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Zbigniew Jerzy Jakubowski, Józef Kamiński

ON SOME CLASSES OF MOCANU-BAZILEVIČ FUNCTIONS

In this paper we investigate some classes of functions generated by different types of relations with the homography $z \to (1 + Az)/(1 + Bz)$, $z \in \Delta = \{z: |z| < 1\}$ where parameters A, B may take complex values. The main results concern certain families of α-convex Mocanu-Bazilevič functions (M o c a n u (1969) [78], B a z i l e v i č (1955) [14]). The results obtained are a continuation of the considerations contained in [40], [41] [55] and [42]. The basic investigations are preceded by a survey of various classes of Carathéodory functions with positive real part.

1. ON VARIOUS CLASSES OF CARATHÉODORY FUNCTIONS

Let $\mathcal{P}$ denote the well-known class of functions

(1.1) $p(z) = 1 + p_1 z + \ldots + p_n z^n + \ldots$

holomorphic and satisfying the condition $\text{Re}\, p(z) > 0$ in the disc $\Delta = \{z: |z| < 1\}$, ([17]). As is known, many classes of functions

(1.2) $f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots, \quad z \in \Delta,$

are generated by functions belonging to $\mathcal{P}$. Here belong, among others, the known classes S^*, S^c, T, C, U, R, K of functions (of the form (1.2)) starlike, convex, typically-real, convex in the direction of the imaginary axis, starlike in the direction of the real axis, possessing a derivative with a positive real part in Δ, close-to-convex (see e.g. [32]). In particular, functions $f \in R$ satisfy the condition $f'(z) = p(z)$, $z \in \Delta$, $p \in \mathcal{P}$. There also hold corresponding relations for the remaining classes of functions.

In the investigations, the so-called classes of functions of order α, $\alpha \in \langle 0, 1)$, appeared comparatively early ([91]; cf. [107]). So, let $\mathcal{P}_\alpha$ denote the family of functions of the form (1.1) satisfying in Δ the condition $\operatorname{Re} p(z) > \alpha$, S_α^* - the class of functions of the form (1.2) starlike of order α ($zf'(z)/f(z) = p(z)$, $z \in \Delta$, $p \in \mathcal{P}_\alpha$). Similarly, the classes S_α^c and other ones are introduced. (In the sequel, we shall apply an analogous system of symbols. So, e.g. $S^*(A, B)$ stands for the class of functions f generated in the same way as starlike functions, but by functions p belonging to a fixed family $\mathcal{P}(A, B)$).

In 1968 R. M. Goel ([28]) investigated the class $P_{[M]}$, $M > \frac{1}{2}$, of functions (1.1) satisfying the condition

(1.3) $|p(z) - M| < M, \quad z \in \Delta,$

and certain two classes generated by $\mathcal{P}_{[M]}$ (cf. also [27]). These investigations were later extended by W. Janowski ([44], [45] - 1969; [46], [47] - 1970) and other authors (e.g. [38], [43], [50], [51], [76], [79], [80], [85], [86], [87], [88], [92], [102], [105], [110], [112]). The idea to replace the half-plane $\operatorname{Re} w > \alpha$ by a disc appeared earlier in the papers by MacGregor ([66] - 1962; [67] - 1963; [68] - 1964) who considered certain implications of condition (1.3) with $M = 1$. This particular case can also be found in later publications (e.g. [57], [93], [96]).

In 1971 ([34]) the author introduced the class $\mathcal{P}_{m,M}$ of functions of the form (1.1) which satisfy the conditions

(1.4) $|p(z) - m| < M, \quad z \in \Delta,$

where the real numbers m, M satisfy the inequality

(1.5) $|1 - m| < M \le m.$

He also introduced some classes of functions of the form (1.2) generated by the family $\mathcal{P}_{m,M}$ ([35]).

Let Ω denote the family of functions

(1.6) $q(z) = q_1 z + \ldots + q_n z^n + \ldots$

holomorphic and satisfying the condition $|q(z)| < 1$ in Δ.

In the investigations carried out among other things, the following fact was applied: $p \in \mathcal{P}_{m,M}$ if and only if

(1.7) $p(z) = (1 + Aq(z))/(1 + Bq(z)), \quad q \in \Omega, \quad z \in \Delta,$

(1.8) $A = (M^2 - m^2 + m)M^{-1}, \ B = (1 - m)M^{-1}, \quad (A, B) \in E_1,$

where

(1.9) $E_1 = \{(A, B): -1 < A \le 1, \quad -1 < B < A\}.$

In papers [34]-[38], certain properties of the family $\mathcal{P}_{m,M}$ were obtained as well as of the classes generated by this family (e.g. $R_{m,M}$, $T_{m,M}$).

In 1973 W. Janowski ([48], [49]) defined the class $\mathcal{P}(A, B)$, $(A, B) \in D$ where

(1.10) $D = \{(A, B): -1 < A \le 1, \quad -1 \le B < A\},$

directly by condition (1.7). It is obvious that the condition $(A, B) \in E_1$ has been extended because the points $(A, B) \in E_2$, where

(1.11) $E_2 = \{(A, B): -1 < A \le 1, \quad B = -1\},$

were added. Therefore the class $\mathcal{P}(A, B)$, $(A, B) \in D$, also comprises Carathéodory functions of order $\alpha = (1 - A)/2$. Moreover, notice that, for example,

$\mathcal{P}(1, B) = \mathcal{P}_{[1/(1+B)]}$, ([28]),

$\mathcal{P}(A, 0) = \mathcal{P}_{(A)} =: \{p \in \mathcal{P}: |p(z) - 1| < A\}$, $A > 0$, ([96]),

$\mathcal{P}(A, -A) = \mathcal{P}^{(A)} =: \{p \in \mathcal{P}: (|p(z) - 1|/|p(z) + 1|) < A\}$, $A > 0$, ([16]).

Obviously, $\mathcal{P}(1, -1) = \mathcal{P}$. The basic results of W. Janowski concern the properties of the classes $\mathcal{P}(A, B)$ and $S^*(A, B)$. Many other problems connected with the classes $\mathcal{P}_{m,M}$ or $\mathcal{P}(A, B)$ and the families generated by them can be found in papers [1], [2], [3], [5], [6], [7], [8], [10], [15], [25], [29], [39], [41], [52], [53], [54], [59], [60], [82], [83], [94], [108], [113].

In papers [34] (1971) and [38] (1973), an attempt was made to replace in (1.4) the number m by the complex number c, replacing simultaneously condition (1.5) by the inequality

(1.12) $|1 - c| < M \le \operatorname{Re} c.$

Functions of the family $\mathcal{P}_{c,M}$ thus defined have the form (1.7), where instead of (1.8) we have

(1.13) $A = (M^2 - |c|^2 + c)M^{-1}, \quad B = (1 - \bar{c})M^{-1}$.

It follows easily from (1.12) and (1.13) that $|A| \le 1$, $|B| < 1$. Among other things, exact estimations of the absolute values of coefficients in this class were obtained. This result has simultaneously been obtained by R. J. Libera and A. E. Livingston in [63].

In the next years, the investigations of the general case, i.e. the case when A and B are complex numbers, were significantly extended. So, A. Szynal, J. Szynal and J. Zygmunt obtained in [106] generalizations of some results known earlier (e.g. [34], [35], [44], [63], [107]) concerning estimations of coefficients. In [101] J. Stankiewicz and J. Waniurski investigated, among other things, the class $\mathcal{P}_n(A, B)$, $|A| \le 1$, $|B| \le 1$, of functions $p(z) = 1 + p_n z^n + \dots$, $z \in \Delta$, $n = 1, 2, \dots$ In both those cases, the functions considered did not have to belong to $\mathcal{P}$. Various classes of functions with complex parameters A, B were also investigated, e.g. in [42], [56], [71], [111].

Notice also that the investigation of the class of functions of the form (1.7) with complex parameters A, B can always be reduced to the case when either A or B are real (e.g. $(A, B) \in \mathbb{C} \times \mathbb{R}$, $|A| \le 1$, $0 < B \le 1$).

It is worth noticing that in all classes of functions of the form (1.1) considered earlier, the values $p(z)$, $z \in \Delta$, always belonged to some convex set (halfplane, disc). Some general propertied of such functions can be found, for instance, in [30]. On the other hand, in all the cases, the definition pattern based on (1.7) was obligatory. Therefore, in these definitions, the term of subordination and its properties can be applied (e.g. as in [101]).

2. ON THE CLASS $M_\alpha(A, B)$

In 1969 P. T. Mocanu ([78]) introduced the class M_α, $0 \le \alpha \le 1$, of functions of the form (1.2) satisfying in the disc Δ the conditions:

(2.1) $f(z)f'(z)z^{-1} \neq 0$,

(2.2) $\operatorname{Re} J(f, z, \alpha) > 0$,

where

(2.3) $$J(f, z, \alpha) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

The class M_α proved to be interesting to many mathematicians ([22], [26], [58], [69], [72], [73] - [75], [77], [103], [104]). It was proved in [74], among other things, that functions belonging to M_α are, for $\alpha > 0$, elements of the known class of Bazilevič functions, [14].

The consecutive investigations developed in several directions. In particular, in papers [4], [9], [18], [20], [21], [23], [40] - [42], [55], [61], [62], [76], [81], [84], [89], [90], [94], [95], [97] - [100], [109], condition (2.2) underwent various modifications.

Let $(A, B) \in \mathbb{C}^2$ be a couple of complex numbers satisfying the conditions

(2.4) $$\begin{cases} A \neq B \\ |B| \leq 1, \\ |A - B| \leq 1 - \operatorname{Re}(A\bar{B}), \\ \text{if } |B| = 1, \quad A = -|A|B, \quad \text{then } |A| \leq 1. \end{cases}$$

Denote by $M_\alpha(A, B)$ the class of functions (1.2) satisfying condition (2.1) as well as the condition

(2.5) $$J(f, z, \alpha) = \frac{1 + Aq(z)}{1 + Bq(z)}, \quad z \in \Delta,$$

for some $q \in \Omega$, where $\alpha \geq 0$, and (A, B) satisfies assumptions (2.4).

It follows from (2.4) and (2.5) that $\operatorname{Re} J(f, z, \alpha) > 0$ in Δ. Obviously, $M_\alpha(1, -1) = M_\alpha$, $M_o(1, -1) = S^*$, $M_1(1, -1) = S^c$, $M_o(A, B) = S^*(A, B)$.

In paper [42], some properties of the families $M_\alpha(A, B)$ were demonstrated. It was also shown that the function f defined by the formulae

$$f(z) = \begin{cases} z(1+Bz)^{\frac{A-B}{B}}, & z \in \Delta, \quad \text{if} \quad B \neq 0, \\ ze^{Az}, & z \in \Delta, \quad \text{if} \quad B = 0 \end{cases}$$

belongs to $M_o(A, B)$, and that it does not belong to $M_\alpha(A, B)$ for no values of $\alpha > 0$. Next, the functions f_k, $k = 1, 2$ defined by the formula

$$J(f_k, z, \alpha) = \frac{1 + Az^k}{1 + Bz^k}, \quad z \in \Delta, \quad (f(0) = f'(0) - 1 = 0),$$

belong to $M_\alpha(A, B)$ for any $\alpha \geq 0$. Moreover, they turned out to be functions realizing the extrema of certain functionals.

The next sections of the present paper are a natural continuation of [42]. The results obtained are generalizations of the respective results obtained earlier by different authors.

In the sequel, unless otherwise stated, we assume that, for the couple (A, B), conditions (2.4) hold.

3. ON SOME RELATION BETWEEN HARDY CLASSES AND THE CLASS $M_\alpha(A, B)$

As is well known, a function f holomorphic in Δ belongs to the Hardy class H^λ $(o < \lambda < +\infty)$ if

$$\lim_{r \to 1^-} \int_o^{2\Pi} |f(re^{it})|^\lambda dt < +\infty.$$

Denote also by H^∞ the class of functions bounded and holomorphic in Δ (e.g. [24], p. 2).

It was shown in [55], among other things, that if $f \in M_\alpha(A, B)$, $(A, B) \in D$ where D is the set of couples of real numbers, defined by (1.10), then f and f' belong to certain Hardy classes H^λ where $\lambda = \lambda(\alpha, A, B)$. Also, for $f \in M_\alpha(A, B)$, $\alpha > 0$, $(A, B) \in D$, the Hardy classes to which f'' belongs were determined. We shall now show that the class $M_\alpha(A, B)$, $(A, B) \in D$, possesses the following property corresponding to the known theorem for $M_\alpha(1, -1)$, ([70]).

THEOREM 3.1. (i) There exists an $f \in M_0(A,B)$, $(A,B) \in D$ such that $f'' \notin H^\lambda$ for any value of $\lambda > 0$.

(ii) There exists an $f \in M_\alpha(A, B)$, $(A, B) \in D$, such that, for no value of $\lambda > 0$, $f^{(n)} \notin H^\lambda$, $n = 3, 4, 5, \ldots$

P r o o f. (i) In [65], a function g was constructed which is holomorphic in Δ, continuous and univalent in the closed disc $\overline{\Delta}$, $g(0) = 0$ and such that

$$(3.1) \quad \overline{\lim_{r\to 1^-}} |g'(re^{it})| = +\infty$$

for almost all $t \in \langle 0, 2\Pi)$. It is clear that there exists a number $b > 0$ such that $|g(z)| < b$ for $z \in \Delta$. Take the functions

$$(3.2) \quad g_1(z) = \frac{g(z)}{b}, \quad z \in \Delta,$$

and

$$(3.3) \quad \begin{cases} P_1(z) = \dfrac{1 + Ag_1(z)}{1 + Bg_1(z)} & \text{if } (A, B) \in E_1, \\ P_2(z) = 1 + \dfrac{1 + A}{2} g_1(z) & \text{if } (A, B) \in E_2 \end{cases}$$

(cf. (1.9) and (1.11)).

As $g_1(0) = 0$, $|g_1(z)| < 1$, $z \in \Delta$, therefore $P_k \in \mathcal{P}(A, B)$ where $(A, B) \in E_k$, $k = 1, 2$. It follows from (3.1)-(3.3) that

$$(3.4) \quad \overline{\lim_{r\to 1^-}} |P_k'(re^{it})| = +\infty \quad \text{a.e. on } \langle 0, 2\Pi).$$

Next, consider the functions f_k defined by

$$(3.5) \quad \frac{zf_k'(z)}{f_k(z)} = P_k(z), \quad (f_k(0) = 0), \quad z \in \Delta, \quad k = 1, 2.$$

Obviously, $f_k \in M_0(A, B)$, $(A, B) \in E_k$, $k = 1, 2$. We shall prove that $f_k'' \notin H^\lambda$ for any $\lambda > 0$.

Assume that there exists $\lambda > 0$ such that $f_k'' \in H^\lambda$. Each function $f \in H^\lambda$, $\lambda > 0$, has a radial limit in almost any direction. Moreover, as $f_k \not\equiv 0$, therefore also $f_k(e^{it}) \not\equiv 0$ on any set of positive measure (e.g. [24], p. 17). Thus it follows from the fact that f and f' belong to certain Hardy classes (cf. [55]) and from (3.5) that $\lim_{r \to 1^-} P'(re^{it})$ exists almost everywhere. This contradicts (3.4).

(ii) Consider the functions f_k defined by the condition

$$J(f_k, z, \alpha) = P_k(z), \quad (f_k(0) = f_k'(0) - 1 = 0), \quad z \in \Delta,$$
$$k = 1, 2,$$

where P_k are defined by (3.3). Obviously, $f_k \in M_\alpha(A, B)$, $(A, B) \in E_k$, $k = 1, 2$. Taking into account the appropriate results of [55], in the same way as in (i), we can prove that $f_k'' \notin H^\lambda$ for any $\lambda > 0$. Hence and from the fact that if $f' \in H^\lambda$, $0 < \lambda < 1$, then $f \in H^{\lambda/(1-\lambda)}$ (e.g. [24], p. 88), we obtain (ii).

4. ON p-VALENT FUNCTIONS

As is well known, a function f is said to be p-valent in the disc Δ if it is holomorphic in Δ and the equation $f(z) = w_o$ possesses p roots in Δ for some w_o and, for any complex number w, the number of solutions of the equation $f(z) = w$ in Δ does not exceed p (e.g. [32], vol. I, p. 89).

Let A, B, α and $J(f, z, \alpha)$ be defined in the same way as in the definition of the family $M_\alpha(A, B)$ (see (2.3), (2.4)), and $p \geq 1$ an arbitrarily fixed positive integer. Denote by $M_\alpha^p(A, B)$ the family of functions f_p

$$(4.1) \qquad f_p(z) = z^p + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \ldots$$

holomorphic in Δ and such that $f_p(z)f_p'(z)z^{1-2p} \neq 0$,

$$(4.2) \qquad \frac{1}{p} J(f_p, z, \alpha) = \frac{1 + Aq(z)}{1 + Bq(z)}$$

for $z \in \Delta$ and for some $q \in \Omega$.

Obviously, $M_\alpha^1(A, B) = M_\alpha(A, B)$. The class $M_\alpha^p(A, B)$ where $(A, B) \in D$ was investigated in [41]. The class $M_\alpha^p(1, -1)$ was introduced in [40]. The families $M_0^p(1, -1)$ and $M_1^p(1, -1)$ are well-known subclasses of p-valent starlike and p-valent convex functions, respectively, investigated in [31].

Notice that, for all (A, B) satisfying (2.4), we have $M_\alpha^p(A, B) \subset M_\alpha^p(1, -1)$. Therefore functions belonging to $M_\alpha^p(A, B)$ are, in particular, p-valent starlike [40]. The structure of the introduced class $M_\alpha^p(A, B)$ is described by the following

THEOREM 4.1. If $[M_{\alpha/p}(A, B)]^p$ denotes the set of the p-th powers of functions of the class $M_{\alpha/p}(A, B)$, then

$$M_\alpha^p(A, B) = [M_{\alpha/p}(A, B)]^p. \tag{4.3}$$

P r o o f. For any function $f_p \in M_\alpha^p(A, B)$, consider the function

$$f(z) = \sqrt[p]{f_p(z)} = z \sqrt[p]{\frac{f_p(z)}{z^p}}, \qquad \sqrt[p]{1} = 1, \ z \in \Delta.$$

Then condition (4.3) follows from the following identity in Δ:

$$\frac{1}{p} J(f_p, z, \alpha) = J(f, z, \frac{\alpha}{p}).$$

By Theorem 4.1, it is obvious that $M_0^p(A, B)$ consists of the p-th powers of starlike univalent functions belonging to $M_0(A, B) = S^*(A, B)$. Simultaneously, $M_1^p(A, B)$, $p > 1$, is not identical with the set of the p-th powers of convex univalent functions in $M_1(A, B) = S^c(A, B)$. It seems to be interesting that a p-valent convex function of $M_1^p(A, B)$ is the p-th power of some α-convex function belonging to the class $M_\alpha(A, B)$ for $\alpha = 1/p$.

THEOREM 4.1 also enables one to obtain the properties of the class $M_\alpha^p(A, B)$ corresponding to certain properties of univalent functions of $M_\alpha(A, B)$. The following Lemmas are well-known.

LEMMA 4.1 [42]. If $0 \le \beta \le \alpha$, then $M_\alpha(A, B) \subset M_\beta(A, B)$.

LEMMA 4.2 [42]. If the function f of the form (1.2) belongs to the class $M_\alpha(A, B)$, then

$$|a_2| \le \frac{|A - B|}{1 + \alpha}, \tag{4.4}$$

(4.5) $$|a_3 - \lambda a_2^2| \le \frac{|A - B|}{2(1 + 2\alpha)} \max(1, s), \quad \lambda \in \mathbb{C},$$

where

$$s = (1 + \alpha)^{-2}|2\lambda(1 + 2\alpha)(A - B) + B\alpha^2 + (5B - 3A)\alpha + 2B - A|.$$

The estimations in (4.4) and (4.5) in the class $M_\alpha(A, B)$ are exact.

Using Lemmas 4.1, 4.2 as well as conditions (4.3), we can prove, for instance,

THEOREM 4.2. If $0 \le \beta \le \alpha$, then $M_\alpha^p(A, B) \subset M_\beta^p(A, B)$.

THEOREM 4.3. If f_p of the form (4.1) belongs to $M_\alpha^p(A, B)$, then, for any $\lambda \in \mathbb{C}$,

(4.6) $$|b_{p+2} - \lambda b_{p+1}^2| \le p^2 \frac{|A - B|}{2(p + 2\alpha)} \max(1, u)$$

where

$$u = (p + \alpha)^{-2}|2p^2\lambda(p + 2\alpha)(A - B) - p^3(A - B) + p^2(B - 2\alpha A + 2\alpha B) + p\alpha(3B - A) + B\alpha^2|.$$

Estimation (4.6) in $M_\alpha^p(A, B)$ is exact.

5. ON k-SYMMETRIC FUNCTIONS

Let $M_\alpha(A, B, k)$ (k - a fixed positive integer) denote the class of functions f_k of the form

(5.1) $$f_k(z) = z + b_{k+1}z^{k+1} + b_{2k+1}z^{2k+1} + \dots, \quad z \in \Delta,$$

belonging to $M_\alpha(A, B)$.

Notice that $M_\alpha(1, -1, k)$ is the class of α-convex k-fold symmetric functions, introduced in [19]. The definitions of the classes $M_\alpha(A, B)$ and $M_\alpha(A, B, k)$ allow one easily to get the following relation between theses classes:

THEOREM 5.1. A function f_k of the form (5.1) belongs to $M_\alpha(A, B, k)$ if and only if the function f of the form

$$f(z) = [f_k(z^{1/k})]^k = z + a_2 z^2 + a_3 z^3 + \dots, \qquad z \in \Delta,$$

belongs to the family $M_\beta(A, B)$ where $\beta = \alpha k$.

The Theorem stated above permits one to formulate certain properties of the class $M_\alpha(A, B, k)$ corresponding to the well-known properties of the class $M_\alpha(A, B)$. In particular, Theorem 5.1 and Lemma 4.2 imply

THEOREM 5.2. If a function f_k of the form (5.1) belongs to the class $M_\alpha(A, B, k)$, then, for any $\lambda \in \mathbb{C}$,

$$(5.2) \qquad |b_{2k+1} - \lambda b_{k+1}^2| \le \frac{|A - B|}{2k(1 + 2\alpha k)} \max(1, v)$$

where

$$v = (1 + \alpha k)^{-2} |B\alpha^2 k^2 + (3B - A)\alpha k + 2(2\lambda - 1)\alpha(A - B) + B + \frac{2\lambda - 1}{k}(A - B)|.$$

For each $\lambda \in \mathbb{C}$, estimation (5.2) is exact.

The following corollary is a direct implication of Theorems 5.1 and 5.2 as well as of Lemma 4.2:

COROLLARY 5.1. If a function f_k of the form (5.1) belongs to $M_\alpha(A, B, k)$, then

$$|b_{k+1}| \le \frac{|A - B|}{k(1 + \alpha k)},$$

$$|b_{2k+1}| \le \frac{|A - B|}{2k(1 + 2\alpha k)} \max(1, v)$$

where

$$v = (1 + \alpha k)^{-2} |\tfrac{1}{k}(B - A) + B\alpha^2 k^2 + \alpha k(3B - A) + 2\alpha(B - A) + B|.$$

6. ON CERTAIN CLASSES OF FUNCTIONS OF TWO COMPLEX VARIABLES

As before, let $\Delta \subset \mathbb{C}$ be the unit disc, $(A, B) \in \mathbb{C}^2$ - a couple satisfying assumptions (2.4), $\alpha \ge 0$. Denote by $U \subset \mathbb{C}^2$ a fixed bounded complete Reinhardt domain with its centre at the origin (e.g. [13]). We shall also apply the following notation:

H^U - the family of holomorphic functions $f\colon U \to \mathbb{C}, f(0, 0) = 1$,

Ω^U - the family of holomorphic functions $w\colon U \to \mathbb{C}, w(0, 0) = 0$,

$|w(z_1, z_2)| < 1$ for $(z_1, z_2) \in U$,

L - the differential-functional operator defined on H^u with values

$$Lf(z_1, z_2) = f(z_1, z_2) + z_1 f_1'(z_1, z_2) + z_2 f_2'(z_1, z_2).$$

We shall consider the family $M_\alpha^U(A, B)$ of functions $f \in H^u$ for which

$$(6.1) \quad f(z_1, z_2)Lf(z_1, z_2) \neq 0,$$

$$(6.2) \quad (1 - \alpha) \frac{Lf(z_1, z_2)}{f(z_1, z_2)} + \alpha \frac{L^2 f(z_1, z_2)}{Lf(z_1, z_2)} = \frac{1 + Aw(z_1, z_2)}{1 + Bw(z_1, z_2)},$$

where $L^2 f = L(Lf)$, $w \in \Omega^U$, $(z_1, z_2) \in U$.

Obviously, if $A = 1$, $B = -1$, then condition (6.2) is equivalent to the inequality

$$\mathrm{Re}\left[(1 - \alpha) \frac{Lf(z_1, z_2)}{f(z_1, z_2)} + \alpha \frac{L^2 f(z_1, z_2)}{Lf(z_1, z_2)}\right] > 0, \quad (z_1, z_2) \in U.$$

Therefore $M_\alpha^U(1, -1)$ is the family considered by P. L i c z b e r s k i [64] (see also [33]). The class $M_\alpha^U(A, B)$ is also a generalization of the known classes $M_0^U(1, -1)$, $M_1^U(1, -1)$ introduced by I. B a v r i n in [12].

Denote by Z_1 the intersection $U \cap \{z_2 = 0\}$ and by Z_2 - the projection of the intersection $U \cap \{z_1 = kz_2\}$, $k \in \mathbb{C}$, onto the plane $z_1 = 0$. Let F_1 and F_2 be functions of one variable with values

$$F_1(z_1) = z_1 f(z_1, 0), \quad z_1 \in Z_1,$$

$$F_2(z_2) = z_2 f(kz_2, z_2), \quad z_2 \in Z_2.$$

Applying the method used in [12], we can give the following interpretation of functions belonging to $M_\alpha^U(A, B)$.

THEOREM 6.1. A function $f \in H^U$ belongs to $M_\alpha^U(A, B)$ if and only if

1° F_1 is holomorphic in Z_1, $F_1(0) = F_1'(0) - 1 = 0$, $z_1^{-1}F_1'(z_1)F_1(z_1) \neq 0$,

$$J(F_1, z_1, \alpha) = \frac{1 + Aw(z_1, 0)}{1 + Bw(z_1, 0)}, \text{ where } z_1 \in Z_1, \quad w \in \Omega^U.$$

2° For any fixed $k \in \mathbf{C}$, the function F_2 is holomorphic in Z_2, $F_2(0) = F_2'(0) - 1 = 0$, $z_2^{-1}F_2(z_2)F_2'(z_2) \neq 0$,

$$J(F_2, z_2, \alpha) = \frac{1 + Aw(kz_2, z_2)}{1 + Bw(kz_2, z_2)}, \quad z_2 \in Z_2, \quad w \in \Omega^U$$

(here $J(F_k, z_k, \alpha)$, $k = 1, 2$, are defined as in (2.3)).

Next, we shall present certain properties of $M_\alpha^U(A, B)$ connected with some theorems concerning the class $M_\alpha(A, B)$ of functions of one variable.

THEOREM 6.2. If $0 \leq \beta \leq \alpha$, then $M_\alpha^U(A, B) \subset M_\beta^U(A, B)$.

P r o o f. Take a function $f \in M_\alpha^U(A, B)$ for $|B| < 1$ and an arbitrarily fixed point $(\overset{o}{z}_1, \overset{o}{z}_2) \in U$. It follows from the properties of U that, for any $\zeta \in \Delta$, also the point $(\zeta\overset{o}{z}_1, \zeta\overset{o}{z}_2) \in U$. We construct a function g of the variable ζ: $g(\zeta) = \zeta f(\zeta\overset{o}{z}_1, \zeta\overset{o}{z}_2)$, $\zeta \in \Delta$. Then the following identity holds in the disc Δ:

$$(1 - \alpha)\frac{Lf(\zeta\overset{o}{z}_1, \zeta\overset{o}{z}_2)}{f(\zeta\overset{o}{z}_1, \zeta\overset{o}{z}_2)} + \alpha\frac{L^2f(\zeta\overset{o}{z}_1, \zeta\overset{o}{z}_2)}{Lf(\zeta\overset{o}{z}_1, \zeta\overset{o}{z}_2)} = J(g, \zeta, \alpha).$$

Hence and from the definition of the class $M_\alpha^U(A, B)$ it follows that, for $|B| < 1$, $|J(g, \zeta, \alpha) - s| < \rho$ where

$$s = \frac{1 - A\bar{B}}{1 - |\bar{B}|}, \quad \rho = \frac{|A - B|}{1 - |B|^2},$$

(see [42]). This means that $g \in M_\alpha(A, B)$, $|B| < 1$ (cf. [42]). Therefore, according to Lemma 4.1, also $g \in M_\beta(A, B)$, $|B| < 1$, $0 \leq \beta \leq \alpha$. Hence

$$\left|(1-\beta)\frac{Lf(\zeta\overset{o}{z}_1, \zeta\overset{o}{z}_2)}{f(\zeta\overset{o}{z}_1, \zeta\overset{o}{z}_2)} + \beta\frac{L^2f(\zeta\overset{o}{z}_1, \zeta\overset{o}{z}_2)}{Lf(\zeta\overset{o}{z}_1, \zeta\overset{o}{z}_2)} - s\right| < \rho.$$

Hence, because of the arbitrary choice of the point $(\overset{o}{z}_1, \overset{o}{z}_2) \in U$, we obtain that, for each $(z_1, z_2) \in U$, the following condition is satisfied:

$$\left|(1-\beta)\frac{Lf(z_1, z_2)}{f(z_1, z_2)} + \beta\frac{L^2f(z_1, z_2)}{Lf(z_1, z_2)} - s\right| < \rho.$$

This implies that $f \in M^U_\beta(A, B)$ for $|B| < 1$.

It can be noticed that $f \in M^U_\alpha(A, B)$ for $|B| = 1$ if and only if condition (6.1) holds and if

$$\mathrm{Re}\left[(1-\alpha)\frac{Lf(z_1, z_2)}{f(z_1, z_2)} + \alpha\frac{L^2f(z_1, z_2)}{Lf(z_1, z_2)}\right] > \frac{1-A_1}{2}$$

$$-1 < A_1 \le 1 \quad \text{where } A_1 = \begin{cases} -|A|, & \text{if } A = |A|B, \quad |A| < 1, \\ |A|, & \text{if } A = -|A|B, \quad |A| \le 1, \end{cases}$$

(cf. |42|). The proof of Theorem 6.2 for the case when $|B| = 1$ can thus be constructed in a similar way as for $|B| < 1$.

Our next step is to show some integral representation of functions of $M^U_\alpha(A, B)$. The proof of the Theorem given below can be carried out similarly as in [64].

THEOREM 6.3. A function $f \in M^U_\alpha(A, B)$, $\alpha > 0$, if and only if it possesses the following representation:

$$f(z_1, z_2) = \left\{\frac{1}{\alpha}\int_0^1 [g(tz_1, tz_2)]^{\frac{1}{\alpha}} t^{\frac{1}{\alpha}-1}\, dt\right\}^\alpha$$

where $g \in M^u_o(A, B)$.

Finally, we apply the following notation:

$$\gamma = \frac{\mathrm{Re}(A\bar{B}) + |A-B||B| - |B|^2}{2|B|^2}$$

$$\gamma' = \frac{\mathrm{Re}(A\bar{B}) - |A-B||B| - |B|^2}{2|B|^2} \qquad (B \ne 0)$$

$$K = K(\alpha, A, B; r)$$

where

$$K = \begin{cases} e^{|A|r} & \text{for } \alpha = 0,\ B = 0 \\ (1 + |B|r)^{\gamma}(1 - |B|r)^{\gamma'} & \text{for } \alpha = 0,\ B \neq 0 \\ [\Phi(\frac{1}{\alpha}, 1 + \frac{1}{\alpha}; \frac{|A|}{\alpha}r)]^{\alpha} & \text{for } \alpha > 0,\ B = 0 \\ [F_1(\frac{1}{\alpha}, \frac{-\gamma}{\alpha}, \frac{\gamma'}{\alpha}, 1 + \frac{1}{\alpha}; -|B|r, |B|r)]^{\alpha} & \text{for } \alpha > 0,\ B \neq 0 \end{cases}$$

In these formulae, $\Phi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^n$ is a degenerate hypergeometric function, whereas

$$F_1(a, b, b', c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n} m!n!} x^m y^n$$

denotes a hypergeometric function of two variables ([11], pp. 219, 237). Let

$$\overline{U}_r = \{(rz_1, rz_2): (z_1, z_2) \in \overline{U}\}, \quad r \in (0, 1).$$

THEOREM 6.4. If $f \in M_\alpha^U(A, B)$, then, for any $(z_1, z_2) \in \overline{U}_r$, $0 < r < 1$,

$$-K(\alpha, A, B; -r) \leq |f(z_1, z_2)| \leq K(\alpha, A, B; r).$$

P r o o f. Let r_o, $0 < r_o < 1$, be an arbitrarily fixed number. Take next an arbitrarily fixed point $(\overset{o}{z}_1, \overset{o}{z}_2) \in \overline{U}_{r_o}$. If the number ρ satisfies the inequality $r_o < \rho < 1$, then $(\overset{o}{z}_1, \overset{o}{z}_2) \in$ $\in U_\rho$ and $(\overset{o}{z}_1\rho^{-1}, \overset{o}{z}_2\rho^{-1}) \in U$. It follows from the properties of the domain U that also $(\zeta\overset{o}{z}_1\rho^{-1}, \zeta\overset{o}{z}_2\rho^{-1}) \in U$ holds for any $\zeta \in \Delta$. Next, consider the function

$$\Phi(\zeta) = \zeta f(\zeta\overset{o}{z}_1\rho^{-1}, \zeta\overset{o}{z}_2\rho^{-1}), \quad f \in M_\alpha^U(A, B).$$

Notice that Φ is a holomorphic function of the variable $\zeta \in \Delta$, $\Phi(0) = \Phi'(0) - 1 = 0$ and, moreover, that in the disc Δ the following conditions are satisfied: $\Phi(\zeta)\Phi'(\zeta)\zeta^{-1} \neq 0$,

$$J(\phi, \zeta, \alpha) = \frac{1 + Aw(\zeta \overset{o}{z}_1 \rho^{-1}, \zeta \overset{o}{z}_2 \rho^{-1})}{1 + Bw(\zeta \overset{o}{z}_1 \rho^{-1}, \zeta \overset{o}{z}_2 \rho^{-1})}, \quad w \in \Omega^U.$$

This implies that $\phi \in M_\alpha(A, B)$. Applying the estimations for $|f(z)|$, $f \in M_\alpha(A, B)$, derived in [42], we obtain

$$-|\zeta| K(\alpha, A, B; -|\zeta|) \le |\zeta f(\zeta \overset{o}{z}_1 \rho^{-1}, \zeta \overset{o}{z}_2 \rho^{-1})|$$
$$\le |\zeta| K(\alpha, A, B; |\zeta|).$$

Putting $\zeta = \rho$ and letting ρ tend to r_o, we shall get

$$-K(\alpha, A, B; -r_o) \le |f(\overset{o}{z}_1, \overset{o}{z}_2)| \le K(\alpha, A, B; r_o).$$

The above inequalities are equivalent to the proposition of Theorem 6.4 because of the arbitrary choice of $(\overset{o}{z}_1, \overset{o}{z}_2) \in \overline{U}_{r_o}$ and r_o, $0 < r_o < 1$.

REMARK. It follows from Theorem 6.3 that

$$Lf(z_1, z_2) = [f(z_1, z_2]^{1 - \frac{1}{\alpha}} [g(z_1, z_2)]^{\frac{1}{\alpha}}, \quad g \in M_o^U(A, B).$$

Hence and from Theorem 6.4 one can derive estimations for

$$|Lf(z_1, z_2)|, \quad f \in M_\alpha^U(A, B), \quad \alpha \ge 1.$$

REFERENCES

[1] A h m a d F., *Starlike integral operators*, Bull. Austral. Math. Soc., 32 (1985), 217-224.

[2] A h m a d F., *An integral operator that maps a subclass of starlike functions into itself*, Matematičeski Vestnik, 39 (1987), 361-365.

[3] A h m a d F., *Jakubowski starlike integral operator*, Journal Natural Sci. and Math., 28(1), (1988), 157-160.

[4] A o u f M. K., *A generalization of starlike functions of complex order*, Houston J. Math., 12(2), (1986), 155-162.

[5] A o u f M. K., *On coefficient bounds of a certain class of p-valent λ-spiral functions of order* α, Internat. J. Mat. and Math. Sci., 10(2) (1987), 259-266.

[6] A o u f M. K., *On a class of p-valent starlike functions of order* α, Internat. J. Math. and Math. Sci., 10(4), (1987), 733-744.

[7] A o u f M. K., *On a class of p-valent close-to-convex functions of order* β *and type* α, Internat. J. Math. and Math. Sci., 11(2),(1988), 259-266.

[8] A o u f M. K., *A generalization of multivalent functions with negative coefficients II*, Bull. Korean Math. Soc., 25(2), (1988), 221-232.

[9] A o u f M. K., Y o n i s Y. M., *A generalization of alpha-close--to-convex functions of order beta*, Bull. Inst. Math. Acad. Sinica, 15(1), (1987), 107-117.

[10] B a j p a i S. K., D w i v e d i S. P., *Certain convexity theorems for univalent analytic functions*, Publ. Inst. Math., 28(42), (1980), 5-11.

[11] B a t e m a n H., E r d é l y i A., *Higher transcendental functions*, Vol. I, Moskva 1973, in Russian.

[12] B a v r i n J. J., *Criteria of the belonging of regular functions to two classes of functions of two complex variables*, Dokl. Akad. Nauk SSSR, 152(2), (1963), 255-258, in Russian.

[13] B a v r i n J. J., *Classes of holomorphic functions of several complex variables and extremal problems for those classes of functions*, Moskva 1976, in Russian.

[14] B a z i l e v i č I. E., *On a case of integrability in quadratures of the Loewner-Kufarev equation*, Mat. Sb., 37 (1955), 471-476, in Russian.

[15] B o g u c k i Z., Z d e r k i e w i c z J., *Le rayon d'univalence de certaines fonctions analytiques*, Ann. UMCS, 31(4), (1977), 27-33.

[16] C a p l i n g e r T. R., C a u s e y W. M., *A class of univalent functions*, Proc. Amer. Math. Soc., 39(2), (1973), 357-361.

[17] C a r a t h é o d o r y C., *Über den Variabilitätsbereich der Fourierschen Konstanten positiven harmonischen Funktionen*, Rend. Circ. Math. (Palermo), 32 (1911), 193-217.

[18] C e r e b i e ż-T a r a b i c k a K., G o d u l a J., Z ł o t k i e w i c z E., *On a class of Bazilevič functions*, Ann. UMCS, 33 (1979), 45-57.

[19] C o o n c e H. B., M i l l e r S. S., *Distortion properties of p-fold symmetric* α*-starlike functions*, Proc. Amer. Mat. Soc., 44 (1974), 336-340.

[20] C h e r n i k o v V. V., *A variational method for spirallike func-*

tions of the Bazilevič class, Ekstr. Zadači Teor. Funkcii (Tomsk), 1 (1980), 110-117, in. Russian.

[21] C h e r n i k o v V. V., S i z h u k P. I., *Domains of values of initial coefficients of some p-symmetric spirallike functions*, Ekstr. Zadači Teor. Funkcii (Tomsk), 3 (1984), 71-84, in Russian.

[22] D i m k o v G. M., *Sur les coefficients des fonctions de Mocanu*, Comptes Rendus de l'Acad. Bulgare Scient., 36(1), (1983), 31-32.

[23] D r o z d a W., S z y n a l A., S z y n a l J. *The Jenkins' type inequality for Bazilevič functions*, Lecture Notes in Math., 1039, Analytic functions, Błażejewko 1982, (1983), 130-141.

[24] D u r e n P. L., *Theory of H^p spaces*, Academic Press, New York, 1970

[25] D w i v e d i S. P., B h a r g a w a G. P., S h u k l a S. L., *On some classes of meromorphic univalent functions*, Rev. Roumaine Math. Pures Appl., 25(2), (1980), 209-215.

[26] E e n i g e n b u r g P. J., M i l l e r S. S., *The H^p classes for α-convex functions*, Proc. Amer. Math. Soc., 38(3), (1983), 558-562.

[27] G o e l R. M., *The radius of univalence of certain analytic functions*, Tôhoku Math. J., 18 (1966), 398-403.

[28] G o e l R. M., *A class of close-to-convex functions*, Czechoslovak Math. J., 18(93), (1968), 104-116.

[29] G o e l R. M., S o h i N. S., *On the order of starlikeness of a subclass of convex functions and some convolution results*, Houston J. Math., 9(2), (1983), 209-216.

[30] G o l u z i n G. M., *Some estimates for bounded functions*, Mat. Sb., 26(68), (1950), 7-18, in Russian.

[31] G o o d m a n A. W., *On the Schwarz-Christoffel transformation and p-valent functions*, Trans. Amer. Math. Soc., 68 (1950), 204-223.

[32] G o o d m a n A. W., *Univalent functions*, Vols. I-II, Mariner Publishing Company, 1983.

[33] H o h l o v J. E., *On Mocanu and Bazilevič functions of several complex variables*, Trudy Sem. Kraev. Zadačam (Kazan), 15 (1978), 132-138, in Russian.

[34] J a k u b o w s k i Z. J., *On the coefficients of Carathéodory functions*, Bull. Acad. Polon. Sci., 19(9), (1971), 805-809.

[35] J a k u b o w s k i Z. J., *On the coefficients of starlike functions of some classes*, Bull. Acad. Polon. Sci., 19 (9), (1971), 811-815.

[36] J a k u b o w s k i Z. J., *On some applications of the Clunie method*, Ann. Polon. Math., 26 (1972), 211-217.

[37] J a k u b o w s k i Z. J., *On the coefficients of starlike functions of some classes*, Ann. Polon. Math., 26 (1972), 305-313.

[38] J a k u b o w s k i Z. J., *On some properties of extremal functions of Carathéodory*, Ann. Soc. Math. Polon., 17 (1973), 71-80.

[39] J a k u b o w s k i Z. J., *On the properties of an integral operator*, Bull. Austral. Mat. Soc., 32 (1985), 55-68.

[40] J a k u b o w s k i Z. J., K a m i ń s k i J., *On some properties of multivalent alpha-starlike functions*, Demonstratio Math., 9(2), (1976), 257-265.

[41] J a k u b o w s k i Z. J., K a m i ń s k i J., *On some properties of Mocanu-Janowski functions*, Rev. Roumaine Math. Pures Appl., 23(10), (1978), 1523-1532.

[42] J a k u b o w s k i Z. J., K a m i ń s k i J., *On some classes of alpha-convex functions*, Mathematica (Cluj), 27(50), (1985), 13-26.

[43] J a k u b o w s k i Z. J., M o l ę d a A., *On the radius of convexity of some classes of holomorphic functions*, Acta Univ. Lodz., Folia math., 34 (1980), 11-33.

[44] J a n o w s k i W., *On the radius of starlikeness of some families of regular functions*, Bull. Acad. Polon. Sci., 17(8), (1969), 503-508.

[45] J a n o w s k i W., *Extremal problems for a family of functions with positive real part and for some related families*, Bull. Acad. Polon. Sci., 17(10), (1969), 633-637.

[46] J a n o w s k i W., *On the radius of starlikeness of some families of regular functions*, Ann. Soc. Math. Polon., 14 (1970), 137-149.

[47] J a n o w s k i W., *Extremal problems for a family of functions with positive real part and for some related families*, Ann. Polon. Math., 23 (1970), 159-177.

[48] J a n o w s k i W., *Some extremal problems for certain families of analytic functions*, Bull. Acad. Polon. Sci., 21(1), (1973), 17-25.

[49] J a n o w s k i W., *Some extremal problems for certain families of analytic functions*, Ann. Polon. Math., 28 (1973), 297-328.

[50] K a c z m a r s k i J., *On the coefficients of some classes of starlike functions*, Bull. Acad. Polon. Sci., 17(8), (1969), 495-501.

[51] K a c z m a r s k i J., *On the radius of* β-μ-*N-spiral-starlikeness of the family* $S^*(\alpha, \lambda\ M)$ *of spiral-starlike functions in the disc* $|z| < 1$, Bull. Acad. Polon. Sci., 18(8), (1970), 467-473.

[52] K a c z m a r s k i J., *Extremal properties of some class of close-to-convex functions*, Folia Sci. Univ. Techn. Resov., 18 (1985), 19-40.

[53] K a c z m a r s k i J., *On the curvature of level-lines of some class of k-symmetrical functions,* Folia Sci. Univ. Techn. Resoviensis, 7(48), (1988), 35-47.

[54] K a c z m a r s k i J., *Extremal properties of starlike functions in the ring* $0 < |z| < 1$, Acta Univ. Lodz., Folia math., 3 (1989), 41-67.

[55] K a m i ń s k i J., *Some growth problems for certain α-convex functions,* Demonstratio Math., 12(1), (1979), 211-230.

[56] K o w a l R., S t a n k i e w i c z J., *On p-valent functions with reference to the Bernardi integral operator,* Folia Sci. Univ. Techn. Resov., 33 (1986), 17-24.

[57] K r z y ż J., R e a d e M. O., *The radius of univalence of certain analytic functions,* Michigan Math. J., 11 (1964), 157-159.

[58] K u l s h r e s t h a P. K., *Coefficient problem for alpha-convex univalent functions,* Arch. Rational Mech. Anal., 54 (1974), 205-211.

[59] K u m a r V., S h u k l a S. L., *Jakubowski starlike integral operators,* J. Austral. Math. Sci., 37 (1984), 117-127.

[60] K u m a r V., S h u k l a S. L., *On an integral operator for convex univalent functions,* Bull. Austral. Math. Soc., 34 (1986), 211-218.

[61] L e a c h R. J., *On some classes of multivalent starlike functions,* Trans. Amer. Math. Soc., 209 (1975), 267-273.

[62] L e w a n d o w s k i Z., M i l l e r S., Z ł o t k i e w i c z E., *Generating functions for some classes of univalent functions,* Proc. Amer. Math. Soc., 56 (1976), 111-117.

[63] L i b e r a R. J., L i v i n g s t o n A. E., *Bounded functions with positive real part,* Czechosl. Math. J., 22(97), (1972), 195-209.

[64] L i c z b e r s k i P., *On a certain family of holomorphic functions of two complex variables,* ZNPŁ, Matematyka 11(301), (1977), 57-64.

[65] L o h w a t e r A. J., P i r a n i a n G., R u d i n W., *The derivative of a schlicht function,* Math. Scand., 3 (1955), 103-106.

[66] M a c G r e g o r T. H., *Functions whose derivative has a positive real part,* Trans. Amer. Math. Soc., 104 (1962), 532-537.

[67] M a c G r e g o r T. H., *The radius of univalence of certain analytic functions II,* Proc. Amer. Math. Soc., 14 (1963), 521-524.

[68] M a c G r e g o r T. H., *A class of univalent functions,* Proc. Amer. Math. Soc., 15 (1964), 311-317.

[69] M i l l e r S. S., *Distortion properties of alpha-starlike functions,* Proc. Amer. Math. Soc., 38(2), (1973), 311-318.

[70] M i l l e r S. S., M o c a n u P. T., *Alpha-convex functions and*

derivatives in the Nevanlinna class, Studia Univ. Babes-Bolyai Math., 20 (1975), 35-40.

[71] M i l l e r S. S., M o c a n u P. T., *Univalent solutions of Briot-Bouquet differential equations,* J. Diff. Equat., 56 (1985), 297-309.

[72] M i l l e r S. S., M o c a n u P. T., R e a d e M. O., *On generalized convexity in conformal mappings II,* Rev. Roumaine Math. Pures Appl., 21 (1976), 219-225.

[73] M i l l e r S. S., M o c a n u P. T., R e a d e M. O., *All α-convex functions are univalent and starlike,* Proc. Amer. Math. Soc., 37(2), (1973), 553-554.

[74] M i l l e r S. S., M o c a n u P. T., R e a d e M. O., *Bazilevič functions and generalized convexity,* Rev. Roumaine Math. Pures Appl., 19(2), (1974), 213-224.

[75] M i l l e r S. S., M o c a n u P. T., R e a d e M. O., *The radius of α-convexity for the class of starlike univalent functions, α-real,* Proc. Amer. Math. Soc., 51(2), (1975), 395-400.

[76] M i l l e r S. S., M o c a n u P. T., R e a d e M. O., *Janowski alpha-convex functions,* Ann. UMCS, 29(11), (1975), 93-98 (1977).

[77] M i l l e r S. S., M o c a n u P. T., R e a d e M. O., *The order of starlikeness of alpha-convex functions,* Mathematica (Cluj), 20(43), (1978), 25-30.

[78] M o c a n u P. T., *Une propriété de convexité généralisée dans la théorie de la répresentation conforme,* Mathematica (Cluj), 11(34), (1969), 127-133.

[79] N a s r M. A., A o u f M. K., *Bounded starlike functions of complex order,* Proc. Indian Acad. Sci., 92(2), (1983), 97-102.

[80] O l e j n i c z a k E., *Extremal problems for certain families of analytic and symmetric functions,* Acta Univ. Lodz., Folia math., 10 (1977), 81-104, in Polish.

[81] P a d m a n a b h a n K. S., P a r v a t h a m R., *Some applications of differential subordination,* Bull. Austral. Math. Soc., 32 (1985), 321-330.

[82] P a n d e y R. K., B h a r g a v a G. P., *On convex and starlike univalent functions,* Bull. Austral. Math. Soc., 28 (1983), 393-400.

[83] P a n d e y R. K., B h a r g a v a G. P., *On the radius of starlikeness of certain analytic functions with integral representation,* Comm. Fac. Sci. Univ. (Ankara), 32 (1983), 31-40.

[84] P a r v a t h a m R., Radha S., *On α-starlike and α-close-to-convex functions with respect to n-symmetric points,* Indian J. Pure Appl. Math., 16(9), (1986), 1114-1122.

[85] P a s c u N. N., *Janowski alpha-starlike-convex functions,* Studia Univ. Babes-Bolyai, Math., (1976), 23-27.

[86] P l a s k o t a W., *On the coefficients of some families of regular functions,* Bull. Acad. Sci. Polon., 17(11), (1969), 715-718.

[87] P l a s k o t a W., *Limitation des coefficients dans une famille de fonctions holomorphes dans le cercle* $|z| < 1$, Ann. Polon. Math., 24 (1970), 65-70.

[88] P l a s k o t a W., *Sur quelques problémes extrémaux dans les familles des fonctions générées par les fonctions de Carathéodory,* Ann. Polon. Math., 25 (1971), 139-144.

[89] P r o k h o r o v D. V., S z y n a l J., *Inverse coefficients for* (α, β)*-convex functions,* Ann. UMCS, 35(15), 125-143, (1981).

[90] P r o k h o r o v D. V., S z y n a l J., *Estimation of the moduli of Mocanu functions with fixed Initial coefficients. In: Theory of functions and approximations,* Izd. Saratov Univ. (Saratov), (1983), 156-163, in Russian.

[91] R o b e r t s o n M. S., *On the theory of univalent functions,* Ann. Math., 37 (1936), 374-408.

[92] S h a f f e r D. B., *Distortion theorems for a special class of analytic functions,* Proc. Amer. Math. Soc., 39(2), (1973), 281-287.

[93] S h a h G. M., *On the univalence of some analytic functions,* Pacific J. Math., 43(1), (1972), 239-250.

[94] S h a u n m u g a m T. N., *Studies on analytic functions with special reference to integral operators,* Anna Univ. Madras, (1987).

[95] S i l v i a E. M., *On a subclass of spiral-like functions,* Proc. Amer. Math. Soc., 44(2), (1974), 411-420.

[96] S i n g h V., *Univalent functions with bounded derivative in the unit disc,* Indian J. Pure and Appl. Math., 8(11), (1977), 1370-1377.

[97] S i z h u k P. I., C h e r n i k o v V. V., N i k i t i n S. V., *The boundary of generalized convexity for starlike mappings,* Ekstr. Zadači Teor. Funkcii (Tomsk), 1 (1980), 95-98, in Russian.

[98] S i z h u k P. I., C h e r n i k o v V. V., *On some extremal properties of spirallike univalent functions,* Ekstr. Zadači Teor. Funkcii (Tomsk), 2 (1983), 62-69, in Russian.

[99] S i z h u k P. I., C h e r n i k o v V. V., *The radius of* α*-convexity of order* β *for a class of starlike functions of order* γ, Ekstr. Zadači Teor. Funkcii (Tomsk), 5 (1986), 49-58, in Russian.

[100] S i z h u k G. I., S i z h u k P. I., *On* α-γ*-spirallike functions of order* β, Ekstr. Zadači Teor. Funkcii (Tomsk), 6 (1988), 76-79, in Russian.

[101] S t a n k i e w i c z J., W a n i u r s k i J., *Some classes of functions subordinate to linear transformation and their applications,* Ann. UMCS, 28(9), (1974), 85-94 (1976).

[102] S z y n a l J., *On certain classes of regular functions,* Ann. UMCS, 25(9), (1971), 109-120 (1973).

[103] S z y n a l J., *Some remarks of coefficients inequality for α-convex functions,* Bull. Acad. Polon. Mat., 20(11), (1972), 917-919.

[104] S z y n a l J., W a j l e r S., *On the fourth coefficient for α-convex functions,* Rev. Roumaine Math. Pures Appl., 19(9), (1974), 1153-1157.

[105] S z y n a l A., S z y n a l J., W a j l e r S., *On the radii of convexity for certain classes of analytic functions,* ZNT WSI,(Lublin) 1972, 21-30, in Polish.

[106] S z y n a l A., S z y n a l J., Z y g m u n t J., *On the coefficients of functions whose real part is bounded,* Ann. UMCS, 26(5), (1972), 63-70 (1974).

[107] T o n i N. E., T r a h a n N. E., *Analytic functions whose real parts are bounded below,* Math. Z., 115 (1970), 252-258.

[108] T u a n P. D., A n h V. V., *Extremal problems for functions of positive real part with a fixed coefficient and applications,* Czechoslovak Math. Journal, 30(105), (1980), 302-312.

[109] T u r z y ń s k i A., *On a subclass of starlike functions,* ZNPŁ, Matematyka, 11(301), (1977), 73-79.

[110] W i a t r o w s k i P., *On the radius of convexity of some family of functions regular in the ring* $0 < |z| < 1$, Ann. Polon. Math., 25 (1971), 85-98.

[111] W ł o d a r c z y k K., *Some properties of analytic maps of operators in* J^**-algebra,* Mh. Math., 96 (1983), 325-330.

[112] Z m o r o v i c h V. A., P o k h i l e v i c h V. A., *On α-close-to-convex functions,* Ukrain. Mat. Z., 33(5), (1981), 670-673, in Russian.

[113] Z y s k o w s k i J., *On some families of functions generated by functions with positive real part,* ZNUŁ, Matematyka, 17 (1977), 155-181.

Chair of Special Functions
University of Łódź
Faculty of Applied Physics and Mathematics
Technical University of Gdańsk

Zbigniew Jerzy Jakubowski, Józef Kamiński

O PEWNYCH KLASACH FUNKCJI MOCANU-BAZYLEWICZA

W pracy zbadano kilka klas funkcji generowanych przez różnego typu związki z homografią $z \to (1 + Az)/(1 + Bz)$, $z \in \Delta = \{z: |z| < 1\}$, przy czym dopuszczono możliwość przyjmowania przez parametry A, B wartości zespolonych. Zasadnicze rezultaty dotyczą pewnych rodzin α-wypukłych funkcji Mocanu-Bazylewicza (M o c a n u (1969) [78], B a z i l e v i č (1955) [14]). Otrzymane wyniki stanowią kontynuację wcześniejszych prac, a w szczególności [40], [41], [55] i [42]. Podstawowe badania poprzedzono przeglądem różnych klas funkcji Carathéodory´ego o części rzeczywistej dodatniej.