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CONCERNING TOPOLOGIZATION OF $P(t)$

To Professor Lech Włodarski on His 80th birthday

We prove that on algebras of polynomials there are at least two vector space topologies making the multiplication separately continuous. This solves a problem posed in [2].

Let A be a real or complex algebra provided with a vector space (Hausdorff) topology τ. We say that it is a semitopological (resp. topological) algebra if its multiplication is separately (resp.jointly) continuous. In [2] it was shown that every uncountably generated algebra has at least two different topologies making of it a complete semitopological algebra. As one of these topologies we can take the maximal locally convex topology τ_{max}^{LC} given by means of all seminorms and as another – the topology τ_{max}^{p} given by means of all p-homogeneous seminorms with a fixed p satisfying $0 < p < 1$ (it is known (see [4], Example on p. 56) that τ_{max}^{LC} is a complete topology and the same was proved in [2] about the topologies τ_{max}^{p}). However, as shown in [2], on a countably generated algebra all topologies τ_{max}^{p} coincide and so there was asked a question whether, in particular, τ_{max}^{LC} is a unique topology making of the algebra $P(t)$ of all (real

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or complex) polynomials a complete semitopological algebra. It was mentioned in "added in proof" that the answer to this question is in negative and the aim of this paper is to provide the reader with details of the construction.

Let Q be the family of all sequences $q = (q_i)_1^\infty$ with entries of the form $q_i = s_i + m$, where s_i are natural numbers satisfying $\frac{s_{i+1}}{s_i} \geq 2$ and m is a non-negative integer depending upon q. Clearly $q \in Q$ implies $q' \in Q$, where $q'_i = q_i + 1$. For a fixed q in Q denote by R_q the family of all sequences $r = (r_i)_0^\infty$ of real numbers such that $r_k \geq 1$ and $r_k = 1$ for $k = 0$ or $k \neq q_i$ for all i. Put $R = \bigcup_{q \in Q} R_q$. The definition of R implies that for each r in R and each natural m there is a natural $k(r,m)$ such that for each $k \geq k(r,m)$ there is at most one j satisfying

$$k \leq j \leq k+m \quad \text{and} \quad r_j > 1. \tag{1}$$

Every sequence r in R defines on the algebra $P(t)$ a norm

$$|x|_r = \sum_{i=0}^{\infty} |a_i(x)|\, r_i, \tag{2}$$

where $x = \sum_0^\infty a_i(x)t^i$ is a polynomial in $P(t)$, so that only finitely many coefficients $a_i(x)$ are different from zero. Denote by τ_R the topology given on $P(t)$ by means of all seminorms (2), $r \in R$. Clearly all maps $x \to a_i(x)$ are linear functionals on $P(t)$, which are continuous in the topology τ_R.

Proposition. *The algebra $A = (P(t), \tau_R)$ is a complete locally convex semitopological algebra and $\tau_R \neq \tau_{max}^{LC}$. Consequently the algebra $P(t)$ has at least two different complete locally convex topologies making of it a semitopological algebra.*

Proof. First we show that A is a complete topological vector space. Let $(x_\alpha)_{\alpha \in \mathfrak{a}}$ be a Cauchy net in A. Thus for each r in R and each positive ε there is an index $\alpha(r,\varepsilon)$ such that $|x_\alpha - x_\beta|_r < \varepsilon$ for all $\alpha, \beta \succeq \alpha(r,\varepsilon)$. Since the functionals $a_i(x)$ are continuous in the topology τ_R, there exist finite limits $a_i = \lim_\alpha a_i(x_\alpha)$, for $i = 0, 1, 2, \ldots$. We shall show that only finitely many numbers a_i can be different from zero. In fact, if $a_{i_k} \neq 0$ for an increasing sequence (i_k) of natural numbers, then there is a subsequence $q_m = i_{k_m}$ with $\frac{q_{m+1}}{q_m} \geq 2$ so that $q = (q_i)$ is in Q. Setting $r_{q_m} = \max\{1, \frac{2m}{|a_{q_m}|}\}$ and $r_i = 1$ for $i \neq q_m$ for all m we obtain a sequence $r = (r_m)$ in R. Since the norm $|\cdot|_r$ is continuous in the topology τ_R there exists a finite limit $M = \lim_\alpha |x_\alpha|_r$. But for any fixed m we have $|a_{q_m}(x_\alpha)| > \frac{|a_{q_m}|}{2}$ for sufficiently large α, what implies $|x_\alpha|_r \geq \frac{2m}{|a_{q_m}|}|a_{q_m}(x_\alpha)|$ for this α. This implies $M > m$, what is impossible, because M is finite and m was an arbitrarily chosen natural number. The contradiction shows that only finitely many numbers a_i are different from zero. Thus setting $x_o = \sum_0^\infty a_i t^i$ we obtain an element of $P(t)$. Put $y_\alpha = x_\alpha - x_o$, it is also a Cauchy net in A and $\lim_\alpha a_i(y_\alpha) = 0$ for all i. The completeness of A will follow if we show that $\lim_\alpha |y_\alpha|_r = 0$ for all r in R, because then $\lim_\alpha x_\alpha = x_o$. Assume then that $M_o = \lim_\alpha |y_\alpha|_{r_o} > 0$ for some r_o in R and try to get a contradiction. Define a support of a non-zero polynomial x setting $supp(x) = \{i : a_i(x) \neq 0\}$ and put $supp(0) = \emptyset$. For each r in R and for all $x, y \in A$ the relation $supp(x) \cap supp(y) = \emptyset$ clearly implies

$$|x + y|_r = |x|_r + |y|_r. \tag{3}$$

Choose an index $\alpha_o \in \mathfrak{a}$ so that $|y_{\alpha_o} - y_\alpha|_{r_o} < \frac{M_o}{2}$ for all $\alpha \succeq \alpha_o$ and put $S_o = supp(y_{\alpha_o})$, it is a finite or empty set of non-negative integers. Define a projection P on A setting

$$Px = \sum_{i \in S_o} a_i(x) t^i,$$

clearly it is a continuous operator on A. Denoting by I the identity operator on A we have the following obvious relation true for all elements x in A

$$\begin{aligned} &supp(Px) \cap supp((I-P)x) = \emptyset \quad \text{and} \\ &supp((I-P)x) \cap supp(y_{\alpha_o}) = \emptyset. \end{aligned} \tag{4}$$

Thus

$$\begin{aligned} |y_\alpha - y_{\alpha_o}|_{r_o} &= |P(y_\alpha) - y_{\alpha_o} + (I-P)y_\alpha|_{r_o} \\ &= |P(y_\alpha) - y_{\alpha_o}|_{r_o} + |(I-P)y_\alpha|_{r_{r_o}}, \end{aligned}$$

which implies $|(I-P)y_\alpha|_{r_o} < \frac{M_o}{2}$ for all $\alpha \succeq \alpha_o$. Since $\lim_\alpha a_i(y_\alpha) = 0$ for all i and S_o is a finite set, we have $\lim_\alpha |Py_\alpha|_{r_o} = 0$. Thus by (3) and (4) we obtain

$$\begin{aligned} M_o = \lim_\alpha |y_\alpha|_{r_o} &= \lim_\alpha |Py_\alpha + (I-P)y_\alpha|_{r_o} \\ &= \lim_\alpha |Py_\alpha|_{r_o} + \lim_\alpha |(I-P)y_\alpha|_{r_o} \\ &= \lim_\alpha |(I-P)y_\alpha|_{r_o} \le \frac{M_o}{2}, \end{aligned}$$

what is a contradiction proving the completeness of A.

To prove that A is a semitopological algebra it is sufficient to show that the operator $x \to tx$ is continuous, because it implies the continuity of the operator of multiplication by any fixed polynomial. Thus it is sufficient to show that $x \to |tx|_r$ is a continuous norm on A for each r in R. But it follows from the relation $|tx|_r = r_1|x|_{r'}$, where $r'_i = \frac{r_{i+1}}{r_1}$, $i = 0, 1, 2, \ldots$. We have $r' \in R$ because for any $q \in Q$ the sequence $(q_i + 1)$ is also in Q.

It remains to be shown that the topology τ_R is different from τ^{LC}_{max}. To this end it is sufficient to indicate a norm $|\cdot|_o$ on $P(t)$ which is not continuous in the topology τ_R. We put

$$|x|_o = \sum_0^\infty (k+1)|a_k(x)|.$$

If it is continuous in the topology τ_R, then there is a finite number of elements $r^{(1)}, \ldots, r^{(s)}$ in R and a positive constant C such that

$$|x|_o \le C \max\{|x|_{r^{(1)}}, \ldots, |x|_{r^{(s)}}\} \tag{5}$$

for all x in A. We shall use now the formula (1) taking there an m with $m > s$ and a k with $k > \max\{C, k(r^{(1)}, m), \ldots, k(r^{(s)}, m)\}$ we obtain an index $j_o \ge k$ with $r^{(n)}_{j_o} = 1$ for $n = 1, 2, \ldots, s$. Setting now in (5) $x = t^{j_o}$ we obtain $|t^{j_o}|_o = (j_o + 1) > k > C$ and $C \max\{|t^{j_o}|_{r^{(1)}}, \ldots, |t^{j_o}|_{r^{(s)}}\} = C$ so that (5) fails to be true. Thus $|\cdot|_o$ is a discontinuous norm on A and so $\tau_R \ne \tau^{LC}_{max}$. The conclusion follows.

By a result in [7] the algebra $(P(t), \tau_{max}^{LC})$ is a topological algebra. We shall show that the constructed above algebra A is not topological, what gives an alternate proof of $\tau_R \neq \tau_{max}^{LC}$. In fact, if A were topological, then for each r in R there would exist a positive constant C and a finite number $r^{(1)}, \ldots, r^{(s)}$ of elements of R such that

$$(6) \qquad |xy|_r \leq C \max\{|x|_{r^{(1)}}, \ldots, |x|_{r^{(s)}}\} \max\{|y|_{r^{(1)}}, \ldots, |y|_{r^{(s)}}\}$$

for all $x, y \in A$ (see [1], [3], [5], or [6]). Suppose that the formula (6) holds true and chose r so that $\limsup r_i = \infty$ and the corresponding sequence (q_i) consists of even numbers. Using again the formula (1) choose an odd m with $m > 2s$. Choose an index j with $r_{2j} > C$ so large that the interval of integers with center at j and length m lies entirely on the right of $k(r, m)$. By (1) this interval must contain two points of the form $j - p$ and $j + p$ such that $r_{j+p}^{(i)} = r_{j-p}^{(i)} = 1$ for $i = 1, 2, \ldots, s$. Setting now $x = t^{j+p}, y = t^{j-p}$ with p as above we have left hand of (6) equal to $r_{2j} > C$ while the right hand equals exactly to C and (6) fails to be true. Thus A is not a topological algebra.

The construcion given in the Proposition can be extended onto some other algebras accordingly to the following pattern: suppose that a (real or complex) algebra A can be decomposed into a direct sum $A = A_o + J$ where A_o is a subalgebra of A and J is its two-sided ideal, so that each element x of A can be uniquely written as $x = x_o + x_1$ with $x_o \in A_o$ and $x_1 \in J$. Suppose that A_o has some complete topology τ_R making of it a semitopological algebra and different from τ_{max}^{LC}, which is given by means of a family of seminorms $(|\cdot|_r)_{r \in R}$. We provide A with the topology τ_1 given by means of seminorms of the form

$$|x|_{(r,\alpha)} = |x_o|_r + |x_1|_\alpha,$$

where $|\cdot|_\alpha$ is an arbitrary seminorm on J, so that τ_1 restricted to J equals τ_{max}^{LC}. For a fixed $y = y_o + y_1 \in A$ we have

$$\begin{aligned} |xy|_{(r,\alpha)} &= |x_o y_o|_r + |x_o y_1 + x_1 y|_\alpha \\ (7) \qquad &\leq |x_o y_o|_r + |x_o y_1|_\alpha + |x_1 y|_\alpha, \end{aligned}$$

and similarly for $|yx|_{(r,\alpha)}$. Since the maps $x \to |x_o y_o|_r$, $x \to |x_o y_1|_\alpha$

and $x \to |x_1 y|_\alpha$ are continuous seminorms on (A, τ_1), the right multiplication by y is a continuous map. Similarly the left multiplication by y is also a continuous map, so that (A, τ_1) is a semitopological algebra. It is not hard to see that the topology τ_1 is complete on A and different from τ_{max}^{LC} (it is different from it on the subalgebra A_o), so that A has two different topologies τ_1 and τ_{max}^{LC} making of it a complete semitopological algebra. This pattern can be used when A is an algebra of polynomials in an arbitrary (finite or not) number of variables, or a free algebra (algebra of polynomials in non-commuting variables) in arbitrary number of variables. To use the pattern we fix one variable t_o and take as A_o the algebra $P(t_o)$ (as topology τ_R we take the topology used in the Proposition), then the ideal J consist of all linear combinations of monomials, each of them containing a variable different from t_o. Unfortunately this method does not work for many algebras. In particular we do not know the answer to the following

Problem. *Suppose that A is an infinite dimensional real or complex algebra each element of with all elements algebraic (over the field of scalars). Is τ_{max}^{LC} the only topology making of it a complete semitopological algebra ?*

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O TOPOLOGIZACJI $P(t)$

Dowodzi się, że w algebrze wielomianów istnieją conajmniej dwie różne topologie zupełne, przy których algebra jest przestrzenią wektorową topologiczną z oddzielnie ciągłym mnożeniem. Rozstrzyga to problem postawiony w pracy [2].

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