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ON SOME GENERALIZATION OF α-CONVEX FUNCTIONS

Let $\alpha \in [0,\infty)$ and $\beta \in (0,1]$ be any numbers. Let $M_\alpha(\beta)$ denote the family of all functions f holomorphic in the unit disc U, such that $z^{-1} f'(z) f(z) \neq 0$ for $z \in U$, and satisfying the condition

$$|\arg \{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)})\}| < \beta\frac{\pi}{2}, \quad z \in U.$$

Certain relations between functions from the particular families $M_\alpha(\beta)$ are obtained; also, some functional defined on $M_\alpha(\beta)$, depending on the coefficients of the expansion of the function in a Taylor series, is estimated.

1. In the present paper the notions of a set, a family, and a class of functions will be regarded as equivalent. By C, U, U_0 we shall denote the complex plane, the disc $\{z \in C : |z| < 1\}$ and the ring $\{z \in C : 0 < |z| < 1\}$, respectively.

Let $\alpha \in [0,\infty)$, $\beta \in (0, 1]$ be fixed numbers. One knows the families $S^*(\beta)$, ([1], [8]), and M_α, ([6]), of functions holomorphic in the disc U, defined, respectively, by the conditions

(1) $$|\arg \frac{z f'(z)}{f(z)}| < \beta\frac{\pi}{2}, \quad z \in U,$$

$$|\arg J(\alpha, z ; f)| < \frac{\pi}{2}, \quad z \in U, \quad z^{-1} f(z) f'(z) \neq 0,$$

where

$$(2) \qquad J(\alpha, z ; f) = (1-\alpha)\frac{z f'(z)}{f(z)} + \alpha\left(1+\frac{z f''(z)}{f'(z)}\right), \quad z \in U.$$

Functions of these classes are called, respectively, strongly starlike functions of order β and α-convex ones.

Let $M_\alpha(\beta)$, $\alpha \in [0,\infty)$, $\beta \in (0, 1]$ stand for the family of all functions f, $f(0) = f'(0) - 1 = 0$, holomorphic in U, such that $z^{-1} f'(z) f(z) \neq 0$ for $z \in U_0$, and satisfying the condition

$$|\arg J(\alpha, z ; f)| < \beta \frac{\pi}{2}, \quad z \in U.$$

Note that this condition is equivalent to the fact that $J(\alpha, z ; f) = p^\beta(z)$ for some Caratheodory functions, i.e., for a function p holomorphic in U, $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in U$.

2. It is obvious that $M_0(\beta) = S^*(\beta)$. Moreover, the following theorem takes place.

Theorem 1. For any fixed $\alpha \geqslant 0$ and $\beta \in (0,1]$, each function of the family $M_\alpha(\beta)$ is a strongly starlike function of order β.

P r o o f. For $\alpha = 0$, the theorem is self-evident. So, let us assume that $\alpha > 0$. Let f be any fixed function of the family $M_\alpha(\beta)$.

Put

$$(3) \qquad q(z) = \left[\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\beta}} - 1\right] \cdot \left[\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\beta}} + 1\right]^{-1}, \quad z \in U,$$

where by $(z f'(z) f^{-1}(z))^{\frac{1}{\beta}}$ we mean that unique branch of the power which, for $z = 0$, takes the value 1.

The function q is meromorphic in U, $q(0) = 0$, and $q(z) \neq \pm 1$. We shall show that $|q(z)| < q$ for $z \in U$. To this effect, suppose that there exists a point $\zeta \in U$ for which

$$\max_{|z|<|\zeta|} |q(z)| = |q(\zeta)| = 1.$$

Then there exists a constant l, $l \geqslant 1$, for which $\zeta q'(\zeta) =$

$= lq(\zeta)$, ([3], lemma 1). Consequently, taking account of (2) and (3), we shall obtain

$$J(\alpha, \zeta ; f) = \left(\frac{1 + q(\zeta)}{1 - q(\zeta)}\right)^{\beta} + 2\alpha\beta l \frac{q(\zeta)}{1 - q^2(\zeta)}.$$

Denote $\Delta_\beta = \{w \in C : |\arg w| < \beta \frac{\pi}{2}\}$. The function q is holomorphic in the disc $|z| < |\zeta|$, and $|q(z)| < 1$ in this disc. So, the function $(1 + q(z))(1 - q(z))^{-1}$ transforms the disc $|z| < |\zeta|$ into the half-plane $\operatorname{Re} w > 0$. In consequence, the values of the function

$$\left[(1 + q(z))(1 - q(z))^{-1}\right]^{\frac{1}{\beta}} \quad \text{for} \quad |z| < |\zeta|$$

belong to Δ_β. At the same time, at the point $z = \zeta$, it takes the values lying on the boundary of the set Δ_β. Since $2\alpha\beta l > 0$, therefore $J(\alpha, \zeta ; f) \notin \Delta_\beta$. Hence we get a contradiction since $f \in M_\alpha(\beta)$. Consequently, $|q(z)| < 1$ for all $z \in U$.

Finally, from (3) we have that the function f satisfies condition (1), which proves that f is a strongly starlike function of order β, q.e.d.

The above theorem will enable us to prove the following

Theorem 2. If, for any fixed $\beta \in (0,1]$, $0 \leqslant \alpha_1 \leqslant \alpha_2$, then $M_{\alpha_2}(\beta) \subset M_{\alpha_1}(\beta)$.

P r o o f. Suppose that there exist α_1, α_2, $0 < \alpha_1 < \alpha_2$, and a function f of the family $M_{\alpha_2}(\beta)$, which does not belong to $M_{\alpha_1}(\beta)$. Consequently, for every $z \in U$, $J(\alpha_2, z ; f) \in \Delta_\beta$, and there exists a point $\zeta \in U$ for which $J(\alpha_1, \zeta ; f) \notin \Delta_\beta$.

Put $w_1 = J(\alpha_2, \zeta ; f)$, $w_2 = J(\alpha_1, \zeta ; f)$ and $w = -\alpha_1 w_1 + \alpha_2 w_2$. It is evident that $w_1 \in \Delta_\beta$, $w_2 \notin \Delta_\beta$, and

$$w = (\alpha_2 - \alpha_1) \frac{\zeta f'(\zeta)}{f(\zeta)}.$$

It follows from theorem 1 that $\zeta f^{-1}(\zeta)f'(\zeta) \in \Delta_\beta$. Consequently, in view of $\alpha_2 - \alpha_1 > 0$, the point $w \in \Delta_\beta$. Since $\alpha_2 w_2 = w + \alpha_1 w_1$ and $\alpha_2 > 0$, therefore $w_2 \in \Delta_\beta$, which contradicts our supposition that $w_2 \notin \Delta_\beta$. Thus, the theorem has been proved.

Note that, if $0 < \beta_2 \leq \beta_1 \leq 1$, then $\Delta_{\beta_2} \subset \Delta_{\beta_1}$. Consequently, for any fixed $\alpha \geq 0$, $M_\alpha(\beta_2) \subset M_\alpha(\beta_1)$. So, the following corollary takes place.

Corollary. If $0 \leq \alpha_1 < \alpha_2$ and $0 < \beta_2 \leq \beta_1 \leq 1$, then

$$M_{\alpha_2}(\beta_2) \subset M_{\alpha_1}(\beta_1).$$

It can be shown, ([7]), that a function f holomorphic in U belongs to the class $M_0(\beta)$ if and only if there exists a Caratheodory function p such that

$$f(z) = z \exp \int_0^z (p^\beta(\zeta) - 1)\zeta^{-1}\, d\zeta.$$

In particular, the function

$$f_*(z) = z \exp \int_0^z \left[\left(\frac{1 + \varepsilon\zeta}{1 - \varepsilon\zeta}\right)^\beta - 1\right] \zeta^{-1}\, d\zeta, \quad |\varepsilon| = 1,$$

belongs to the family $M_0(\beta)$. At the same time, for $z = -\bar{\varepsilon}\rho$, $0 < \rho < 1$,

$$J(\alpha, -\bar{\varepsilon}\rho\, ;\, f_*) = \left(\frac{1 - \rho}{1 + \rho}\right)^\beta + 2\alpha\beta \frac{-\rho}{1 - \rho^2}.$$

Hence it appears that

$$\lim_{\rho \to 1^-} J(\alpha, -\bar{\varepsilon}\rho\, ;\, f_*) = = -\infty,$$

which means that f_* does not belong to any family $M_\alpha(\beta)$ for $\alpha > 0$ and $0 < \beta \leq 1$. This example shows that $M_0(\beta)$ is a family essentially wider than $M_\alpha(\beta)$, $\alpha > 0$.

3. Our further considerations will concern mutual relations between functions of the classes $M_\alpha(\beta)$, $\alpha > 0$, and $M_0(\beta)$. We shall prove

Lemma. For any fixed $\alpha > 0$, $\beta \in (0,1]$ and every function $f \in M_\alpha(\beta)$, the function

$$F_\gamma(z) = f(z) \cdot \left(\frac{zf'(z)}{f(z)}\right)^\gamma$$

belong to the class $M_0(\beta)$ for all γ, $0 < \gamma \leqslant \alpha$.

P r o o f. Let $f \in M_\alpha(\beta)$, $\alpha > 0$. Then

$$\frac{zF_\gamma'(z)}{F_\gamma(z)} = (1 - \gamma)\,\frac{zf'(z)}{f(z)} + \gamma\,\left(1 + \frac{zf''(z)}{f'(z)}\right) = J(\gamma,\, z\,;\, f).$$

Since $0 < \gamma \leqslant \alpha$, it follows from *Theorem 2* that $f \in M_\gamma(\beta)$, i.e., $J(\gamma, z\,;\, f) = p^\beta(z)$ for some Caratheodory function p. Consequently, F_γ belongs to $M_0(\beta)$, q.e.d.

The following theorem is also true.

Theorem 3. For any fixed $\beta \in (0,1]$ and for every function $F \in M_0(\beta)$, the function

$$(4) \qquad f(z) = \left[\frac{1}{\alpha}\int_0^z F^{\frac{1}{\alpha}}(\zeta)\,\zeta^{-1}\,d\zeta\right]^{-1}, \quad \alpha > 0,$$

being the solution to the equation

$$(5) \qquad F(z) = f(z) \cdot \left(\frac{zf'(z)}{f(z)}\right)^\alpha$$

with the initial condition $f(0) = 0$, belongs to the family $M_\alpha(\beta)$.

P r o o f. Since $F \in M_0(\beta)$, it is a univalent and starlike function. From this fact and from the well-known result ([5], theorem 5) we shall obtain that, for the given function $F \in M_0(\beta)$, the solution f of the form (4) of equation (5) is a function holomorphic in U,

$$f(0) = f'(0) - 1 = 0, \quad f(z) \cdot f'(z) \neq 0 \quad \text{for} \quad z \in U_0.$$

Since

$$z\,F^{-1}(z)F'(z) = p^{\beta}(z)$$

for some Caratheodory function p, therefore

$$J\,(\alpha,\ z\ ;\ f) = \frac{zF'(z)}{F(z)} = p^{\beta}(z).$$

So, indeed, the function $f \in M_{\alpha}(\beta)$.

4. We shall now deal with the problem of estimation of coefficients in the family $M_{\alpha}(\beta)$ and, in particular, with the Goluzin functional. We shall prove

Theorem 4. If a function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

belongs to the family $M_{\alpha}(\beta)$, then, for any fixed $\mu \in \mathbf{C}$,

$$(6)\qquad |a_3 - \mu a_2^2| \leqslant \begin{cases} \dfrac{\beta}{1+2\alpha} & \text{when} \quad |\mu - c(\alpha)| \leqslant r(\alpha,\ \beta), \\[2ex] \dfrac{\beta}{1+2\alpha}|\nu| & \text{when} \quad |\mu - c(\alpha)| > r(\alpha,\ \beta), \end{cases}$$

where

$$\nu = \frac{\beta}{(1+\alpha)^2}\left[\alpha^2 + 8\alpha + 3 - 4\mu\,(1+2\alpha)\right]$$

and

$$r(\alpha,\beta) = \frac{(1+\alpha)^2}{4\beta(1+2\alpha)}, \qquad c(\alpha) = \frac{\alpha^2 + 8\alpha + 3}{4(1+2\alpha)}.$$

Equality in estimate (6) holds, respectively, for functions being the solution to the equations

$$J(\alpha,\ z\ ;\ f) = \left(\frac{1+\varepsilon z}{1-\varepsilon z}\right)^{\beta}, \qquad |\varepsilon| = 1,$$

$$J(\alpha, z; f) = \left(\frac{1 + \varepsilon z^2}{1 - \varepsilon z^2}\right)^{\beta}, \quad |\varepsilon| = 1.$$

P r o o f. Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be any function of the family $M_\alpha(\beta)$.

Then

$$J(\alpha, z; f) = \left[(1 + q(z))(1 - q(z))^{-1}\right]^{\beta}$$

for some function

$$q(z) = \sum_{n=1}^{\infty} c_n z^n$$

holomorphic in U, $|q(z)| < 1$. In consequence, comparing the coefficients and determining the functional $a_3 - \mu a_2^2$ for any fixed $\mu \in C$, we shall get that

$$|a_3 - \mu a_2^2| = \frac{\beta}{1 + 2\alpha}\left| c_2 + \frac{\beta[\alpha^2 + 8\alpha + 3 - 4\mu(1 + 2\alpha)]}{(1 + \alpha)^2} c_1^2 \right| \leqslant$$

$$\leqslant \frac{\beta}{1 + 2\alpha}\left(|c_2| + |\nu|\,|c_1|^2\right).$$

Since, for the function q, the estimates $|c_1| \leqslant 1$ and $|c_2| \leqslant 1 - |c_1|^2$ are true, therefore

$$|a_3 - \mu a_2^2| \leqslant \frac{\beta}{1 + 2\alpha}\left[1 + (|\nu| - 1)|c_1|^2\right] \leqslant \frac{\beta}{1 + 2\alpha}\max\{1, |\nu|\}.$$

It can easily be verified that $|\nu| \leqslant 1$ if and only if

$$|\mu - c(\alpha)| \leqslant r(\alpha, \beta).$$

In consequence, we shall get estimate (6).

From *Theorem 4* one can obtain estimates of $|a_2|$ and $|a_3|$ in the class $M_\alpha(\beta)$ by taking $\mu = 0$ and considering the limit case $\mu \to \infty$, respectively. These estimates, with particular values of the parameters α and β, yield the already-known results in the families $M_0(\beta)$ ([2]), $M_\alpha(1)$, ([4], [9]).

REFERENCES

[1] D. A. Brannan, W. E. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc., 1 (1969), 431-443.

[2] D. A. Brannan, J. Clunie, W. E. Kirwan, *Coefficient estimates for a class of starlike functions*, Canad. J. Math., 22 (1970), 476-485.

[3] L. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc., 2 (1971), 469-474.

[4] P. K. Kulshrestha, *Coefficient problem for a class of Mocanu-Bazilevič functions*, Ann. Polon. Math., XXXI (1976), 291-299.

[5] S. S. Miller, P. T. Mocanu, M. O. Reade, *Bazilevič functions and generalized convexity*, Rev. Roum. Math. Pures at Appl., XIX, 2 (1974), 213-224.

[6] P. T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Math. Cluj, 11, 34 (1969), 127--133.

[7] J. Stankiewicz, *Some extremal problems for the class S_α*, Ann. Univ. Mariae Curie-Skłodowska, Sec. A, 25 (1971), 101-107.

[8] J. Stankiewicz, *Quelques problémes extrémaux dans les classes des fonctions α-angulairemente étoilées*, Ann. Univ. Mariae Curie--Skłodowska, Sec. A, 20 (1966), 59-75.

[9] J. Szynal, *Some remarks on coefficients inequality for α-convex functions*, Bull. Acad. Polon. Sci., Ser. Math. Astronom. Phys., 20, 11 (1972), 917-919.

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O PEWNYM UOGÓLNIENIU α-WYPUKŁYCH FUNKCJI

Niech $\alpha \in [0, \infty)$ i $\beta \in (0,1]$ będą ustalonymi liczbami. Przez $M_\alpha(\beta)$ oznaczmy rodzinę wszystkich funkcji f, holomorficznych w kole U = {z ∈

$\in C : |z| < 1\}$, takich, że $z^{-1}f'(z)f(z) \neq 0$ dla $z \in U_0$, spełniających warunek

$$|\arg\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)})\}| < \beta\frac{\pi}{2}, \quad z \in U.$$

W przedstawionej pracy podane są pewne związki między funkcjami z poszczególnych rodzin $M_\alpha(\beta)$. Ponadto oszacowany jest pewien funkcjonał określony na $M_\alpha(\beta)$ zależny od współczynników rozwinięcia funkcji w szereg Taylora.