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ON ŚWIĄTKOWSKI FUNCTIONS

This paper is connected with basic properties of Świątkowski functions of one and two variables. Among other things, it has been shown that, in the class of a.e. continuous functions of class $\mathcal{M}_1$ of Zahorski, the possession of the Świątkowski property by a function is equivalent to the fact that the function belongs to class $\mathcal{M}_2$ of Zahorski. The last theorem of the paper is connected with the generalisation of the notion of Świątkowski function to the case of function of two variables, in the context that, for this class of functions, the fundamental theorem of Mańk-Świątkowski is true.

In paper [1] T. Mańk and T. Świątkowski defined a new class of functions whose elements (according to the terminology adopted in [3]) will be called Świątkowski functions (or functions possessing the property of Świątkowski). It turns out that, in some cases, the class of Świątkowski functions comprises a number of important classes of functions. In particular, it can be shown that, in the class of almost everywhere continuous functions of the first class of Zahorski, the fact that the function f belongs to the second class of Zahorski is equivalent to the possession of the property of Świątkowski by a function f. (see *Theorem 2)*. Hence it easily appears that, for instance, if f is a.e. continuous derivative, then it is Świątkowski function (see *Corollary 1)*. *Theorem 3* is connected with generalisations of the notion of Świątkowski functions to the case of functions of two variables in the context that, for this class of functions, the fundamental theorem of Mańk-Świątkowski is true.

We use the standard notions and notation. By R we shall denote the set of real numbers with its natural topology.

Throughout the paper, we consider real functions defined on the closed segment $I \subset R$ or on the plane R^2. Let f be an arbitrary function, then: by C_f (D_f) we denote the set of all continuity (discontinuity) points of f and write $E^{\alpha}(f) = f^{-1}((-\infty, \alpha))$ and $E_{\alpha}(f) = f^{-1}((\alpha, +\infty))$ for $\alpha \in R$. The symbol $\mathcal{M}^*$ denotes class of such functions $f : I \to R$, that: $E_{\alpha}(f_{|K}) = \emptyset$ or $E_{\alpha}(f_{|K}) \cap C_f \neq \emptyset$ and $E^{\alpha}(f_{|K}) = \emptyset$ or $E^{\alpha}(f_{|K}) \cap C_f \neq \emptyset$ for every $\alpha \in R$ and every nondegenerate segment $K \subset I$.

By $[a, b]$ and (a, b) respectively, we shall denote closed and open segment joining a and b (for segments lying on the real line we shall use these symbols also in the case when $a > b$). To simplify denotations, we shall write (a, b) and $[a, b]$ instead of $(a, b) \cap I$ and $[a, b] \cap I$, respectively. Image and inverse image of these segments will be denoted by $f(a, b)$, $f[a, b]$, $f^{-1}(a, b)$ and $f^{-1}[a, b]$ respectively, to avoid double brackets. By $m(a, b)$ we denote the middle-point of the interval (a, b) and by $\rho(a, b)$ we denote the distance of points a and b on the plane.

Let $D(B_1)$ denote the class of Darboux function (resp. functions in Baire class 1). According to the definition in [3] we assume, for a function f of two variables, that $f \in D$ if for every closed segment $[a, b] \subset R^2$ the image $f[a, b]$ is a connected set. Moreover we assume $D \cap B_1 = DB_1$.

The closure, interior, diameter and Lebesgues measure of the set A we denote by: $\bar{A}$, Int A, $\delta(A)$ and $\mu(A)$ respectively.

The uniformly convergence of a sequence of functions $\{f_n\}$ to f we shall denote by $f_n \rightrightarrows f$.

The term Zahorski classes denote the classes $\mathcal{M}_k$ ($k = 0, 1, \ldots, 5$) defined by Zahorski in [7]. In particular $\mathcal{M}_0 = \mathcal{M}_1 = DB_1$ and $\mathcal{M}_2$ denotes the class of such functions f, that for every $\alpha \in R$ the sets $E^{\alpha}(f)$ and $E_{\alpha}(f)$ are of type F_{σ} and every one-sided neighbourhood of each $x \in E^{\alpha}(f)$ ($x \in E_{\alpha}(f)$) intersects $E^{\alpha}(f)$ ($E_{\alpha}(f)$) in a set of the positive measure.

To make precise the denotations of the terms used in the present paper, we accept the following definitions.

Definition 1 [1]. We say, that a function $f : I \to R$ possesses the property of Świątkowski (or f is Świątkowski function) if for every two points $x, y \in I$ such that $f(x) \neq f(y)$ there exists a point z of continuity of f, such that $z \in (x, y)$ and $f(z) \in (f(x), f(y))$.

Definition 2 [6]. We say, that a function f possesses the property of Young if for each x, there exist sequences $\{x_n^-\}$, $\{x_n^+\}$ such that $x_n^- \nearrow x$, $x_n^+ \searrow x$ and $\lim_{n \to \infty} f(x_n^-) = f(x) = \lim_{n \to \infty} f(x_n^+)$.

In [1] it has been shown that the class of Świątkowski functions is not contained in the class DB_1. So, the question of finding a necessary and sufficient condition for a function $f \in DB_1$ to be a Świątkowski one seems interesting. The answer to this question is given by *Theorem 1* which will be preceded by the following lemma:

Lemma 1 [6]. Let $f \in B_1$. Then $f \in D$ if and only if f possesses the property of Young.

Theorem 1. Let $f : I \to R$ and let $f \in \mathcal{M}_1$. Then f possesses the property of Świątkowski if and only if $f \in \mathcal{M}^*$.

P r o o f. Necessity. Let K be an arbitrary nondegenerate interval and let $\alpha \in R$. Suppose that $x \in E^\alpha(f_{|K})$. Now, if $f(K) = \{f(x)\}$, then $E^\alpha(f_{|K}) \cap C_f \neq \emptyset$. In the opposite case, according to the fact that $f \in D$, we infer that there exists an element x' of $E^\alpha(f_{|K})$ such that $f(x) \neq f(x')$. Let $x'' \in C_f \cap (x, x') \cap f^{-1}(f(x), f(x'))$. Of course $x'' \in C_f \cap E^\alpha(f_{|K})$. This ends the proof of necessary condition.

Sufficiency. Let $x_1, x_2 \in I$ and let $f(x_1) \neq f(x_2)$. Since $f \in D$, then there exist points $x_1', x_2' \in (x_1, x_2)$ such that $\emptyset \neq f(x_1'), f(x_2')) \subset [f(x_1'), f(x_2')] \subset (f(x_1), f(x_2))$. Assume, for instance, that: $f(x_1) < f(x_1') < f(x_2') < f(x_2)$.

Put $g = f_{|[x_1, x_2]}$ and let $A = g^{-1}(g(x_1'), g(x_2'))$. It is suffice to show, that

$$g^{-1}(g(x_1), g(x_2)) \notin 2^{D_f}. \tag{1}$$

Assume, to the contrary, that:

$$g^{-1}(g(x_1), g(x_2)) \in 2^{D_f}. \tag{1'}$$

Put $h = g_{|\bar{A}} = f_{|\bar{A}}$. Since $f \in B_1$, then there exists the point $x_o \in \bar{A}$ such that $x_o \in C_h$.

Now, we remark that, if $\{x_n\}$ is a sequence of elements of A and $\lim_{n \to \infty} x_n = x_o$, then $\{h(x_n)\}$ is the sequence of elements of the segment $(g(x_1'), g(x_2'))$. Hence, according to the continuity of h at x_o, we infer that

$h(x_o) \in [g(x_1'), g(x_2')] \subset (g(x_1), g(x_2))$.

Of course $g(x_o) \neq g(x_1')$ or $g(x_o) \neq g(x_2')$. Assume, for instance that $g(x_o) \neq g(x_2')$. According to the continuity of h at x_o, we deduce that there exists an open segment $(a, b) \subset [x_1, x_2]$ such that

$$x_o \in (a, b) \text{ and } g((a, b) \cap \bar{A}) \subset (g(x_1), g(x_2')) \tag{2}$$

It is easy to see that:

$$(a, x_o) \cap A \neq \emptyset \text{ or } (x_o, b) \cap A \neq \emptyset \tag{2'}$$

If $x_o \in A$, then (2') follows from *Lemma 1*, if $x_o \in \bar{A} \setminus A$ then (2') is, of course, true too.

Now, let

$$x_* = \begin{cases} \text{an arbitrary element of the set } (a, x_o) \cap A, & \text{if } (a, x_o) \cap A \neq \emptyset \\ x_o, & \text{if } (a, x_o) \cap A = \emptyset \end{cases}$$

$$x^* = \begin{cases} \text{an arbitrary element of the set } (x_o, b) \cap A, & \text{if } (x_o, b) \cap A \neq \emptyset \\ x_o, & \text{if } (x_o, b) \cap A = \emptyset \end{cases}$$

Remark that $x_o \in [x_*, x^*] \subset [x_1, x_2]$ and $[x_*, x^*]$ is the non-degenerate segment. Denote $A^* = \bar{A} \cap [x_*, x^*]$. Thus $x_o \in A^*$ and, according to (2) and (1'), we infer that

$$A^* \in 2^{D_f}. \tag{3}$$

Put $B^* = [x_*, x^*] \setminus A^*$. Of course $x_*, x^* \notin B^*$. Moreover (2) implies, that $f(A^*) \subset (-\infty, g(x_2'))$. Thus, according to the fact that $f \in \mathcal{U}^*$, we deduce that $E^{g(x_2')}(f_{|[x_*, x^*]}) \cap C_f \neq \emptyset$, which means, according to (3), that $B^* \neq \emptyset$. Now, let (α, β) be an arbitrary component of B^*. Then

$$g[\alpha, \beta] \subset (-\infty, g(x_1')]. \tag{4}$$

In fact. Assume to the contrary that:

$$g[\alpha, \beta] \cap (g(x_1'), +\infty) \neq \emptyset. \tag{4'}$$

First, we remark, that since $g \in D$, then:

$\{y \in [\alpha, \beta] : g(y) \in (g(x_1'), g(x_2'))\} = \emptyset$.

It is easy to see (according to (2) and (*)) that $g(\alpha) \in (g(x_1), g(x_1')]$, which, according to (4') is impossible, because $g \in D$. The obtained contradiction ends the proof of (4).

We can remark that:

$$E_{f(x_1')}(f_{|[x_*, x^*]}) \subset A^* \tag{5}$$

From (2′) we deduce that $x_* \in A$ or $x^* \in A$. Assume, for instance, that $x_* \in A$. Hence $f(x_*) \in (g(x_1'), g(x_2'))$, which means that $E_{f(x_1')}(f_{|[x_*, x^*]}) \neq \emptyset$. Since $f \in \mathcal{M}^*$, then $E_{f(x_1')}(f_{|[x_*, x^*]}) \notin 2^{D_f}$. This, according to (5), contradicts (3).

It is easy to see, that the class of Świątkowski functions is not contained in the class $\mathcal{M}^* \cap D$ as well as in $\mathcal{M}^* \cap B_1$.

The above theorem leads us to a number of interesting corollaries. Below, we shall present some of them.

Theorem 2. Let f be a.e. continuous functions and $f \in \mathcal{M}_1$. Then $f \in \mathcal{M}_2$ if and only if f possesses the property of Świątkowski.

P r o o f. Necessity. We shall show that $f \in \mathcal{M}^*$, which according to Theorem 1 denotes that f possesses the property of Świątkowski.

Let α be an arbitrary real number and K an arbitrary nondegenerated interval. Suppose that $E^\alpha(f_{|K}) \neq \emptyset$ ($E_\alpha(f_{|K}) \neq \emptyset$). Then, according to our assumptions we have that $C_f \cap E^\alpha(f_{|K}) \neq \emptyset$ ($C_f \cap E_\alpha(f_{|K}) \neq \emptyset$).

Sufficiency. Let α be an arbitrary real number and let $x \in E^\alpha(f)$ (proof in the case if $x \in E_\alpha(f)$ is similar). Let K be an arbitrary one-sided neighbourhood of x. If $f(K) = \{f(x)\}$ then $\mu(E^\alpha(f) \cap K) = \mu(K) > 0$.

In the opposite case, we infer that there exists a point $y \in (K \cap E^\alpha(f)) \setminus f^{-1}(\{f(x)\})$ and so there exists a point $z \in (x, y) \cap C_f$ such that $f(z) \in (f(x), f(y)) \subset (-\infty, \alpha)$. Then there exists a positive real number $\delta < \min(|z - x|, |z - y|)$ such that $f(z - \delta, z + \delta) \subset (-\infty, \alpha)$, which means that $\mu(E^\alpha(f) \cap K) \geqslant 2\delta > 0$.

Corollary 1. Let f be a.e. continuous derivative. Then f possesses the property of Świątkowski.

The next corollary is the simple consequence of *Theorem 2* and the theorem of M u k h o p a d h y a y [2].

Corollary 2. Let f be a.e. continuous and $f \in \mathcal{M}_1$. Then if f maps no zero measure perfect set onto an interval, then f possesses the property of Świątkowski.

Corollary 3. Let f be a.e. continuous and aproximately continuous function. Then f possesses the property of Świątkowski.

In [1] it has been shown that, if f_n possesses the property of Świątkowski and $f_n \in DB_1$ for n = 1, 2... and moreover $f_n \rightrightarrows f$ then f possesses the property of Świątkowski (and of course $f \in DB_1$). This theorem is the fundamental theorem of Mańk-Świątkowski.

In the present paper it will be demonstrated that the fundamental theorem of Mańk-Świątkowski is true also for functions of two variables (if we assume the definition of Świątkowski function in such a way as in *Definition 3*). By K(x, r) we shall denote an open circle having the centre at x and the radius r.

Definition 3 [4]. We say, that a function $f : R^2 \to R$ possesses the property of Świątkowski if for every two points $x, y \in R^2$ such that $f(x) \neq f(y)$ there exists a point z of continuity of f such that $z \in K(m(x, y), \frac{1}{2}\varrho(x, y))$ and $f(z) \in (f(x), f(y))$.

Lemma 2 [3]. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real functions of two variables such that $f_n \in DB_1$ for n = 1, 2, ... Then if $f_n \rightrightarrows f$ then $f \in DB_1$.

Theorem 3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real functions of two variables such that $f_n \in DB_1$ and f_n possesses the property of Świątkowski for n = 1, 2, ... Then, if $f_n \rightrightarrows f$ then $f \in DB_1$ and f possesses the property of Świątkowski.

P r o o f. According to *Lemma 2*, it is suffice to show that f possesses the property of Świątkowski.

Let $x_1, x_2 \in R^2$ and $f(x_1) \neq f(x_2)$. Assume, for instance, that $f(x_1) < f(x_2)$. Put $A_1 = K(m(x_1, x_2), \frac{1}{2}\varrho(x_1, x_2))$. We shall show that there exists a point $z \in A_1 \cap C_f$ such that $f(z) \in (f(x_1), f(x_2))$. Put

$$\alpha_1' = f(x_1) + \frac{1}{3}(f(x_2) - f(x_1)); \quad \alpha_1'' = f(x_1) + \frac{2}{3}(f(x_2) - f(x_1))$$

and let

$$\beta_1' = \alpha_1' + \frac{1}{3}(\alpha_1'' - \alpha_1'); \quad \beta_1'' = \alpha_1' + \frac{2}{3}(\alpha_1'' - \alpha_1').$$

Now, we shall show that there exists an open circle $A_2 = K(z_1, r_1)$ such that $r_1 < \frac{1}{4}\varrho(x_1, x_2)$, $\overline{A}_2 \subset A_1$ and $f(A_2) \subset (\alpha_1', \alpha_1'')$. Let N_1 denotes such natural number that for $x \in R^2$ and $n \geqslant N_1$: $|f_n(x) - f(x)| < \frac{1}{12}(\alpha_1'' - \alpha_1')$.

Since $f \in D$, then there exist points $z_1', z_1'' \in (x_1, x_2)$ such that $f(z_1') = \beta_1'$ and $f(z_1'') = \beta_1''$. Hence, we can choose from $K(m(z_1', z_1''), \frac{1}{2} \varrho(z_1', z_1''))$ a point $z_1 \in C_{f_{N_1}}$ such that $f_{N_1}(z_1) \in$ $\in (\beta_1', \beta_1'')$. Of course $z_1 \in A_1$. Let $r_1 > 0$ be such real number that $r_1 < \frac{1}{4} \varrho(x_1, x_2)$ and if $A_2 = K(z_1, r_1)$, then $\overline{A}_2 \subset A_1$ and $\delta(f_{N_1}(A_2)) \leq \frac{1}{12} (\alpha_1'' - \alpha_1')$.

It is easy to see that:

$$f(A_2) \subset (\alpha_1', \alpha_1'') \tag{6}$$

In fact. Remark that $|f(x) - f_{N_1}(z_1)| < \frac{1}{3} (\alpha_1'' - \alpha_1')$ for $x \in A_2$, which means that (6) is true.

If $f(A_2) = \{f(z_1)\}$, then we put $z = z_1$ and so z is the required point.

In the opposite case, let α_2', α_2'' denote such elements of (α_1', α_1''), that $\alpha_2' < \alpha_2''$, $\alpha_2'' - \alpha_2' < \frac{1}{2} (\alpha_1'' - \alpha_1')$ and there exist points $y_2', y_2'' \in A_2$ such that y_2', y_2'', z_1 are colineare and $f(y_2') = \alpha_2'$, $f(y_2'') = \alpha_2''$. Moreover, we put $\beta_2' = \alpha_2' + \frac{1}{3} (\alpha_2'' - \alpha_2')$, $\beta_2'' = \alpha_2' + \frac{2}{3} (\alpha_2'' - \alpha_2')$ and let $z_2', z_2'' \in (y_2', y_2'')$ be such points that $f(z_2') = \beta_2'$, $f(z_2'') = \beta_2''$. In analogously way as above, we can prove that there exists an open circle $A_3 = K(z_2, r_2)$ such that $r_2 <$ $< \frac{1}{2^3} \varrho(x_1, x_2)$, $\overline{A}_3 \subset A_2$ and $f(A_3) \subset (\alpha_2', \alpha_2'')$.

In this way we have defined two sequences $\{\overline{A}_n\}$ and $\{[\alpha_n', \alpha_n'']\}$ where $A_n = K(z_{n-1}, r_{n-1})$.

If for some n_o, $f(A_{n_o}) = \{f(z_{n_o-1})\}$, then we put $z = z_{n_o-1}$, and so z is the required point.

Now suppose that $f(A_n)$ is a nondegenerate interval for $n = 1, 2, \ldots$ Hence, $\{\overline{A}_n\}$ and $\{[\alpha_n', \alpha_n'']\}$ are infinite, decreasing sequence of closed sets and

$$\lim_{n \to \infty} \delta(\overline{A}_n) = 0, \quad \lim_{n \to \infty} \delta([\alpha_n', \alpha_n'']) = 0, \quad f(A_n) \subset (\alpha_{n-1}', \alpha_{n-1}'')$$

for $n = 1, 2, \ldots$

Let $\{z\} = \bigcap_{n=1}^{\infty} \overline{A}_n$.

Of course $z \in A_1$ and $f(z) \in (f(x_1), f(x_2))$.

Remark that $\{f(z)\} = \bigcap_{n=1}^{\infty} [\alpha'_n, \alpha''_n]$. In fact. Assume, to the contrary, that $\bigcap_{n=1}^{\infty} [\alpha'_n, \alpha''_n] = \{\alpha\}$ and $\alpha \neq f(z) = \alpha'$. Let, for instance, $\alpha' < \alpha$. Let n^* be such natural number that $\alpha' < \alpha'_{n^*} < \alpha$. Then $z \in A_{n^*+1}$ and $f(z) \in f(A_{n^*+1}) \subset (\alpha'_{n^*}, \alpha''_{n^*})$, which contradicts our assumption that $\alpha' < \alpha'_{n^*}$.

Finally, we remark that $z \in C_f$. This ends the proof.

H. Nonas and W. Wilczyński in [3] have assumed the following definition of Świątkowski function of two variables: a function $f : R^2 \to R$ possesses the property of Świątkowski if for every two points $x, y \in R^2$ such that $f(x) \neq f(y)$ there exists a point z of continuity of f, such that $z \in (x, y)$ and $f(z) \in (f(x), f(y))$. H. Nonas and W. Wilczyński have shown the example of a sequence of functions $\{f_n\}$ such that for every n, $f_n \in D$ and f_n possesses the property of Świątkowski, as well as $f_n \rightrightarrows f$, but f fails to possesses the property of Świątkowski. If we additionally assume that $f_n \in B_1$ $(n = 1, 2, \ldots)$ then it is not known the answer to the question: does f possesse the property of Świątkowski? (if we assume the definition of Świątkowski function in such a way as in paper [3]).

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O FUNKCJACH ŚWIĄTKOWSKIEGO

W pracy tej zostały omówione pewne własności funkcji Świątkowskiego. Pokazane zostało m. in., że w zakresie prawie wszędzie ciągłych funkcji pierwszej klasy Zahorskiego, klasa funkcji posiadających własność Świątkowskiego jest równa klasie $\mathcal{M}_2$ Zahorskiego. Pracę zamykają rozważania dotyczące własności Świątkowskiego funkcji dwóch zmiennych.