A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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LOWER BOUNDS OF A DETERMINANTAL EFFICIENCY MEASURE FOR L-S ESTIMATORS

1. INTRODUCTION

The main task of this paper is an analysis of the chosen efficiency lower bounds for least squares estimators (l-s estimators) of parameters of the linear model given by

(1) $\mathcal{M}_o \stackrel{df}{=} (\mathcal{R}^{n\times k}, S, Y, X\beta, \Omega, k_o = k,\ n_o = n,\ \mathcal{P}(Y) = \mathcal{N}_n(X\beta, \Omega))$

where:

$\mathcal{R}^{n\times k}$ - the set of $(n\times k)$ real matrices;

S - a probability space, i.e. $S = (U, \mathcal{F}, \mathcal{P})$ where U denotes the sample space, $\mathcal{F}$ is a Borel σ-field of U subsets, $\mathcal{P}$ is a measure satisfying the condition $\mathcal{P}(U) = 1$, and $Y = X\beta + \Xi$; $k_o = \text{rank}(X)$, $n_o = \text{rank}(\Omega)$, $Y \in S$; $\Xi \in S$; $X\beta = \mathcal{E}(Y)$, $\Omega = \mathcal{D}(Y)$, $\beta \in \mathcal{R}^{k\times 1}$, $X \in \mathcal{R}^{n\times k}$; $k < n$;

$\mathcal{E}$, $\mathcal{D}$, rank (A) - denote expectation, dispersion and rank operators;

$\mathcal{P}(Y) = \mathcal{N}_n(X\beta, \Omega)$ - denotes that the probability distribution of Y is the n-dimensional normal distribution with $\mathcal{E}(Y) = X\beta$, and $\mathcal{D}(Y) = \Omega$.

The definitions of the analytical form of bounds are taken from W a t s o n [6], B l o o m f i e l d, W a t s o n [2],

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and Knott [3]. The analysis of the range of five efficiency lower bounds is based on our own numerical results [4, 5]. The range of these bounds has been made dependent on the form of matrix Ω, values of autocorrelation coefficient ϱ, the sample size n and the number of parameters k.

2. CHARACTERIZATION OF THE EFFICIENCY LOWER BOUNDS FOR L-S ESTIMATORS. FORMS OF MATRIX Ω

Lower bounds of infimum of the determinantal efficiency measure also called "efficiency lower bounds" were derived for the model $\mathcal{M}_o$ under constraint $X'X = I$. This constraint, according to the arguments given by Watson [6], Bloomfield, Watson [2], and Anderson [1] (§ 10.2), does not influence the lower bounds range. They are given by the following relations:

$$\text{(2)} \qquad \inf \; e_B(X,\Omega) \geq e_1(n,k,\lambda_j) = \prod_{j=1}^{k} \frac{4\lambda_j\lambda_{n-j+1}}{(\lambda_j + \lambda_{n-j+1})^2},$$

$$\text{(3)} \qquad \inf \; e_B(X,\Omega) \geq e_2(n,k,\lambda_j) = \frac{4\prod_{j=1}^{n}\lambda_j}{\left(\prod_{j=1}^{k}\lambda_j + \prod_{j=1}^{k}\lambda_{n-j+1}\right)^2},$$

$$\text{(4)} \qquad \inf \; e_B(X,\Omega) \geq e_3(n,\lambda_j) = \frac{4\lambda_1\lambda_n}{(\lambda_1 + \lambda_n)^2},$$

$$\text{(5)} \qquad \inf \; e_B(X,\Omega) \geq e_4(n,k,\lambda_j) = \left(\frac{4\lambda_1\lambda_n}{(\lambda_1 + \lambda_n)^2}\right)^k,$$

$$\text{(6)} \qquad \inf \; e_B(X,\Omega) \geq e_5(n,k,\lambda_j) = \frac{1}{4}\sum_{i=1}^{4} e_i,$$

where: $\inf e_B(X,\Omega) = \inf \det(\mathcal{D}(\hat{B}))/\det(\mathcal{D}(B))$ denotes infimum of the determinantal efficiency measure of the estimator $B = (X'X)^{-1}X'Y$ in relation to the most efficient estimator $\hat{B} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}Y$ as a function of X and Ω. $\mathcal{D}(\hat{B})$, $\mathcal{D}(B)$ are dispersion matrices of $\hat{B}$ and B. The analytical form of $\inf e_B(X,\Omega)$ as an exact function of n,k, and λ_j (where λ_j is "j" eigen value of Ω, "n" is the sample size, k is the number of parameters in the model $\mathcal{M}_o$) is not known. We can yet define some lower bounds for $\inf e_B(X,\Omega)$. The values of these bounds are the values of the functions $e_i(n,k,\lambda_j)$, $i = 1, \dots, 5$, defined in the relations (2)-(6). The similar ranges of the $e_1, \dots, e_4$ with respect to the values of n, ϱ and the form of matrix Ω, on the one hand and the lack of the strong arguments for preferring one of these bounds caused, that we have treated the lower bound e_5 as a mean representative of the former bounds. This is why the bound e_5 will be the basis for our further analysis of the range of lower bounds of the determinantal efficiency measure (after describing the specific properties of the bounds $e_1, \dots, e_4$).

In the analysis we have taken into account the four forms of the matrix Ω:

a) variance-covariance matrix of the random component (satisfying the first order autoregressive scheme) of the form

(7) $$\Omega_1 = \sigma^2 \begin{bmatrix} 1 & \varrho & \varrho^2 & \dots & \varrho^{n-1} \\ \varrho & 1 & \varrho & \dots & \varrho^{n-2} \\ \varrho^2 & \varrho & 1 & \dots & \varrho^{n-3} \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ \varrho^{n-1} & \varrho^{n-2} & \varrho^{n-3} & \dots & 1 \end{bmatrix}$$

b) variance-covariance matrix of the random component (satisfying the scheme $\Xi_t = \varrho_1 \Xi_{t-1} + \varrho_2 \Xi_{t-2} + U_t$; $\varrho_1 = \varrho$, $\varrho_2 = \varrho^2$) of the form

$$(8) \qquad \Omega_2 = \sigma^2 \begin{bmatrix} 1 & h(1)\rho & \dots & h(n-1)\rho^{n-1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ h(n-1)\rho^{n-1} & h(n-2)\rho^{n-2} & \dots & 1 \end{bmatrix},$$

where

$$h(s) = \frac{\sum_{r=0}^{\infty} k_{r+s}k_r\rho^{2r}}{\sum_{r=0}^{\infty} k_r^2}, \qquad k_o, k_1 = 1, \quad k_r = k_{r-1} + k_{r-2}, \quad s = 1,2, \dots, n-1, \ r = 2,3, \dots$$

$k(r)$ - coefficient satisfying recursive Fibonacci relation,

c) matrix $\Omega_3 = \sigma^2(A_1'A_1)^{-1}$, where A_1 has the form

$$(9) \qquad A_1 = A_1(\rho, \rho) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\rho & 1 & 0 & \dots & 0 \\ -\rho & -\rho & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

d) matrix $\Omega_4 = \sigma^2(A_2'A_2)^{-1}$, where A_2 has the form

$$(10) \qquad A_2 = A_2(\rho) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\rho & 1 & 0 & \dots & 0 \\ 0 & -\rho & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

While generating Ω_i $i = 1, \dots, 4$ it was assumed that $\sigma = 1$. This does not change the generality of further results (due to the properties of the determinants and the definition of $e_B(X, \Omega)$).

3. ANALYSIS OF THE RANGE OF EFFICIENCY LOWER BOUNDS FOR L-S ESTIMATORS (THE CASE OF MODEL $\mathcal{M}_o$)

Further analysis is based on the results of the numerical Monte-Carlo experiments. They depend on:

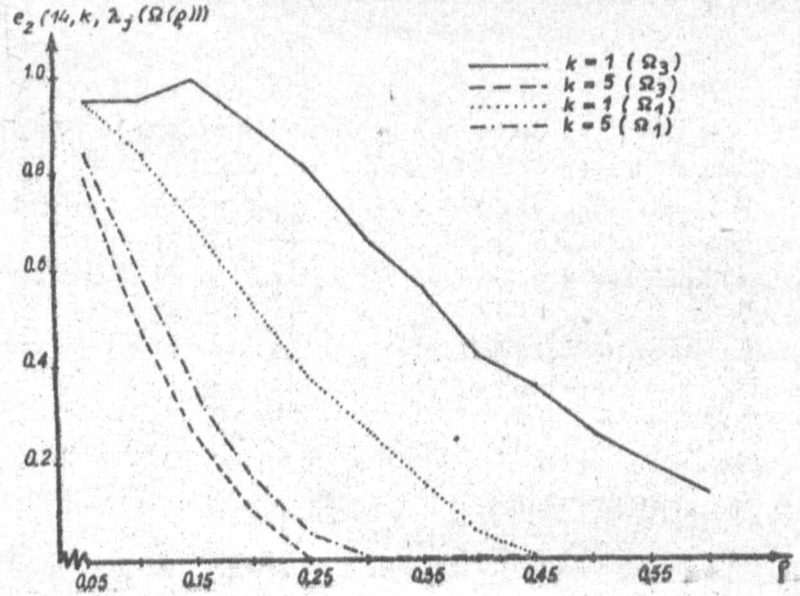
- generation of the values $\varrho : |\varrho| < 1$ and matrix $\Omega(\varrho)$ according to the definitions of the Ω_i, $i = 1, \ldots, 4$,
- calculation (by the Jaccobi algorithm) of eigen values of eeach matrix Ω_i,
- calculation for each value of ϱ, n, k and each matrix Ω_i, the values of the function e_j, $j = 1, \ldots, 5$, $(n = 8(2), 20$, $k = 1(1)5$, $\varrho = 0.05\ (0.05)\ 0.99)$.

The experiments carried out in such a way gave us an opportunity not only to analyse the range of the efficiency lower bounds for l-s estimator in relation to ϱ, n, k and the form of Ω, but also to compare the analysed bounds.

3.1. Comparison of the bounds $e_1, \ldots, e_5$.

The results of our experiments show that despite the same direction in the behaviour of the range of the bounds $e_1, \ldots, e_5$ the increase of the value of the coefficient ϱ and the change of the form of the matrix Ω_i, $i = 1, \ldots, 4$, cause some differences in this range behaviour. The behaviour of the range of the bound e_2 deviates the most from the behaviour of the range of the bounds e_1, e_3, e_4 (when the values n,k and the form of matrix Ω are fixed and ϱ is changed). This fact follows, among others, from a different definition of e_2. The behaviour of the range of the e_2 conditioned on the changes in ϱ for Ω_1, Ω_2 differs substantially from the run of range of the e_2 for Ω_3 and Ω_4 (see: Fig. 1a, b).

It means (see: def. Ω_3, Ω_4 and Ω_1, Ω_2) that there is a considerable influence of the heteroscedasticity of the model on the range of the e_2. The bounds e_3 and e_4 are functionally related, i.e. $e_4 = e_3^{\ k}$ and are equal only if k=1 (for $k = 1$ the relation $e_4 = e_3 = e_1$ holds as well). The bound a)



b)

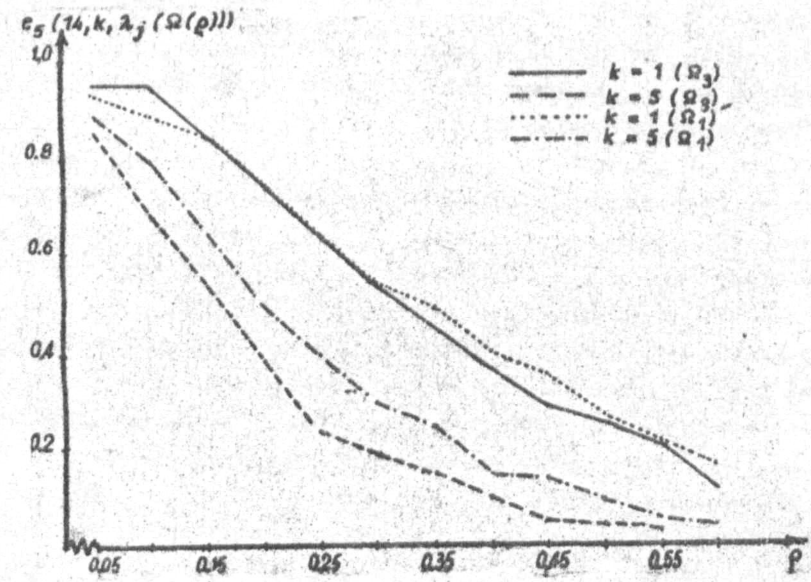


Fig. 1. The run of range of the efficiency lower bounds e_2, e_5 in dependence on k and the form of Ω_i (n = 14)

e_4 has slightly smaller values which decrease further with the increase in the value of ϱ for $k > 1$, in comparison with the respective values of the e_1, yet the bound e_3 (independently of k) reaches the biggest values among e_1, e_2, e_3, e_4.

The further analysis (because of the similarity in the evolution of the range of e_5 and e_1, e_2, e_3, e_4 in other cases) will be limited to the description of the behaviour of the range of e_5, taking into account the differences between the analysed bounds in dependence on the levels of n, k, ϱ and the form of the matrix Ω_i.

3.2. The dependence of the behaviour of the range of the efficiency lower bounds on the structure of the matrix Ω

The structure of the matrix Ω is usually omitted in the investigations of the efficiency of the estimator B in the case of the model $\mathcal{M}_0$. Generally, it is assumed that the variance-covariance matrix is of the form Ω_1 (such a form of matrix Ω is for instance used as the matrix of weights in the generalized l-s). The experiments carried out confirmed (on the studied structures of Ω) that there are some differences in the values of the lower bounds of the efficiency caused by the changes in the structure of matrix Ω (see: Fig. 1a, b).

The biggest difference is between $e_5\ (20,2,\ \lambda_j\ (\Omega_4\ (\varrho)))$ and $e_5\ (20,2,\ \lambda_j\ (\Omega_3\ (\varrho)))$ for $\varrho \in (0.20,\ 0.35)$, i.e.

$$e_5(20,\ 2, \lambda_j(\Omega_4(\varrho))) - e_5\ (20,\ 2, \lambda_j(\Omega_3(\varrho))) \approx 0.16.$$

The values of e_5 decline slightly (about 0.03 for $\varrho \in (0.05,\ 0.30)$, and about 0.1 for $\varrho \in (0.35,\ 0.55)$) with the change of the form of the variance-covariance matrix from Ω_1 to Ω_2. It means a small influence of the change (from 1 to 2) of the degree of the autoregressive process generating Ξ_t on the values of the efficiency lower bounds in the case of $n = 20$ and $k = 2$.

3.3. The dependence of the efficiency lower bounds on the number of the parameters k

The influence of the k on the behaviour of the range of the efficiency lower bounds is, in comparison with the structure of the matrix Ω, quite meaningful. As we stated in § 3.1

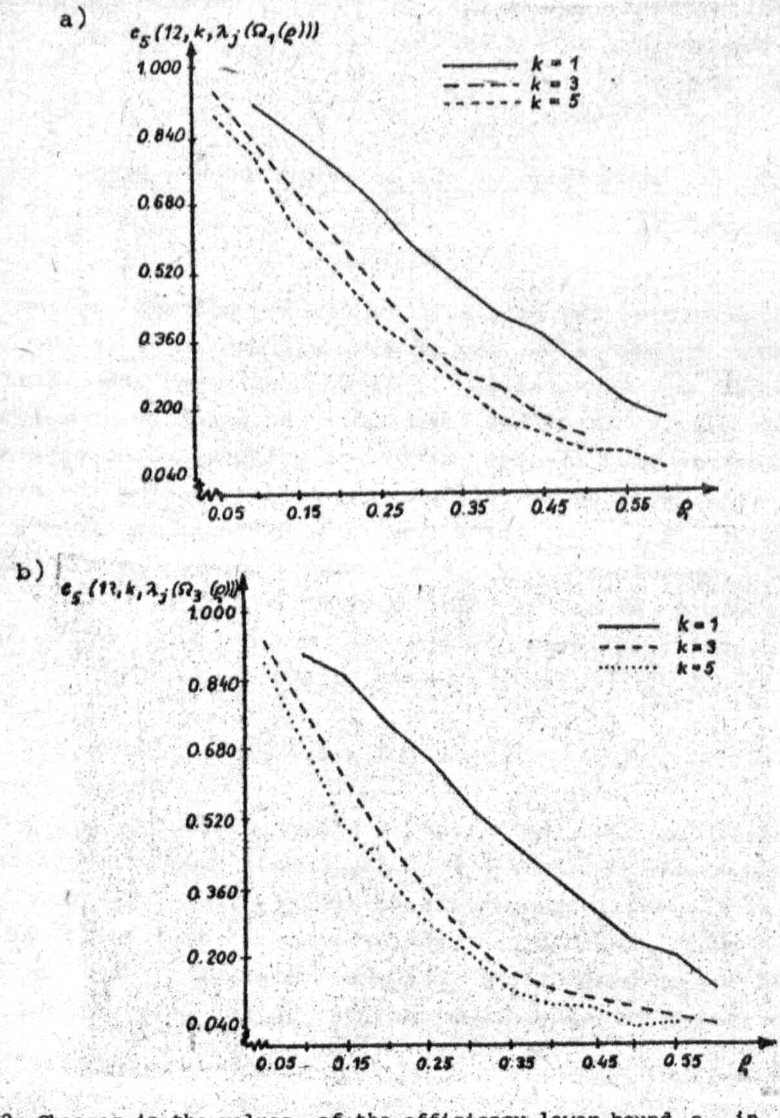


Fig. 2. Changes in the values of the efficiency lower bound e_5 in dependence on the number of the parameters k = 1, 3, 5

(see also Fig. 1a, b) the magnitude of this influence depends on the form of the lower bound. Besides these differences we can note some regularity in the behaviour of the range of all lower bounds. It is expressed in the behaviour of the e_5. The results of the experiments show a distinct decrease of the value of the e_5 with the increase in k from 1 to 3 and the very small decrease of the value of the e_5 corresponding to the growth in the number of parameters from 3 to 5 (see Fig. 2a, b). The magnitude of these changes is dependent on the structure of the matrix Ω (being greater for Ω_3 and Ω_4), i.e.

$$e_5(12,\ 1,\lambda_j(\Omega_1(\varrho))) - e_5(12,\ 3,\lambda_j(\Omega_1(\varrho))) \approx 0.19$$

while

$$e_5(12,\ 1,\lambda_j(\Omega_3(\varrho))) - e_5(12,\ 3,\lambda_j(\Omega_3(\varrho))) \approx 0.30$$

for $\varrho \in (0.2,\ 0.5)$.

3.4. The influence of the sample size on the run of the range of the efficiency lower bounds

The increase of the sample size n caused a slight decrease of the values of the efficiency lower bounds. This decrease appears slightly stronger in the case of the greater number of parameters as well as matrices Ω_1 and Ω_3 and is the biggest for $\varrho \in (0.15,\ 0.45)$. Denoting

$$\Delta e_5(\cdot,\ k,\Omega_i) = e_5(8,\ k,\lambda_j(\Omega_i(\varrho))) - e_5\ (20,\ k,\lambda_j(\Omega_i(\varrho))),$$

$\varrho \in (0.15,\ 0.45)$ we have

$$\Delta e_5(\cdot,\ 2,\Omega_1) \in (0.0677,\ 0.0894),$$

$$\Delta e_5(\cdot,\ 4,\Omega_1) \in (0.0744,\ 0.1442),$$

$$\Delta e_5(\cdot,\ 2,\Omega_2) \in (0.0640,\ 0.0875),$$

$$\Delta e_5(\cdot,\ 4,\Omega_2) \in (0.0734,\ 0.1584),$$

$$\Delta e_5(\cdot,\ 2, \Omega_4) \in (0.0330,\ 0.0694),$$

$$\Delta e_5(\cdot,\ 4, \Omega_4) \in (0.0803,\ 0.1318).$$

The run of the range of $e_5(\cdot, 4, \Omega_2)$ for $n = 8$ and $n = 20$ is given in the following Fig. 3.

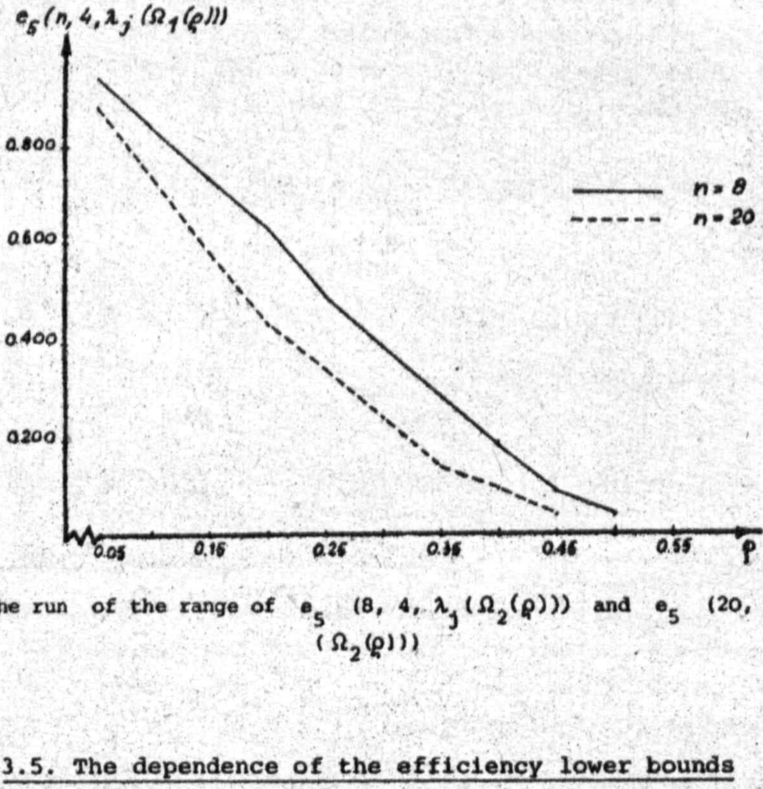


Fig. 3. The run of the range of e_5 $(8, 4, \lambda_j(\Omega_2(\rho)))$ and e_5 $(20, 4, \lambda_j(\Omega_2(\rho)))$

3.5. The dependence of the efficiency lower bounds on the values of the coefficient ρ

In § (3.1)-(3.4) we have given the analysis of the influence of the n,k and the form of the matrix Ω on the shape of the run of the range of $e_1, \ldots, e_5$. All these factors changed the shape of the dependence of the efficiency lower bounds on the value of ρ. Table 1 presents the run of the range of e_i $(10, 3, \lambda_j(\Omega_1(\rho)))$ $i = 1, \ldots, 5$, $\rho = 0.05\ (0.05)\ 0.55,\ 0.61$ in the case when $n = 10$ and $k = 3$.

It follows from it that the relation e_5 $(10, 3, \lambda_j(\Omega_1(\rho)))$

in ρ is straight-line relation with the direction coefficient near unity.

Table 1

The values of the lower bounds $e_1, \ldots, e_5$ in dependence on the values of ρ

ρ	e_1	e_2	e_3	e_4	e_5
0.05	0.979651	0.928180	0.990838	0.972764	0.967858
0.10	0.921201	0.746880	0.963872	0.895485	0.881859
0.15	0.831857	0.529521	0.920610	0.780239	0.765557
0.20	0.721893	0.337525	0.863387	0.643602	0.641602
0.25	0.602622	0.197099	0.795134	0.502713	0.524392
0.30	0.484451	0.106975	0.719102	0.371854	0.420590
0.35	0.375500	0.054470	0.638600	0.260428	0.332250
0.40	0.280958	0.026142	0.556749	0.172576	0.259100
0.45	0.203147	0.011834	0.476307	0.108059	0.199837
0.50	0.142060	0.005039	0.399548	0.063783	0.152600
0.55	0.096117	0.002006	0.328216	0.035357	0.115424
0.61	0.057556	0.000599	0.251445	0.015897	0.081374

It also results from Tab. 1 that the increase of ρ in the interval (0.05, 0.15) has small influence on the decrease of the value of e_5 $(10, 3, \lambda_j(\Omega_1(\rho)))$; it follows that the autocorrelation of 0.05-0.15 has no real influence on the efficiency of l-s estimator.

4. FINAL REMARKS

In the paper (in the range limited by the number of considered levels of the values of n, k, ρ and forms of the matrix Ω) we present the analysis of the run of the range of the efficiency lower bounds of the l-s estimators. These results (see also Fig. 4, ..., Fig. 13) can be easily translated into statements (for empirical and theoretical research works) concerning the estimate of the probable losses in efficiency of l-s estimator in the case of the model with autocorrelation.

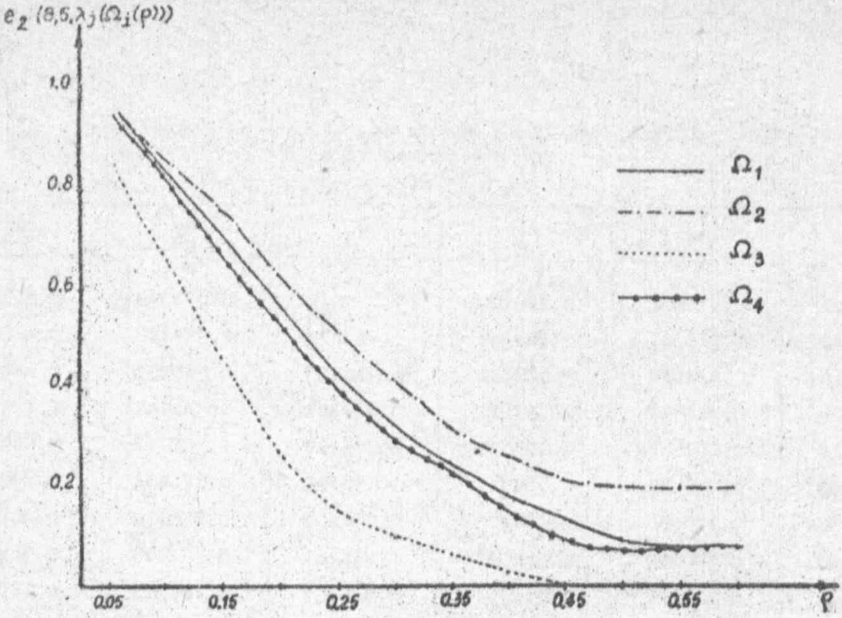


Fig. 4. The run of the range of e_2 (8, 5, $\lambda_j(\Omega_i(\rho))$)

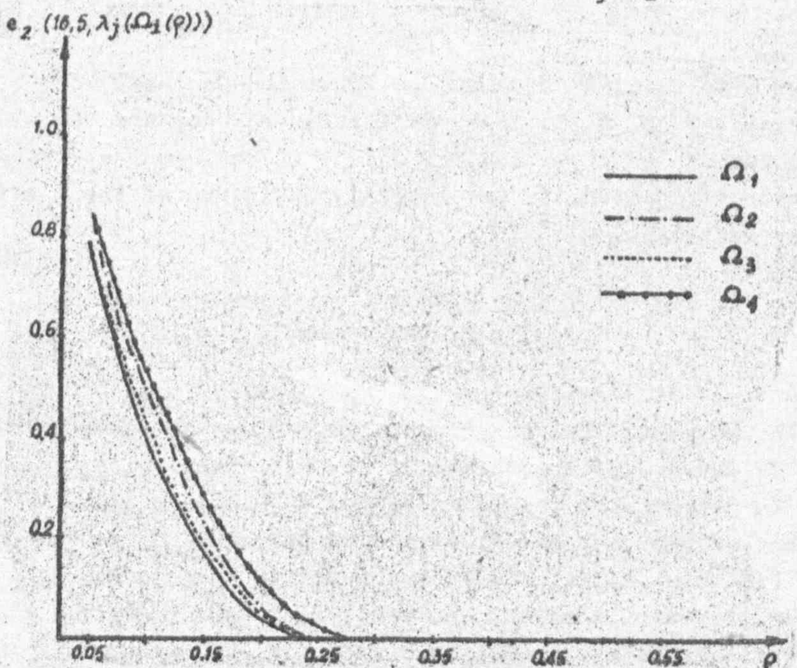


Fig. 5. The run of the range of e_2 (16, 5, $\lambda_j(\Omega_i(\rho))$)

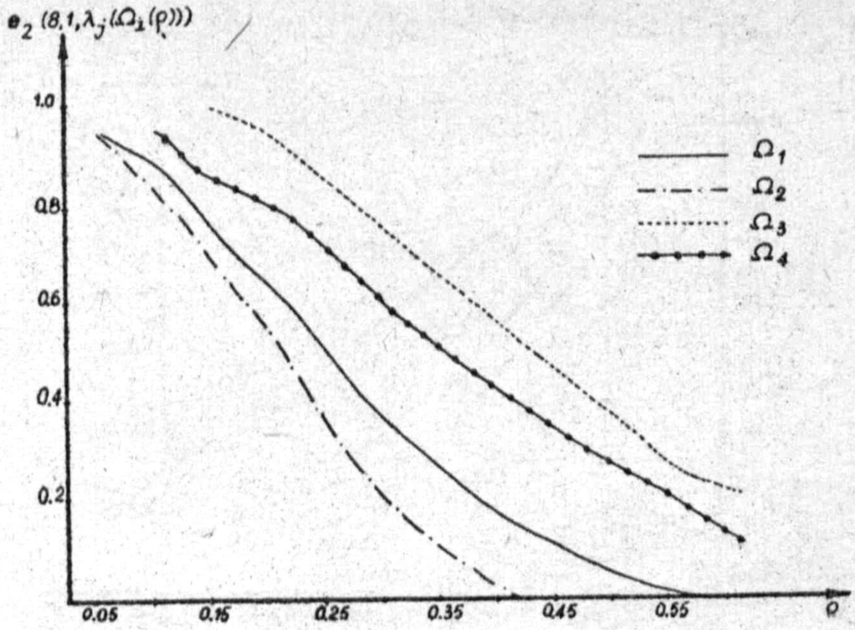


Fig. 6. The run of the range of e_2 $(8, 1, \lambda_j(\Omega_i(\rho)))$

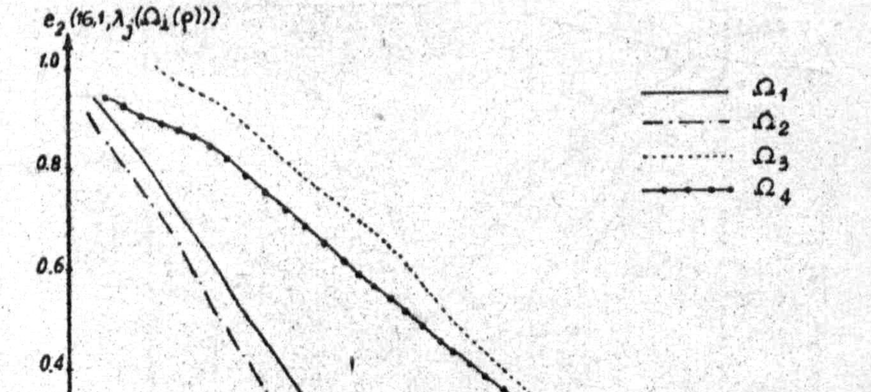


Fig. 7. The run of the range of e_2 $(16, 1, \lambda_j(\Omega_i(\rho)))$

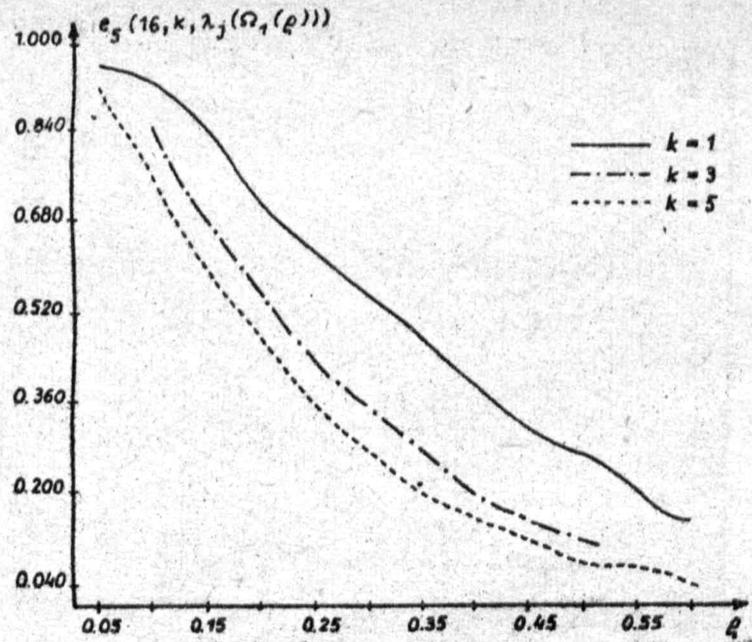


Fig. 8. The run of the range of e_5 (16, k, $\lambda_j(\Omega_1(\varrho))$)

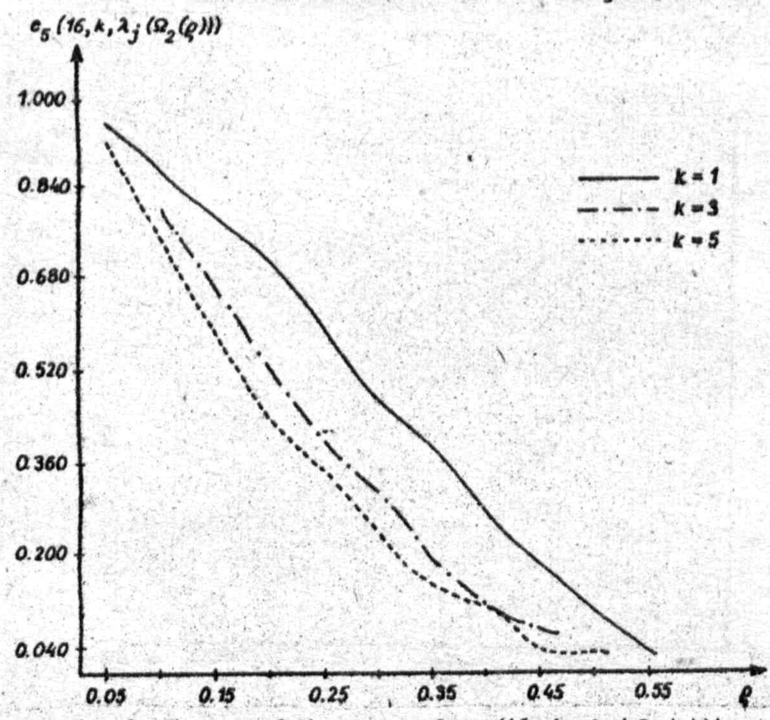


Fig. 9. The run of the range of e_5 (16, k, $\lambda_j(\Omega_2(\varrho))$)

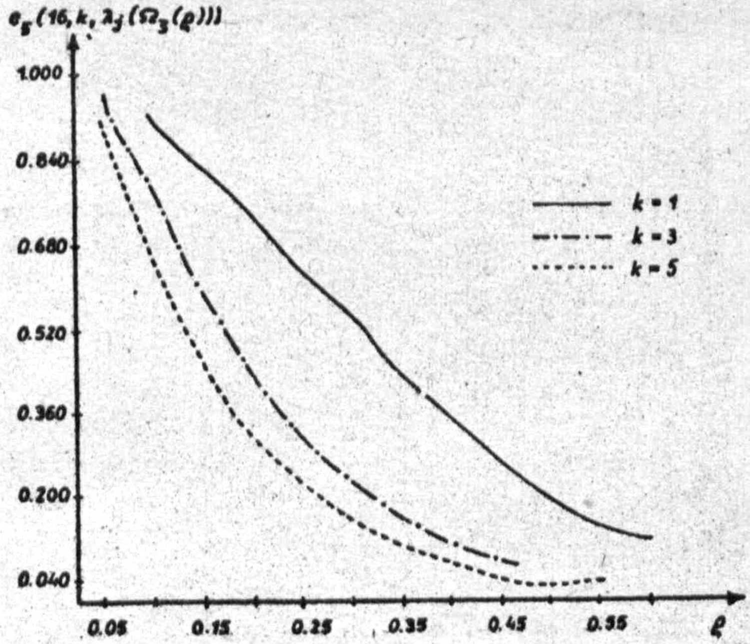


Fig. 10. The run of the range of e_5 $(16,\ k,\ \lambda_j(\Omega_3(\varrho)))$

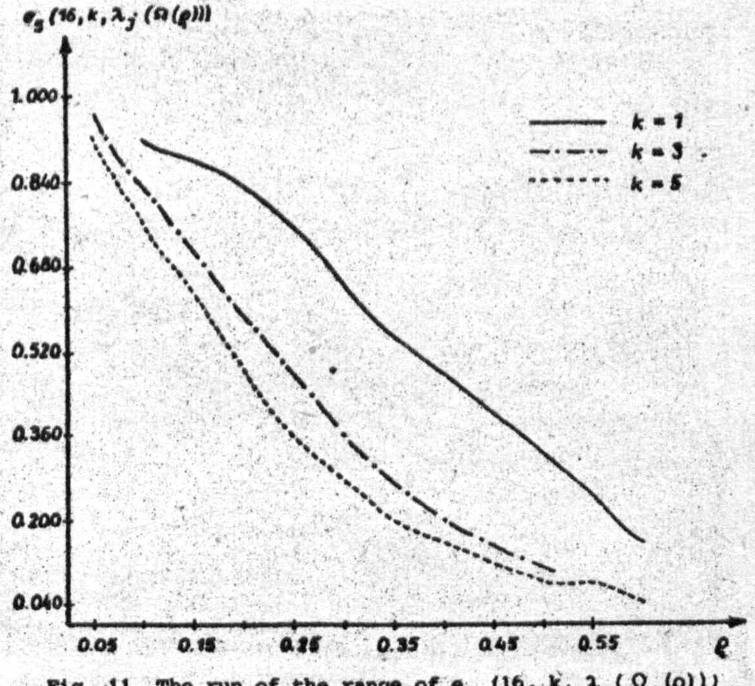


Fig. 11. The run of the range of e_5 $(16,\ k,\ \lambda_j(\Omega_4(\varrho)))$

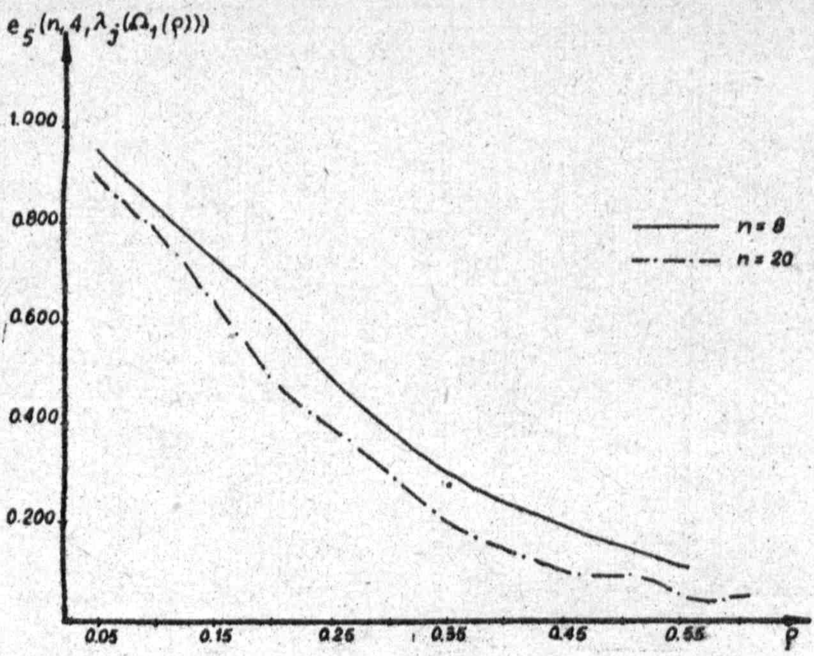


Fig. 12. The run of the range of e_5 (n, 4, $\lambda_j(\Omega_1(\rho))$)

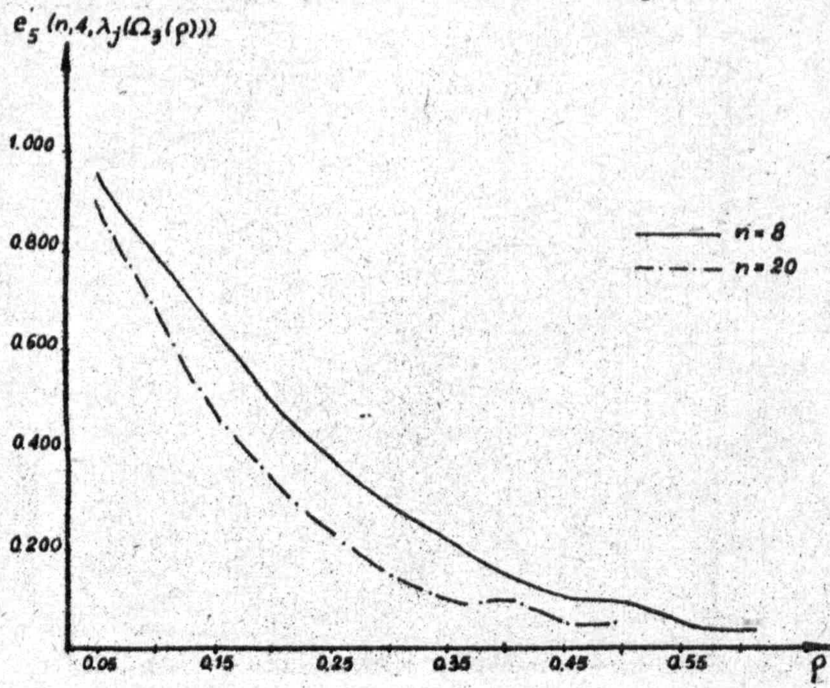


Fig. 13. The run of the range of e_5 (n, 4, $\lambda_j(\Omega_3(\rho))$)

REFERENCES

[1] A n d e r s o n T. (1971): *The Statistical Analysis of Time Series*, New York, Wiley.

[2] B l o o m f i e l d P., W a t s o n G. (1975): *Inefficiency of Least-Squares*, "Biometrika", 62(1), p. 121-128.

[3] K n o t t M. (1975): *On the Minimum Efficiency of Least-Squares*, "Biometrika", 62(1), p. 129-132.

[4] M i l o W. (1977): *Efektywność estymatorów parametrów modeli liniowych z autokorelacją*, Przegl. Statyst., 24(4), p. 443-454.

[5] M i l o W., W a s i l e w s k i Z. (1979): *Efektywność estymatorów parametrów ogólnych modeli liniowych*. Cz. I, work within the contract R. III.9.5.7.

[6] W a t s o n G. (1967): *Linear Least-Squares Regression*, Ann. of Math. Statist., 38, p. 1679-1699.

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DOLNE OGRANICZENIA WYZNACZNIKOWEJ MIARY EFEKTYWNOŚCI DLA ESTYMATORÓW M.N.K.

Celem artykułu jest analiza przebiegu zmienności pięciu dolnych ograniczeń wyznacznikowej miary efektywności estymatora metody najmniejszych kwadratów parametrów ogólnego modelu liniowego z autokorelacją. Opierając się na własnych wynikach numerycznych, zbadano zależność przebiegu zmienności tych ograniczeń od czterech postaci macierzy Ω dyspersji składników losowych, wartości współczynnika autokorelacji $\rho \in (-1,1)$, liczebności próbki n, liczby parametrów k, pięciu postaci analitycznych dolnych ograniczeń. Część wyników podano w formie wykresów. Otrzymane wyniki można wykorzystać do oceny maksymalnych górnych ograniczeń strat na efektywności estymatora m.n.k. w przypadku różnych schematów autokorelacji.