A C T A U N I V E R S I T A T I S L O D Z I E N S I S

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ON SOME CLASS OF CARATHÉODORY FUNCTIONS

Let $\mathcal{P}$ denote the well-known class of functions

$$P(z) = 1 + Q_1 z + \ldots + Q_n z^n + \ldots$$

holomorphic in the disc $\Delta = \{z: |z| < 1\}$ and satisfying in this disc the condition $\operatorname{Re} P(z) > 0$. Let

$$k_a(z) = 1 + \frac{a}{a+1} z + \ldots + \frac{a}{a+n} z^n + \ldots, \quad z \in \Delta,$$

$a \in \mathbb{C} \setminus \{-1, -2, \ldots\}$.

In the paper we examine the properties of the class $\mathcal{P}_a$ of functions of the form $p = P * k_a$, $P \in \mathcal{P}$, where $P * k_a$ stands for the Hadamard convolution of the functions P and k_a. Of course, $\mathcal{P}_\infty = \mathcal{P}$. We also give a few applications and formulate some problems to be solved. The idea of the paper has arisen in connection with the investigations concerning the well-known class T_α ([5], [6]) and with the realization of M. Sc. thesis [10]. Certain general questions concerning applications of the Hadamard convolution can be found, for instance, in [4].

1. INTRODUCTION

Let $\mathcal{P}$ denote the well-known ([1]) class of Carathéodory functions P holomorphic in the unit disc $\Delta = z: [z] < 1$, with the expansion

(1.1) $P(z) = 1 + Q_1 z + \ldots + Q_n z^n + \ldots, \quad z \in \Delta,$

satisfying the condition

(1.2) $\operatorname{Re} P(z) > 0, \quad z \in \Delta.$

Let $a \in \mathbb{C} \setminus \{-1, -2, \ldots\}$. For the above values of the parameter a, let us define a function $k_a(z)$ by the formula

$$(1.3) \quad k_a(z) = 1 + \sum_{n=1}^{\infty} \frac{a}{a+n} z^n, \quad z \in \Delta.$$

DEFINITION. Denote by $\mathcal{P}_a$ the class of functions p of the form

$$(1.4) \quad p = P * k_a$$

where $P \in \mathcal{P}$, k_a is defined by formula (1.3), while $P * k_a$ stands for the Hadamard convolution of the functions P and k_a.

In this paper we examine various properties of the classes $\mathcal{P}_a$ for $a \in \mathbb{C} \setminus \{-1, -2, \ldots\}$. We also give a few applications and formulate problems to be solved.

And so, if

$$(1.5) \quad p(z) = 1 + q_1 z + \ldots + q_n z^n + \ldots, \quad z \in \Delta,$$

$p \in \mathcal{P}_a$, then from (1.1), (1.3), (1.4) and (1.5) we have

$$(1.6) \quad q_n = \frac{aQ_n}{a+n}, \quad n = 1, 2, \ldots,$$

and vice versa.

Since, as is known (e.g. [12], p. 7), $|Q_n| \leq 2$, $n = 1, 2, \ldots$, therefore in the class $\mathcal{P}_a$ the estimates

$$(1.7) \quad |q_n| \leq \frac{2|a|}{|a+n|}, \quad n = 1, 2, \ldots,$$

are true, with that they are sharp.

Note that the function $P_1(z) = \frac{1}{1-z}$, $z \in \Delta$, belonging to the Carathéodory class $\mathcal{P}$ may be treated as the identity with respect to the Hadamard convolution. In view of (1.4), this means that, for each $a \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, $k_a \in \mathcal{P}_a$. Besides, for $a = 0$, $k_0(z) \equiv 1$, $z \in \Delta$, thus

$$\mathcal{P}_0 = \{p(z) \equiv 1, \quad z \in \Delta\}.$$

The definition of the classes $\mathcal{P}_a$ can be extended to the case $a = \infty$. Since each coefficient of expansion (1.3) of the function k_a tends to 1 as a tends to ∞, we shall adopt $k_\infty(z) = P_1(z)$, $z \in \Delta$. In consequence, we shall get

$$\mathcal{P}_\infty = \{p: p = k_\infty * P, \quad P \in \mathcal{P}\} = \mathcal{P}.$$

The respective properties of the class $\mathcal{P}$ and definition (1.4) imply directly the following propositions:

PROPOSITION 1.1. If $p \in \mathcal{P}_a$, then, for $t \in \mathbb{R}$, $p(e^{it}z) \in \mathcal{P}_a$.

PROPOSITION 1.2. If $p \in \mathcal{P}_a$, $r \in \langle 0, 1)$, then $p(rz) \in \mathcal{P}_a$.

PROPOSITION 1.3. If $p \in \mathcal{P}_a$, then $\overline{p(\bar{z})} \in \mathcal{P}_{\bar{a}}$.

2. STRUCTURE FORMULAE

From (1.3) we obtain, for each $a \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$,

$$\frac{1}{a} zk_a'(z) + k(z) = \frac{1}{1 - z}, \quad z \in \Delta.$$

Hence and in view of the definitions of the classes $\mathcal{P}_a$ we obtain

THEOREM 2.1. If $p \in \mathcal{P}_a$, $a \neq 0, -1, \ldots$, then there exists a function $P \in \mathcal{P}$ such that

$$\text{(2.1)} \quad \frac{1}{a} zp'(z) + p(z) = P(z), \quad z \in \Delta,$$

and conversely, for any function $P \in \mathcal{P}$, the solution of form (1.5) of equation (2.1) belongs to the class $\mathcal{P}_a$.

From Theorem 2.1 and (1.2) we immediately get:

COROLLARY 2.1. A function p of form (1.5) belongs to the class $\mathcal{P}_a$, $a \neq 0, -1, \ldots$, if and only if it satisfies the inequality

$$\text{(2.2)} \quad \mathrm{Re}\,\{\frac{1}{a} zp'(z) + p(z)\} > 0, \quad z \in \Delta.$$

REMARK 2.1. In paper [10] M. Orciuch considered the class $\mathcal{P}_{\frac{1}{b}}$ of functions of form (1.1), defined directly by condition (2.2), in the special case when $b = \bar{b} \geq 0$. Hence it appears that the basic results obtained here constitute a natural generalization of those from [10].

Let Ω stand for the Schwarz class of functions ω holomorphic in the disc Δ and such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \Delta$.

Formula (2.1) implies:

COROLLARY 2.2. A function $p \in \mathcal{P}_a$ if and only if there exists a function $\omega \in \Omega$ such that

$$\frac{1}{a}\, zp'(z) + p(z) = \frac{1+\omega(z)}{1-\omega(z)}, \qquad z \in \Delta.$$

In view of the definition of the class $\mathcal{P}_a$ and formula (2.1), we can easily prove:

THEOREM 2.2. Let $a \in \mathbb{C}$, $\operatorname{Re} a > 0$. If $P \in \mathcal{P}$, then the function p defined by the formula

$$(2.3) \qquad p(z) = a \int_0^1 t^{a-1} P(zt)dt, \qquad z \in \Delta,$$

belongs to the class $\mathcal{P}_a$. Conversely, if $p \in \mathcal{P}_a$, then there exists a function $P \in \mathcal{P}$ such that p is of form (2.3).

Making use of the Herglotz formula in the class $\mathcal{P}$ (e.g. [12], p. 9) and the Fubini theorem on the change of succession of integrating in a double Stieltjes integral, we obtain a structure formula in the class $\mathcal{P}_a$ for $a \in \mathbb{C}$, $\operatorname{Re} a > 0$.

THEOREM 2.3. Let $a \in \mathbb{C}$, $\operatorname{Re} a > 0$. A function $p \in \mathcal{P}_a$ if and only if

$$(2.4) \qquad p(z) = \int_0^{2\pi} a \left[\int_0^1 t^{a-1} \frac{e^{i\tau} + tz}{e^{i\tau} - tz} dt\right] d\mu(\tau)$$

where $\mu(\tau)$ is a real non-decreasing function normalized by the condition

$$\int_0^{2\pi} d\mu(\tau) = 1.$$

Formula (2.4) and a suitable theorem of Carathéodory imply, for example,

COROLLARY 2.3. Let $z \neq 0$ be a fixed point of the disc Δ. Then the set of values of the functional $J(p) = p(z)$, $p \in \mathcal{P}_a$, $\operatorname{Re} a > 0$, is the closed convex hull of a curve Γ with the parametric description

$$\gamma(\tau) = a \int_0^1 t^{a-1} \frac{e^{i\tau} + tz}{e^{i\tau} - tz} dt, \qquad \tau \in \langle 0, 2\pi).$$

In turn, taking account of the expansion of the function $P_\tau(z) = \dfrac{e^{i\tau} + tz}{e^{i\tau} - tz}$ in the disc Δ for $t \in (0, 1)$, after transforming formula (2.4) we shall obtain

COROLLARY 2.4. Let a function p of form (1.5) belong to the class $\mathcal{P}_a$ for $a \in \mathbb{C}$, $\operatorname{Re} a > 0$. Then its coefficients are defined by the formulae

$$p_n = \frac{2a}{a+n} \int_0^{2\pi} e^{-ni\tau}\, d\mu(\tau), \quad n = 1, 2, \ldots$$

In consequence, the set V_n of the system of coefficients $(p_1, \ldots, p_n)$, $p \in \mathcal{P}_a$, $\operatorname{Re} a > 0$, is the closed convex hull of the respective curve.

We also have (e.g. [12], p. 27):

PROPOSITION 2.1. Let k_a be the function defined by formula (1.3). If $p \in \mathcal{P}_a$, then

$$p(z) = P(z) * k_a(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho<1} P(\zeta)\cdot k_a(z\cdot\zeta^{-1})\zeta^{-1} d\zeta, \ |z| < \rho,$$

where $P \in \mathcal{P}$ and vice versa.

Of course, in this structure formula one may also use the Herglotz formula and, next, apply the result obtained to various problems.

3. THE PROPERTIES OF THE CLASSES $\mathcal{P}_a$

We shall give a few further - including topological - properties of the classes $\mathcal{P}_a$. They are consequences of the properties of the class $\mathcal{P}$ and those of the Hadamard convolution.

Since the class $\mathcal{P}$ is convex, compact and arcwise connected, and in the disc Δ condition (2.1) is satisfied, therefore we have (the justification as, for example, in [4]).

PROPOSITION 3.1. For any $a \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, the class $\mathcal{P}_a$ is convex, compact and arcwise connected.

Note that the function k_a for $a \neq 0$ has all the coefficients of expansion (1.3) different from zero, thus Hadamard convolution (1.4) is one-to-one with respect to the function P for $a \in \mathbb{C} \setminus \{0, -1, \ldots\}$. This and the fact that the extreme points of the class $\mathcal{P}$ have the well-known form ([3] and, for instance, [12], p. 3)

$$P_\eta(z) = (1 + \eta z)/(1 - \eta z), \quad |\eta| = 1, \quad z \in \Delta,$$

imply

PROPOSITION 3.2. All the extreme points of the class $\mathcal{P}_a$ are of the form

$$p_\eta = P_\eta * k_a$$

where P_η is an extreme point of the class $\mathcal{P}$, that is,

$$p_\eta(z) = 1 + \sum_{n=1}^{\infty} \frac{2a}{a + n} (\eta z)^n, \quad |\eta| = 1, \quad z \in \Delta.$$

The well-known theorem on support points of the class $\mathcal{P}$ ([2]) and the linearity of the Hadamard product (see [4]) imply

PROPOSITION 3.3. The set supp $\mathcal{P}_a$ of support points of the class $\mathcal{P}_a$ consists of functions of the form $p = k_a * P$ where

$$P(z) = \sum_{k=1}^{m} \lambda_k \frac{1 + \chi_k z}{1 - \chi_k z}, \quad z \in \Delta,$$

where $\lambda_k \geq 0$, $\sum_{k=1}^{m} \lambda_k = 1$ and $|\chi_k| = 1$ $(m = 1, 2, \ldots)$.

It is known that [13]: If $P_1(z) = 1 + \sum_{k=1}^{\infty} Q_k^{(1)} z^k$ and $P_2(z) = 1 + \sum_{k=1}^{\infty} Q_k^{(2)} z^k$, $z \in \Delta$, belong to the class $\mathcal{P}$, then $P(z) = 1 + \frac{1}{2} \sum_{k=1}^{\infty} Q_k^{(1)} Q_k^{(2)} z^k$, $z \in \Delta$, belongs to $\mathcal{P}$, too.

Hence we have

PROPOSITION 3.4. If p_1 of form (1.5) belongs to $\mathcal{P}_a$, and $P_1(z) = 1 + \sum_{k=1}^{\infty} Q_k^{(1)} z^k$, $z \in \Delta$, belongs to $\mathcal{P}$, then $p(z) = 1 + \frac{1}{2} \sum_{k=1}^{\infty} Q_k^{(1)} q_k z^k$, $z \in \Delta$, belongs to the class $\mathcal{P}_a$, too.

Indeed, since $p_1 \in \mathcal{P}_a$, there exists $P_2(z) = 1 + \sum_{k=1}^{\infty} Q_k^{(2)} z^k$, $z \in \Delta$, $P_2 \in \mathcal{P}$, such that $p_1 = k_a * P_2$. Hence

$$p_1 * P_1 = k_a * (P_1 * P_2),$$

thus

$$p_1 * (\frac{1}{2}P_1) = k_a * [\frac{1}{2}(P_1 * P_2)] = k_a * (P - \frac{1}{2}) \text{ where } P \in \mathcal{P},$$

therefore there exists $P \in \mathcal{P}$ such that

$$p(z) = \frac{1}{2} + [p_1 * (\frac{1}{2}P_1)](z) = (k_a * P)(z), \ z \in \Delta,$$

that is, $p \in \mathcal{P}_a$.

The class $\mathcal{P}_a$ is therefore invariant with respect to the convolution * considered in the Schur theorem on functions of the family $\mathcal{P}$.

4. ON SOME INCLUSIONS

As is known, $\mathcal{P}_\infty = \mathcal{P}$. Let $a \in \mathbb{C}$ and $\operatorname{Re} a \geq 0$. Let further $p \in \mathcal{P}_a$. Since in the disc Δ condition (2.2) is satisfied, therefore, in virtue of a suitable lemma ([9]), we obtain that $\operatorname{Re} p(z) > 0$ for all $z \in \Delta$. So, $p \in \mathcal{P}$. In consequence, the following proposition is true.

PROPOSITION 4.1. If $a \in \mathbb{C}$, $\operatorname{Re} a \geq 0$, then the inclusion

(4.1) $\mathcal{P}_a \subset \mathcal{P} = \mathcal{P}_\infty$

takes place.

REMARK 4.1. In paper [8], a general theorem of the type: $\operatorname{Re} \Psi(P(z), zP'(z)) > 0 \Rightarrow \operatorname{Re} P(z) > 0$, $z \in \Delta$, was obtained. This theorem implies, among others, inclusion (4.1) for $a \geq 0$.

Directly from (1.7) it follows that a necessary condition for a function p of the class $\mathcal{P}_a$ to belong to the class $\mathcal{P}$ is that $|a| \leq |a + n|$ for each $n = 1, 2, \ldots$ Then, after performing some simple calculations, we shall obtain that $\operatorname{Re} a \geq -\frac{1}{2}n$, $n = 1, 2, \ldots$. Consequently, for $a \in \mathbb{C}$ such that $\operatorname{Re} a < -\frac{1}{2}$ and $a \notin \{-1, -2, \ldots\}$, inclusion (4.1) does not hold.

An example of a function belonging to the class $\mathcal{P}_a \setminus \mathcal{P}$ for $\operatorname{Re} a < -\frac{1}{2}$ is the function

$$p_a(z) = 1 + \frac{a}{a+1} z, \quad z \in \Delta.$$

The question whether $\mathcal{P}_a \setminus \mathcal{P} \neq \emptyset$ for $a \in \mathbb{C}$, $-\frac{1}{2} \le \operatorname{Re} a < 0$, remains open.

We have (the simple proof [10] is omitted):

THEOREM 4.1. Let $a, b \in \mathbb{R}$, $0 \le a < b$. Then

(4.2) $\mathcal{P}_a \subset \mathcal{P}_b$.

REMARK 4.2. The problem concerning the investigations analogous as in Theorem 4.1, for the remaining admissible a's, seems to be interesting. It is open.

However, we have:

THEOREM 4.2. Let $a, b \in \mathbb{R}$ be admissible (that is, $a \neq -1, -2, \ldots$). If $a < b < 0$, then

$$\mathcal{P}_b \cap \mathcal{P} \subset \mathcal{P}_a \cap \mathcal{P}.$$

The proof is carried out by means of the "reductio ad absurdum" method. We make use of condition (2.2) and inequality (1.2). Since there exist $a < b < 0$ and a function $p \in \mathcal{P}_b \cap \mathcal{P}$ such that $p \notin \mathcal{P}_a$, therefore

$$\operatorname{Re}\{\tfrac{1}{b} z_o p'(z_o) + p(z_o)\} > 0,$$

$$\operatorname{Re}\{\tfrac{1}{a} z_o p'(z_o) + p(z_o)\} \le 0,$$

for some $z_o \in \Delta$. Consequently, $(b - a)\operatorname{Re} p(z_o) < 0$, which is not possible in view of our assumptions.

5. ON THE CLASSES $\mathcal{P}[a]$

Let P be any fixed function of the class $\mathcal{P}$. We also know from (4.2) that if $P \in \mathcal{P}_a$, $a \ge 0$, then $P \in \mathcal{P}_b$ for each $b \ge a$. So, denote (see, for instance, [6], [10])

$$a_P = \inf\{b \ge 0 : P \in \mathcal{P}_b\}$$

and put

$$\mathcal{P}[a] = \{P \in \mathcal{P} : a_P = a\}.$$

Note that the classes $\mathcal{P}[a]$ are non-empty for each $0 \le a \le +\infty$.

Indeed, let $a = +\infty$. The function $P_o(z) = \frac{1+z}{1-z}$ is a function of the Carathéodory class, thus $P_o \in \mathcal{P}_\infty$. Let $b > 0$. Then

$$\mathrm{Re}\,\{\frac{1}{b} zP_o'(z) + P_o(z)\} \longrightarrow -\frac{1}{2b} < 0 \quad \text{as} \quad z \longrightarrow -1.$$

So, there exists a point $z_o \in \Delta$ such that $\mathrm{Re}\,\{\frac{1}{b}z_oP_o'(z_o) + P_o(z_o)\} < 0$. Thus $P_o \notin \mathcal{P}_b$ for any $b > 0$. In consequence, in virtue of the definitions of the lower bound and the class $\mathcal{P}[a]$, we have that $P_o \in \mathcal{P}[+\infty]$. Analogously we can show that, for each $0 < a < +\infty$, the function $P_a(z) = 1 + \frac{a}{a+1}z$, $z \in \Delta$, belongs to the class $\mathcal{P}[a]$, and that $P_1 \equiv 1$ belongs to the class $\mathcal{P}[0]$.

The following theorem ([10]) is true.

THEOREM 5.1. Let $P \in \mathcal{P}$. Then $P \in \mathcal{P}[a]$, $0 < a < +\infty$, if and only if $P \in \mathcal{P}_b$ for any $b \geq a$ and $P \notin \mathcal{P}_b$ for any $b \in \langle 0, a)$. Besides, $P \in \mathcal{P}[0]$ if and only if $P \in \mathcal{P}_a$ for any $a \geq 0$. What is more, $P \in \mathcal{P}[+\infty]$ if and only if $P \in \mathcal{P}_\infty$ and $P \notin \mathcal{P}_a$ for any $a \in \langle 0, +\infty)$.

P r o o f. In view of the definitions of the bound a_P and the class $\mathcal{P}[a]$ and by Theorem 4.1, the above theorem is obvious when $a = 0$ or $a = +\infty$. Let $a \in (0, +\infty)$. Assume that $P \in \mathcal{P}_b$ for any $b \geq a$ and $P \notin \mathcal{P}_b$ for $b \in \langle 0, a)$. Then we shall get $a_P = a$, which means that $P \in \mathcal{P}[a]$.

To prove the converse, suppose that $P \in \mathcal{P}[a]$. Then, in virtue of the definitions of the lower bound and the class $\mathcal{P}[a]$, there must exist a sequence $\{b_n\}_{n\in N}$ of numbers converging to a, such that $P \in \mathcal{P}_{b_n}$, $n = 1, 2, \ldots$. Then, from (2.2) we have $\mathrm{Re}\,\{\frac{1}{b_n} zP'(z) + P(z)\} > 0$, $z \in \Delta$, $n = 1, 2, \ldots$. Passing with n to $+\infty$, we shall obtain in the limit: $\mathrm{Re}\,\{\frac{1}{a} zP'(z) + P(z)\} \geq 0$, $z \in \Delta$. Put $u(z) = \mathrm{Re}\,\{\frac{1}{a} zP'(z) + P(z)\}$, $z \in \Delta$. It is a harmonic function in Δ, and $u(0) = 1$, therefore, on the basis of the maximum principle for harmonic functions, we get $\mathrm{Re}\,\{\frac{1}{a} zP'(z) + P(z)\} > 0$, $z \in \Delta$. By this and (2.2), $P \in \mathcal{P}_a$. Consequently, from (4.2) we have $P \in \mathcal{P}_b$ for $b > a$ and, of course, $P \notin \mathcal{P}_b$ for $b < a$ because, otherwise, we would obtain a contradiction with the definition of $a = a_P$ as the lower bound, which ends the proof.

It is evident that the classes $\mathcal{P}[a]$ are disjoint and

$$\mathcal{P} = \bigcup_{0 \le a \le +\infty} \mathcal{P}[a].$$

REMARK 5.1. An open problem is the performance of analogous investigations for the remaining values of the parameter a. In particular, the determination of consequences of Theorem 4.2.

6. ON SOME RELATIONS BETWEEN THE CLASSES $\mathcal{P}_a$

As was mentioned earlier, Theorem 4.1 establishes detailed relationships between the classes $\mathcal{P}_a$ in the case $a \ge 0$. Similar relations for $a < 0$, $a \ne -1, -2, \dots$, are determined by Theorem 4.2. The case $a \ne \bar{a}$ is the most difficult. It turned out, however, that some other inclusions between the classes under consideration are true.

And so, condition (2.1) and the convexity of the class $\mathcal{P}$ imply:

THEOREM 6.1. Let $a, b \in \mathbb{C}\setminus\{0, -1, -2, \dots\}$, $\varphi = \arg a = \arg b \in (-\pi, \pi)$. Then

$$\mathcal{P}_a \cap \mathcal{P}_b \subset \mathcal{P}_{\frac{a+b}{2}}.$$

We also have

THEOREM 6.2. Let $a, b \in \mathbb{C}\setminus\{0, -1, -2, \dots\}$, $a \ne b$,

$$U(\lambda) = \frac{ab}{\lambda(a - b) + a}, \quad \lambda \in \langle 0, 1\rangle.$$

Then

$$\mathcal{P}_a \cap \mathcal{P}_b \subset \mathcal{P}_{U(\lambda)}$$

1° for each $\lambda \in \langle 0, 1\rangle$ if $\arg a = \arg b \in (-\pi, \pi)$;

2° for each $\lambda \in \langle 0, 1\rangle$, $\lambda \ne \lambda_n$, if $\arg a = \arg b = \pi$, where

$$\lambda_n = \frac{|a|(|b| - n)}{n(|b| - |a|)};$$

3° for each $\lambda \in \langle 0, 1\rangle$ if $\varphi = \arg a = \arg b + \pi$, $\varphi \in (-\pi, \pi)$, $\varphi \ne 0$;

4° for each $\lambda \in \langle 0, 1\rangle$, $\lambda \neq \lambda_n$, if $\arg a = 0$, $\arg b = \pi$;

5° for each $\lambda \in \langle 0, 1\rangle$ if the points a, b, 0 do not lie on one line.

The above inclusions are obtained after applying equation (2.1) and examining the image of the segment $\langle 0, 1\rangle$ under the mapping $U(\lambda)$.

Let us still notice that $U(\lambda) = \infty$ when $\lambda = \lambda_\infty = \frac{a}{a - b}$. Consequently, we have

COROLLARY 6.1. Let $\arg a = \arg b + \pi$ and $\arg a \neq 0$. Then $\lambda_\infty = \frac{|a|}{|a| + |b|} \in \langle 0, 1\rangle$, thus $\mathcal{P}_a \cap \mathcal{P}_b \subset \mathcal{P}_\infty = \mathcal{P}$.

In the special case when $a > 0$, $b < 0$, we have $\lambda_\infty \in \langle 0, 1\rangle$, therefore the evident inclusion $\mathcal{P}_a \cap \mathcal{P}_b \subset \mathcal{P}$ holds (see Theorem 4.1).

Let $a = \infty$, $b \in \mathbb{C}\setminus\{0, -1, \ldots\}$, $b \neq \infty$. Using again equation (2.1) and the convexity of the class $\mathcal{P} = \mathcal{P}_\infty$, we obtain

PROPOSITION 6.1. For $b \in \mathbb{C}\setminus\{0, -1, \ldots\}$, we have

$$\mathcal{P}_b \cap \mathcal{P} \subset \mathcal{P}_{\frac{b}{1-\lambda}}, \quad \lambda \in \langle 0, 1\rangle, \quad \lambda \neq \lambda_n = 1 + \frac{b}{n}.$$

REMARK 6.1. Since, only for $\operatorname{Re} a \geq 0$, the inclusion $\mathcal{P}_a \subset \mathcal{P}$ has been determined, it seems interesting to ask about the general properties of the classes

$$\mathcal{P}\{a\} = \mathcal{P}_a \cap \mathcal{P}, \quad \operatorname{Re} a < 0.$$

Theorem 4.2 and Corollary 6.1 concern the very question.

Consider another problem of a similar type. Let

$$(6.1) \qquad \mathcal{P}_{|r} = \{P_{|\Delta_r} : P \in \mathcal{P}\}, \quad \Delta_r = \{z: |z| < r\}.$$

We have:

THEOREM 6.3. Let $\mathcal{P}$ and $\mathcal{P}_a$, $a \in \mathbb{C}\setminus\{0, -1, -2, \ldots\}$ denote the classes of functions, defined earlier, whereas $\mathcal{P}_{|r}$ - the set of restrictions of functions, defined by rule (6.1). Then each function $P \in \mathcal{P}_{|r(a)}$, where

$$r(a) = \sqrt{1 + \frac{1}{|a|^2}} - \frac{1}{|a|} \leq 1,$$

satisfies in the disc $\Delta_{r(a)}$ inequality (2.2). Moreover, the disc

$\Delta_{r(a)}$ for $a \in \mathbb{R}$ cannot be enlarged. In other words, each function $P \in \mathcal{P}$ "belongs" to $\mathcal{P}_a$ in $\Delta_{r(a)}$.

P r o o f. (cf. [4]). Let $P \in \mathcal{P}$ and $H(P) = \frac{1}{a} zP'(z) + P(z)$, $0 \neq z \in \Delta$. Then from [11], (6.2) we have

$$\frac{|zP'(z)|}{\operatorname{Re} P(z)} \le \frac{2|z|}{1 - |z|^2},$$

consequently,

$$\operatorname{Re} H(P) \ge \operatorname{Re} P(z) - \frac{1}{|a|} |zP'(z)|$$

$$\ge \frac{\operatorname{Re} P(z)}{1 - |z|^2} \left[1 - \frac{2}{|a|} |z| - |z|^2\right].$$

Hence it appears that $\operatorname{Re} H(P) > 0$ in $\Delta_{r(a)}$, thus, really $P_{|\Delta_{r(a)}}$ satisfies condition (2.2). Since $P_1(z) = \frac{1 + \varepsilon z}{1 - \varepsilon z}$, $|\varepsilon| = 1$, therefore, for $a = \bar{a}$, the disc $\Delta_{r(a)}$ cannot be enlarged.

7. ON PASSAGES TO THE LIMIT

We shall still deal with some "limit" cases $\operatorname{Re} a \longrightarrow +\infty$ and $|a| \longrightarrow 0$.

Let p be a function of the class $\mathcal{P}_a$ when $\operatorname{Re} a > 0$. Then, as follows from Theorem 2, there exists a function P of the Carathéodory class, such that in the disc Δ we have

$$p(z) = P(z) - z \int_0^1 t^a P'(zt)dt.$$

Since

$$P'(\zeta) = 2 \int_0^{2\pi} \frac{e^{i\tau}}{(e^{i\tau} - \zeta)^2} d\mu(\tau),$$

therefore

$$|p(z) - P(z)| \le 2|z| \int_0^1 t^{\operatorname{Re} a} \frac{1}{(1 - t|z|)^2} dt, \quad |z| = r < 1.$$

Hence in the disc Δ_r we have

$$|p(z) - P(z)| \le \frac{1}{\operatorname{Re} a + 1} \frac{2r}{(1 - r)^2}.$$

This means that if $p \in \mathcal{P}_a$, Re $a > 0$ and the function P satisfies condition (2.3), then in each disc $|z| \le r < 1$ the difference $p(z) - P(z)$ is arbitrarily small when Re a is sufficiently large.

In turn, from representation (2.3) it follows that the function $P_o(z) = 1$, $z \in \Delta$, belongs to each class $\mathcal{P}_a$ for Re $a > 0$. Besides, for any function $p \in \mathcal{P}_a$, Re $a > 0$, we shall get

$$|p(z) - 1| \le |a| \int_0^1 t^{\mathrm{Re}\, a-1} |P(zt) - 1| dt, \quad z \in \Delta,$$

$P \in \mathcal{P}$. Since in the Carathéodory class the inequality

$$|P(zt) - 1| \le \frac{2|z|t}{1 - |z|t}, \quad z \in \Delta, \quad t \in (0, 1\rangle,$$

is satisfied, we obtain

$$|p(z) - 1| \le \frac{|a|}{\mathrm{Re}\, a + 1} \frac{2r}{1 - r}, \quad |z| \le r.$$

Consequently, for any $\varepsilon > 0$ and $r \in (0, 1)$, there exists a' such that if $0 < \mathrm{Re}\, a < |a| < a'$, $p \in \mathcal{P}_a$, then $|p(z) - 1| < \varepsilon$ in Δ_r.

8. CONCLUDING REMARKS

Let us first observe that function (1.3) is a special case of the hypergeometric series ([7]), p. 240)

$$(8.1) \quad G(a, b, c; z) = 1 + \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \ z \in \Delta.$$

Namely, form (1.3) and (8.1) we have

$$(8.2) \quad k_a(z) = G(1, a, 1 + a; z), \quad z \in \Delta.$$

Since, as we know, $k_a \in \mathcal{P}_a$, therefore inclusion (4.1) proves that special form (8.2) of hypergeometric series (8.1) is a Carathéodory function with positive real part if only Re $a \ge 0$. Similar properties of series (8.2) follow also from other theorems proved above.

Since $P_1 * P_2 = P_2 * P_1$, one can easily obtain various properties of a new two-parameter family $\mathcal{P}_{a,b}$, $a, b \in \mathbb{C} \setminus \{-1, -2, \dots\}$, of functions p defined by the formula

$$p = k_a * k_b * P$$

where $P \in \mathcal{P}$ (cf. [5]). Of course, $\mathcal{P}_{\infty,\infty} = \mathcal{P}, \mathcal{P}_{a,b} = \mathcal{P}_{b,a}, \mathcal{P}_{a,b} \subset \mathcal{P}$ if only Re a, Re b ≥ 0, $\mathcal{P}_{a,\infty} = \mathcal{P}_a$.

Proceeding analogously as in the case of Theorem 1 (cf. [5]), we obtain

THEOREM 8.1. If $p \in \mathcal{P}_{a,b}$, $a, b \in \mathbb{C} \setminus \{-1, -2, \dots\}$, then there exists a function $P \in \mathcal{P}$ such that

$$(8.3)\qquad \frac{1}{ab} z^2 p''(z) + \frac{a+b+1}{ab} zp'(z) + p(z) = P(z), \quad z \in \Delta,$$

and conversely, for any function $P \in \mathcal{P}$, solution (1.5) of equation (8.3) belongs to the class $\mathcal{P}_{a,b}$.

P r o o f. If $p \in \mathcal{P}_{a,b}$, then there exists $\tilde{p} \in \mathcal{P}_b$ such that $p = k_a * \tilde{p}$, $\tilde{p} = k_b * P$, $P \in \mathcal{P}$. Consequently, form (2.1) we have

$$\frac{1}{b} z\tilde{p}'(z) + \tilde{p}(z) = P(z), \quad z \in \Delta,$$

and

$$\frac{1}{a} zp'(z) + p(z) = \tilde{p}(z), \quad z \in \Delta.$$

Hence we get equation (8.3). Comparing the coefficients, one can verify that if p is of form (1.5) and satisfies equation (8.3) for some $P \in \mathcal{P}$, then

$$q_n = \frac{a}{a+n} \frac{b}{b+n} Q_n, \quad n = 1, 2, \dots,$$

thus $p \in \mathcal{P}_{a,b}$, which concludes the proof.

In particular, from Theorem 8.1, proceeding as in Section 2, one can obtain many properties of the classes $\mathcal{P}_{a,b}$, among others, an analogue of condition (2.2), and the like.

In Proceedings [5], two more general problems were formulated:

1^o Determine the set of all $(a, b, c) \in \mathbb{C}^3$, $c \neq 0, -1, \dots$, such that hypergeometric series (15) be a Carathéodory function.

2^o For any admissible points $(a, b, c) \in \mathbb{C}^3$, examine the extremal properties of the class $\mathcal{P}(\frac{ab}{c})$ of functions $p = P * G$ where the functions P belong to the class $\mathcal{P}$, while G is series (8.1).

In general, both these problems are open. The results of our paper give only partial solutions. In paper [15], the author investigates a somewhat different question, namely, the problem of the univalence of series (8.1). Applications of the properties of generalized hypergeometric series can be found, for instance, in [14].

To finish with, let us only make mention of other possible applications. One can examine, for example, new classes of functions, generated by functions of the class $\mathcal{P}_a$. In particular, it seems purposeful to investigate the families R_a, S^*_a, S^c_a of functions of the form

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots, \quad z \in \Delta,$$

satisfying the conditions

$$f' \in \mathcal{P}_a, \quad zf'(z)/f(z) \in \mathcal{P}_a, \quad 1 + zf''(z)/f'(z) \in \mathcal{P}_a,$$

respectively. This is not, however, the object of the considerations of this paper.

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Chair of Special Functions
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O PEWNEJ KLASIE FUNKCJI CARATHÉODORY´EGO

Niech $\mathcal{P}$ oznacza znaną klasę funkcji

$$P(z) = 1 + Q_1 z + \dots + Q_n z^n + \dots$$

holomorficznych w kole $\Delta = \{z\colon |z| < 1\}$ i spełniających w tym kole warunek $\operatorname{Re} P(z) > 0$. Niech

$$k_a(z) = 1 + \frac{a}{a+1} z + \dots + \frac{a}{a+n} z^n + \dots, \quad z \in \Delta, \quad a \in \mathbb{C} \setminus \{-1, -2, \dots\}.$$

W pracy badane są własności klasy $\mathcal{P}_a$ funkcji $p = P * k_a$, $P \in \mathcal{P}$, gdzie $P * k_a$ oznacza splot Hadamarda funkcji P oraz k_a. Oczywiście $\mathcal{P}_\infty = \mathcal{P}$. Ponado też kilka zastosowań i sformułowano zadania do rozwiązania. Idea pracy powstała w związku z badaniami dotyczącymi znanej klasy T_α ([5], [6]) oraz realizacją pracy dyplomowej [10].

Pewne ogólne zagadnienia dotyczące zastosowań splotu Hadamarda można znaleźć np. w [4].