. This set will be denoted by R. Since u^i are non-decreasing, therefore, without loss of generality, we may assume that they are continuous on the left. We shall prove

Theorem 2. If u^o is an optimal control in problem (1-4), where $\mathcal{U} = R$, then there exist a constant $\lambda_o \geq 0$ and an absolutely continuous function Ψ, such that conditions (5) are satisfied. Moreover, if a component a_k, $1 \leq k \leq m$, of a switching function

$$a(t) = -\lambda_o \, f_u^o(x^o(t), u^o(t), t) + f_u^*(x^o(t), u^o(t), t)\, \Psi(t)$$

is of the constant sign on the intervals (t_k^i, t_k^{i+1}), $i = 0,1, \dots, r_k - 1$, $k = 1,2, \dots, m$, where $0 = t_k^o < t_k^1 < \dots < t_k^{r_k} = 1$, then the component u_k^o of the optimal control u^o is constant on each interval (t_k^i, t_k^{i+1}), that is, u_k^o is a step function and the number of its jumps does not exceed $r_k + 1$.

P r o o f. The first part of the proposition follows directly from *Theorem 1*. Let $L = (t_k^i, t_k^{i+1})$ be a fixed interval. At first, consider the case when the function a_k is negative on this interval. It can be easily seen that a function $\tilde{u}_k$ defined by the formula

$$\tilde{u}_k(t) = \begin{cases} u_k^o(t_k^i - 0) & \text{for} \quad t \in (t_k^i, t_k^{i+1}], \\ u_k^o(t) & \text{for} \quad t \notin (t_k^i, t_k^{i+1}], \end{cases}$$

satisfies the conditions $\tilde{u}_k(0) = 0$, $\tilde{u}_k(t) \in [0,M^k]$ and $\tilde{u}_k$ is a non-decreasing function. So the control

$$\tilde{u} = (u_1^o, \ldots, u_{k-1}^o, \tilde{u}_k, u_{k+1}^o, \ldots, u_m^o)$$

is an admissible control, i.e. $\tilde{u} \in R$ (see (12)). In view of condition (5), we have

$$\int_0^1 a(t)\tilde{u}(t)dt \le \int_0^1 a(t)u^o(t)dt,$$

where

$$a(t) = -\lambda_o f_u^o(x^o(t),u^o(t),t) + f_u^*(x^o(t),u^o(t),t)\psi(t).$$

Hence

$$(13) \qquad \int_L a_k(t)u_k^o(t_k^i - 0)dt \le \int_L a_k(t)u_k^o(t)dt.$$

The function a_k is negative on the interval L, whereas u_k^o - non-decreasing. Consequently,

$$(14) \qquad a_k(t)u_k^o(t_k^i - 0) \ge a_k(t)\, u_k^o(t) \qquad \text{for} \quad t \in (t_k^i, t_k^{i+1}).$$

Hence it appears that $u_k^o(t) = u_k^o(t_k^i - 0)$ on the entire interval L. Indeed, if, at some point $\tau \in L$, $u_k^o(\tau) > u_k^o(t_k - 0)$, then also $u_k^o(t) > u_k^o(t_k^i - 0)$ on the entire interval (τ, t_k^{i+1}). In view of inequality (14), we get

$$\int_L a_k(t)\, u_k^o(t_k^i - 0)dt > \int_L a_k(t)u_k^o(t)dt.$$

The last inequality contradicts (13).

In the case when $a_k(\cdot)$ is positive on L we adopt

$$\tilde{u}_k(t) = \begin{cases} u_k^o(t_k^{i+1} + 0) & \text{for} \quad t \in (t_k^i, t_k^{i+1}] \\ u_k^o(t) & \text{for} \quad t \notin (t_k^i, t_k^{i+1}]. \end{cases}$$

An analogous reasoning leads to the conclusion that

$$u_k^o(t) = u_k^o\ (t_k^{i+1} + 0) \quad \text{for} \quad t \in L.$$

Further, let us consider a linear system of the form

$$I(x,u) = \int_0^1 (ax + bu)\,dt,$$

$$\text{(15)} \qquad \dot{x} = Ax + Bu, \quad x(0) = x_o$$

$$u(\cdot) \in R,$$

where A,B,a,b are constant matrices of dimensions $n \times n$, $n \times m$, $1 \times n$, $1 \times m$, respectively. It is known that, if system (15) is regularly controllable. (see [1]) and the eigenvalues of A are real than the switching function $a_k(t) = (-B^* \Psi(t) + \lambda_o b)_k$ alternates its sign at most n times. Consequently, each component u_k^o of the optimal control in problem (15) is a step function and possesses at most $n + 1$ jumps.

PHYSICAL INTERPRETATION

The optimal control problem in the class of monotone controls, investigated above, can be interpretated physically in a natural way. Namely, let us consider an object Q supplied with m engines serving to drive and direct the object. Each engine possesses $M^i \geq 0$ of fuel. Assume that the motion of the object is described by the equation $\dot{x} = f(x,u,t)$, $x(0) = x_o$, $t \in [0,1]$ and that we control the quantity of the fuel used up, i.e. $u^i(t)$ is the quantity of fuel used up by the i-th engine in the time interval $[0,t]$, $i = 1, \ldots, m$. We want to determine a control u^0, so that the cost functional

$$I(x,u) = \int_0^1 f^0(x,u,t)dt$$

should attain a minimal value. From *Theorem 2* it follows that u^0 satisfies conditions (5). Besides, if the sign of the switching function is a piecewise constant function, then the optimal control of fuel consists in its explosive use. It the motion of the object is described by the regularly controllable linear system (15) and the matrix A possesses only real eigenvalues, then the number of "explosions" under the optimal control does not exceed n+1.

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O PEWNYM PROBLEMIE STEROWANIA

W pracy rozważane jest zadanie sterowania optymalnego w dowolnej wypukłej klasie sterowań dopuszczalnych. Udowodniona jest całkowa zasada maksimum oraz lokalny warunek konieczny optymalności dla sterowań monotonicznych.