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ON SOME GENERALIZATIONS OF SYMMETRIC CONTINUITY

In this paper we shall give definitions of generalizations of the symmetric continuity of a function $f: R \to R$. We shall prove several properties of these generalizations.

1. Throughout the paper, $\mathbf{B}$ will denote the family of all subsets of R (the real line) having the Baire property; $\mathbf{I}$ will denote the σ-ideal of sets of the first category. For $a \in R$ and $A \subset R$, we denote $a \cdot A = \{a \cdot x: x \in A\}$ and $A - a = \{x - a: x \in A\}$. Recall [7] that 0 is an $\mathbf{I}$-density point of a set $A \in \mathbf{B}$ if and only if $\chi_{n \cdot A \cap [-1, 1]} \xrightarrow[n\to\infty]{\mathbf{I}} 1$, i.e. if and only if, for each increasing sequence $\{n_m\}_{m \in N}$ of natural numbers, there exists a subsequence $\{n_{m_p}\}_{p \in N}$ such that $\chi_{n_{m_p} \cdot A \cap [-1, 1]} \xrightarrow[p\to\infty]{} 1$ except on a set belonging to $\mathbf{I}$ (abbr. $\mathbf{I}$ - a.e.). A point $x_o \in R$ is an $\mathbf{I}$-density point of $A \in \mathbf{B}$ if and only if 0 is an $\mathbf{I}$-density point of $A - x_o$. A point $x_o \in R$ is an $\mathbf{I}$-dispersion point of $A \in \mathbf{B}$ if and only if x_o is an $\mathbf{I}$-density point of $R \setminus A$. The set of all $\mathbf{I}$-density points of A will be denoted by $\phi(A)$. In the obvious manner we can define a right-hand $\mathbf{I}$-density point. The set of all right-hand $\mathbf{I}$-density points of A will be denoted by $\phi^+(A)$.

Further, the family $\mathbf{T_I} = \{A \in \mathbf{B}: A \subset \phi(A)\}$ is a topology on the real line which we call the $\mathbf{I}$-density topology (see [7]). Real functions continuous with respect to the $\mathbf{T_I}$-topology are called $\mathbf{I}$-approximately continuous functions.

DEFINITION [5]. Let $f\colon R \to R$ have the Baire property in a neighbourhood of x_o. The upper I-approximate limit of f at x_o ($\text{I-}\limsup_{x\to x_o} f(x)$) is the greatest lower bound of the set

$$\{y\colon \{x\colon f(x) > y\} \text{ has } x_o \text{ as an I-dispersion point}\}.$$

The lower I-approximate limit, right-hand and left-hand upper and lower I-approximate limits are defined similarly. If

$$\text{I-}\limsup_{x\to x_o} f(x) = \text{I-}\liminf_{x\to x_o} f(x),$$

their common value is called the I-approximate limit of f at x_o and denoted by $\text{I-}\lim_{x\to x_o} f(x)$.

We shall say that f is I-approximately continuous at x_o if and only if f has the Baire property in a neighbourhood of x_o and $\text{I-}\lim_{x\to x_o} f(x) = f(x_o)$. It is known that f is I-approximately continuous if and only if, for each $x \in R$, f is I-approximately continuous at x [7].

The family of all I-approximately continuous functions will be denoted by $I_{ap}-C$, and the family of all I-approximately continuous functions except on a set belonging to I by $I_{ap}-C_{I\text{-a.e.}}$.

In the paper we shall need the following lemmas:

LEMMA [7]. If 0 is an I-density point of $A \in \mathbf{B}$, then, for each natural number n, there exists $\delta_n > 0$ such that, for each $0 < h < \delta_n$ and for each integer $k \in [-n, n-1]$, we have

$$A \cap \left[\frac{k}{n}h, \frac{k+1}{n}h\right] \neq \emptyset.$$

LEMMA [4]. Let $G \subset R$ be an open set. A point 0 is an I-dispersion point of G if and only if, for each natural number n, there exist a natural number k and a real number $\delta > 0$ such that, for any $0 < h < \delta$ and $i \in \{1, \ldots, n\}$, there exist $j_r, j_1 \in \{1, \ldots, k\}$ such that

$$G \cap \left(\frac{(i-1)k + j_r - 1}{n \cdot k} h, \frac{(i-1)k + j_r}{n \cdot k} h\right) = \emptyset$$

and

$$G \cap \left(-\frac{(i-1)k + j_1}{n \cdot k}h, -\frac{(i-1)k + j_1 - 1}{n \cdot k}h\right) = \emptyset.$$

Throughout the paper, cl(A), int(A) will denote the closure and the interior of the set A with respect to the natural topology. Except where a topology is specifically mentioned, all topological notations are considered with respect to the natural topology.

For any $x \in R$, we denote by $\mathbf{P}(x)$ the collection of all intervals $[a, b]$ such that $x \in (a, b)$ and of all sets of the form

$$E = \bigcup_{n=1}^{\infty} [a_n, b_n] \cup \bigcup_{n=1}^{\infty} [c_n, d_n] \cup \{x\} \text{ where, for every } n,$$

$$a_n < b_n < a_{n+1} < x < d_{n+1} < c_n < d_n \text{ and } x \in \phi(E).$$

In [5], there was introduced a topology τ which consists of all sets $U \in \mathbf{T_I}$ such that if $x \in U$, then there exists a set $P \in \mathbf{P}(x)$ included in $\{x\} \cup \text{int } U$. It was proved that τ is the coarsest topology for which all **I**-approximately continuous functions are continuous.

2. In the paper we shall consider real functions of a real variable and denote:

C - the family of all continuous functions,

$\mathbf{C}_{\mathbf{I}\text{-a.e.}}$ - the family of all continuous functions except on a set belonging to **I** (abbr. **I**-almost everywhere),

SC - the family of all symmetrically continuous functions,

$\mathbf{SC}_{\mathbf{I}\text{-a.e.}}$ - the family of all symmetrically continuous functions **I**-almost everywhere.

DEFINITION 1. Let $x_o \in R$. We shall say that a function $f: R \to R$ is symmetrically **I**-continuous at x_o if and only if f has the Baire property in a neighbourhood of x_o and

$$\mathbf{I}\text{-}\lim_{h \to 0+} (f(x_o + h) - f(x_o - h)) = 0.$$

DEFINITION 2. We shall say that a function $f: R \to R$ is symmetrically **I**-continuous if and only if, it is so at each point of its domain.

We shall denote:

I-SC - the family of all symmetrically I-continuous functions,

$\text{I-SC}_{\text{I-a.e.}}$ - the family of all symmetrically I-continuous functions I-almost everywhere.

COROLLARY 1. If a function $f: R \to R$ is a symmetrically I-continuous function I-almost everywhere, then f has the Baire property.

COROLLARY 2. If a function $f: R \to R$ is symmetrically I-continuous at $x_o \in R$, then, for each $\varepsilon > 0$,

$$0 \in \phi^+(\{h > 0: |f(x_o + h) - f(x_o - h)| < \varepsilon\})$$

(or, for each $\varepsilon > 0$, $x_o \in \phi^+(\{x > x_o: |f(x) - f(x')| < \varepsilon$ where $x' = x_o - (x - x_o)\}))$.

THEOREM 1. If a function f is I-approximately continuous at $x_o \in R$, then f is symmetrically I-continuous at x_o.

P r o o f. By the assumption, we infer that

$$\sup \{\alpha: x_o \in \phi(\{x: f(x) \geq \alpha\})\} = f(x_o)$$

and

$$\inf \{\alpha: x_o \in \phi(\{x: f(x) \leq \alpha\})\} = f(x_o),$$

and therefore, for each $\varepsilon > 0$, there exist $\alpha_1 > f(x_o) - \varepsilon$ and $\alpha_2 < f(x_o) + \varepsilon$ such that

$$x_o \in \phi(\{x: f(x) \geq \alpha_1\})$$

and

$$x_o \in \phi(\{x: f(x) \leq \alpha_2\}).$$

Thus, for each $\varepsilon > 0$,

(1) there exists $\alpha_1 > f(x_o) - \varepsilon$ such that $0 \in \phi^+(A_1') \cap \phi^+(A_1'')$

where $A_1' = \{h > 0: f(x_o + h) \geq \alpha_1\}$,

$A_1'' = \{h > 0: f(x_o - h) \geq \alpha_1\}$,

and

(2) there exists $\alpha_2 < f(x_o) + \varepsilon$ such that $0 \in \phi^+(A_2') \cap \phi^+(A_2'')$

where $A_2' = \{h > 0: f(x_o + h) \leq \alpha_2\}$,

$A_2'' = \{h > 0: f(x_o - h) \leq \alpha_2\}$.

Therefore, we have,

(3) $\quad 0 \in \phi^{+}(B_1') \cap \phi^{+}(B_1'')$, where $B_1' = \{h>0: f(x_o+h) > f(x_o)-\varepsilon\}$

$B_1'' = \{h > 0: f(x_o - h) > f(x_o) - \varepsilon\}$, for each $\varepsilon > 0$,

and

(4) $\quad 0 \in \phi^{+}(B_2') \cap \phi^{+}(B_2'')$ where $B_2' = \{h>0: f(x_o+h) < f(x_o) + \varepsilon\}$

$B_2'' = \{h > 0: f(x_o - h) < f(x_o) + \varepsilon\}$, for each $\varepsilon > 0$.

Now, we shall show that

(5) $\quad \inf \{\alpha: 0 \in \phi^{+}(\{h > 0: f(x_o + h) - f(x_o - h) \le \alpha\})\} = 0.$

Let $\alpha \in R$ and $\alpha < 0$. For $\varepsilon = -\frac{\alpha}{2}$, by (3) and (4), we have that $0 \in \phi^{+}(B_1' \cap B_2'') = \phi^{+}(\{h > 0: f(x_o + h) - f(x_o - h) > \alpha\})$, and therefore, 0 is a right-hand I-dispersion point of a set $\{h > 0: f(x_o + h) - f(x_o - h) \le \alpha\}$. Thus

(6) $\quad$ if $0 \in \phi^{+}(\{h > 0: f(x_o + h) - f(x_o - h) \le \alpha\}$, then $\alpha \ge 0$.

Let $\eta \in R$ and $\eta > 0$. By (1) and (2), for $\varepsilon = \frac{\eta}{2}$ and $\alpha_3 = \alpha_2 - \alpha_1$ we have $\alpha_3 < \eta$ and

$$0 \in \phi^{+}(A_2' \cap A_1'') = \phi^{+}(\{h > 0: f(x_o + h) - f(x_o - h) \le \alpha_3\}).$$

Then, by the above and by (6), we have (5).

In a similar way we can show

(7) $\quad \sup \{\alpha: 0 \in \phi^{+}(\{h > 0: f(x_o + h) - f(x_o - h) \ge \alpha\})\} = 0.$

Then, by (5) and (7), we have

(8) $\quad \displaystyle \operatorname*{I-lim}_{h\to 0+} (f(x_o + h) - f(x_o - h)) = 0,$

and the proof of Theorem 1 is completed.

PROPOSITION 1. If a function f is symmetrically continuous at $x_o \in R$, then f is symmetrically I-continuous at x_o.

P r o o f. Let f be a symmetrically continuous function at $x_o \in R$. We shall show that

(9) $\quad \displaystyle \operatorname*{I-lim\,sup}_{h\to 0+} (f(x_o + h) - f(x_o - h)) \le 0.$

Let $\alpha \in R$ and $\alpha > 0$. By the symmetric continuity of f at x_o, we have that there exists $h_o > 0$ such that $(0, h_o) \subset A_\alpha$, where $A_\alpha =$

$= \{h > 0: f(x_o + h) - f(x_o - h) \le \alpha\}$. Since $0 \in \phi^+((0, h_o))$, therefore $0 \in \phi^+(A_\alpha)$. Thus $\inf \{\alpha: 0 \in \phi^+(A_\alpha)\} \le 0$ and condition (9) is true.

In a similar way we can show

(10) $$\text{I-}\liminf_{h\to 0+} (f(x_o + h) - f(x_o - h)) \ge 0.$$

It is easy to see that

(11) $$\text{I-}\liminf_{h\to 0+} (f(x_o + h) - f(x_o - h)) \le \text{I-}\limsup_{h\to 0+} (f(x_o + h) - f(x_o - h))$$

and, by the above,

$$\text{I-}\lim_{h\to 0+} (f(x_o + h) - f(x_o - h)) = 0.$$

Therefore, the function f is symmetrically I-continuous at x_o.

THEOREM 2. If f is defined on an open interval I and f is symmetrically I-continuous and monotone on I, then f is symmetrically continuous on I.

P r o o f. Now, we observe that, for each real function g of a real variable, we have

(12) $$\liminf_{h\to 0+} g(h) \le \text{I-}\liminf_{h\to 0+} g(h)$$

and

(13) $$\text{I-}\limsup_{h\to 0+} g(h) \le \limsup_{h\to 0+} g(h).$$

Let $\delta > 0$ and $m_\delta = \inf_{0<h<\delta} g(h)$. Then $(0, \delta) \subset \{h > 0: g(h) \ge m_\delta\}$ and $0 \in \phi^+(\{h > 0: g(h) \ge m_\delta\})$. Thus $m_\delta \le \sup \{\alpha: 0 \in \phi^+(\{h > 0: g(h) \ge \alpha\})$ and $\sup_\delta m_\delta \le \sup \{\alpha: 0 \in \phi^+(\{h > 0: g(h) \ge \alpha\})\}$. Therefore

$$\liminf_{h\to 0+} g(h) \le \text{I-}\liminf_{h\to 0+} g(h).$$

In a similar way we can prove condition (13).

Now, we shall prove that

(14) $$\lim_{h\to 0+} (f(x_o + h) - f(x_o - h)) = \text{I-}\lim_{h\to 0+} (f(x_o + h) - f(x_o - h))$$

Condition (14) will be followed from

(15) $$\liminf_{h\to 0+} (f(x_o + h) - f(x_o - h)) \geq \text{I-}\liminf_{h\to 0+} (f(x_o + h) - f(x_o - h))$$

and

(16) $$\text{I-}\limsup_{h\to 0+} (f(x_o + h) - f(x_o - h)) \geq \limsup_{h\to 0+} (f(x_o + h) - f(x_o - h)).$$

We assume that f is nondecreasing and suppose that there exists $x_o \in R$ such that (15) does not hold; let

$$k_1 = \liminf_{h\to 0+} (f(x_o + h) - f(x_o - h)) < \text{I-}\liminf_{h\to 0+} (f(x_o + h) - f(x_o - h)) = k_2.$$

Let $0 < \varepsilon < \frac{1}{2}(k_2 - k_1)$ and $B = \{h > 0\colon f(x_o + h) - f(x_o - h) \geq k_2 - \varepsilon\}$. Since $\text{I-}\liminf_{h\to 0+} (f(x_o + h) - f(x_o - h)) = k_2$, there exists $\alpha > k_2 - \varepsilon$ such that $0 \in \phi^+(\{h > 0\colon f(x_o + h) - f(x_o - h) \geq \alpha\}$ and $0 \in \phi^+(B)$. Thus, by lemma [7], we have

(17) for each natural n, there exists $\delta_n > 0$ such that, for each natural $0 \leq l \leq n - 1$, $[\frac{l}{n}h, \frac{l+1}{n}h] \cap B \neq \emptyset$.

By $\liminf_{h\to 0+} (f(x_o + h) - f(x_o - h)) = k_1 < k_2 - 2\varepsilon$, we infer that there exists a sequence $\{h_n\}_{n\in N}$ such that $h_n \xrightarrow[n\to\infty]{} 0$, and, for each natural n, $h_n > 0$, $x_o + h_n \in I$ and

(18) $f(x_o + h_n) - f(x_o - h_n) < k_2 - 2\varepsilon$.

We consider intervals $J_n = [0, h_n]$. Then

(19) for each n, $B \cap J_n = \emptyset$.

Indeed, if $h \in J_n$, then, by the assumption and by (18), we have

$$f(x_o + h) \leq f(x_o + h_n) < f(x_o - h_n) + k_2 - 2\varepsilon \leq f(x_o + h) + k_2 - 2\varepsilon,$$

$$f(x_o + h) - f(x_o - h) < k_2 - 2\varepsilon, \text{ so } h \notin B.$$

Let n_o be a natural number. By (17), we infer that there exists $\delta_{n_o} > 0$ such that, for $h_{n_1} \in \{h_n\}_{n\in N}$ and $h_{n_1} < \delta_{n_o}$,

$$[\frac{n_o - 1}{n_o}h_{n_1}, h_{n_1}] \cap B \neq \emptyset$$

holds which gives a contradiction. Thus, for each $x_o \in I$, (15) holds.

Now, we suppose that condition (16) does not hold. Let $x_o \in R$ be such that

$$k_1 = \text{I-}\limsup_{h\to 0+} (f(x_o + h) - f(x_o - h))$$
$$< \limsup_{h\to 0+} (f(x_o + h) - f(x_o - h)) = k_2.$$

Let $0 < \varepsilon < \frac{1}{2}(k_2 - k_1)$ and $C = \{h > 0\colon f(x_o + h) - f(x_o - h) \leq$ $\leq k_1 + \varepsilon\}$. Since $\text{I-}\limsup_{h\to 0+} (f(x_o + h) - f(x_o - h)) = k_1$ therefore $0 \in \phi^+(C)$. By lemma [7], we have

(20) for each natural n, there exists $\delta_n > 0$ such that, for each $0 < h < \delta_n$ and for each natural $0 \leq l \leq n - 1$, $[\frac{l}{n}h, \frac{l+1}{n}h] \cap C \neq \emptyset$ holds.

By $\limsup_{h\to 0+} (f(x_o + h) - f(x_o - h)) = k_2 > k_1 + 2\varepsilon$, we have that there exists a sequence $\{h_n\}_{n\in N}$ such that $h_n \xrightarrow[n\to\infty]{} 0$ and, for each natural n, $h_n > 0$, $x_o + h_n \in I$ and $(f(x_o + h_n) - f(x_o - h_n) >$ $> k_1 + 2\varepsilon$. Thus, for each $h > h_n$, if $x_o + h \in I$ and $x_o - h \in I$, then $f(x_o + h) \geq f(x_o + h_n) > f(x_o - h_n) + k_1 + 2\varepsilon > f(x_o - h) +$ $+ k_1 + 2\varepsilon$. Therefore $f(x_o + h) - f(x_o - h) > k_1 + 2\varepsilon$ and

(21) for each natural n and for each $h > h_n$, we hace $h \notin C$.

Let n_o be a natural number. By (20), we have that there exists $\delta_{n_o} > 0$ such that, for each $h < \delta_{n_o}$, $[\frac{n_o - 1}{n_o}h, h] \cap C \neq \emptyset$, which gives a contradiction since, by (21), for each $h' > h_{n_1}$ where $h_{n_1} \in \{h_n\}_{n\in N}$ and $h_{n_1} < \frac{h_o - 1}{n_o}h$, $h' \notin C$. Thus, for each

$x_0 \in I$, condition (16) holds. Now, it is easy to see that; by (12), (13), (15) and (16), we have (14).

PROPOSITION 2. $\mathbf{I\text{-}SC}\setminus\mathbf{SC} \neq \emptyset$ and $\mathbf{I\text{-}SC}\setminus\mathbf{I}_{ap}\text{-}\mathbf{C} \neq \emptyset$.

P r o o f. Let $P = \bigcup_{n=1}^{\infty} [a_n, b_n]$ be such that, for each natural n, $0 < b_{n+1} < a_n < b_n$ and $0 \in \phi^+(P)$. Let f be a continuous function at all $x \neq 0$ such that $f(0) = 1$, $f(x) = 0$ at $x \in P \cup (-P)$, and for each natural n, $f(\frac{1}{2}(a_n + b_{n-1})) = 1$ and $f(-\frac{1}{2}(a_n + b_{n+1})) = -1$. It is easy to see that $f \in \mathbf{I\text{-}SC}$ and $f \notin \mathbf{SC} \cup \mathbf{I}_{ap}\text{-}\mathbf{C}$.

H. F r i e d showed in [2] that every symmetrically continuous I-almost everywhere function is continuous I-almost everywhere. Therefore, by the above and by the theorem of R. D. M a u l d i n [6], we have the following

THEOREM 3. $\mathbf{C}_{\mathbf{I}\text{-a.e.}} = \mathbf{SC}_{\mathbf{I}\text{-a.e.}}$. Further, $f \in \mathbf{C}_{\mathbf{I}\text{-a.e.}}$ if and only if, for each $a \in R$, there exist open sets A, A_1 and $B, B_1 \in \mathbf{I}$ such that $\{x \in R: f(x) < a\} = A \cup B$ and $\{x \in R: f(x) > a\} = A_1 \cup B_1$.

COROLLARY 3. There exists a function f nonmeasurable in the sense of Lebesgue, such that $f \in \mathbf{SC}_{\mathbf{I}\text{-a.e.}}$ (S i e r p i ń s k i [9], $f \in \mathbf{C}_{\mathbf{I}\text{-a.e.}}$).

It is known that if a function $f \in \mathbf{SC}$, then f is a Lebesgue measurable function (P r e i s s [8]).

THEOREM 4. A function $f \in \mathbf{I\text{-}SC}_{\mathbf{I}\text{-a.e.}}$ if and only if, for each $a \in R$, there exist $A, A_1 \in \mathbf{T_I}$ and $B, B_1 \in \mathbf{I}$ such that $\{x \in R: f(x) > a\} = A \cup B$ and $\{x \in R: f(x) < a\} = A_1 \cup B_1$.

P r o o f. Let $a \in R$ and put $A = \mathbf{T_I}\text{-int}(\{x \in R: f(x) > a\})$, $A_1 = \mathbf{T_I}\text{-int}(\{x \in R: f(x) < a\})$, $B = \mathbf{T_I}\text{-}\mathrm{Fr}(\{x \in R: f(x) > a\}) \cap \{x \in R: f(x) > a\}$ and $B_1 = \mathbf{T_I}\text{-}\mathrm{Fr}(\{x \in R: f(x) < a\}) \cap \{x \in R: f(x) < a\})$. Thus $A, A_1, B, B_1 \in \mathbf{I}$ fulfil the required conditions.

Now, we assume that, for each $a \in R$, there exist $A, A_1 \in \mathbf{T_I}$ and $B, B_1 \in \mathbf{I}$ such that $\{x \in R: f(x) > a\} = A \cup B$ and $\{x \in R: f(x) < a\} = A_1 \cup B_1$. Therefore, for each $a \in R$, $\{x \in R: f(x) > a\} \in$

$\in \mathbf{B}$ and $\{x \in R: f(x) < a\} \in \mathbf{B}$. Thus the function f has the Baire property and, by [7], $f \in I_{ap}C_{I\text{-a.e.}}$. Then, by Theorem 1, we have that $f \in I\text{-}SC_{I\text{-a.e.}}$.

PROPOSITION 3. There exist functions f and g such that $f \in I\text{-}SC \setminus SC_{I\text{-a.e.}}$ and $g \in SC_{I\text{-a.e.}} \setminus I\text{-}SC$.

P r o o f. Let f be the Dirichlet function. Then, for each $x \in R$, $\{h > 0: f(x+h) - f(x-h) = 1\} \subset W$, where W is a set of all rational numbers.

Thus $0 \in \phi^{+}(\{h > 0: f(x+h) - f(x-h) = 0\})$ and $f \in I\text{-}SC$. By Theorem 3, we know that $f \notin SC_{I\text{-a.e.}} = C_{I\text{-a.e.}}$. Let g be a function such that $g(x) = 0$ at $x \in (-\infty, 0\rangle$ and $g(x) = 1$ at $x \in (0, \infty)$. Then $g \in SC_{I\text{-a.e.}} \setminus I\text{-}SC$.

By Theorems 3, 4 and by Proposition 1, we have

$$SC_{I\text{-a.e.}} = C_{I\text{-a.e.}} \subset I\text{-}SC_{I\text{-a.e.}} = \{f\colon f \text{ has the Baire property}\}.$$

DEFINITION 3. We shall say that $f: R \to R$ is a symmetrically τ-continuous function at x_o if and only if, for each $\varepsilon > 0$, there exists a set $P \in \mathbb{P}(0)$ such that

$$P \cap (0, +\infty) \subset \operatorname{int}\{h > 0: |f(x_o + h) - f(x_o - h)| < \varepsilon\}.$$

We shall say that $f: R \to R$ is symmetrically τ-continuous if and only if it is symmetrically τ-continuous at all $x \in R$.

We shall denote:

$\tau\text{-}SC$ - the family of all symmetrically τ-continuous functions,

$\tau\text{-}SC_{I\text{-a.e.}}$ - the family of all symmetrically τ-continuous functions I-almost everywhere.

LEMMA 1. Let $f \in \tau\text{-}SC_{I\text{-a.e.}}$. Then, for each $x_o \in R$ such that x_o is a point of the symmetric τ-continuity of f and, for any $\gamma > 0$, $\delta > 0$, there exists a set $F = \bigcup_{n=1}^{\infty}[a_n, b_n]$ such that, for each natural n, the set

$$\{x \in x_o + [a_n, b_n]\colon \operatorname{int}(\{t\colon |f(t + (x - x_o)) - f(t - (x - x_o))| < \gamma\}) \cap (x_o, x_o + \delta) \neq \emptyset\}$$

is a residual subset of $x_o + [a_n, b_n]$, and $0 \in \phi^{+}(F)$.

P r o o f. First assume that x_o fulfils conditions $\mathrm{int}(\{t > 0: |f(x_o + t) - f(x_o - t)| < \frac{\gamma}{2}\}) \neq \emptyset$ and $(c, d) \subset \mathrm{int}(\{t > 0: |f(x_o + t) - f(x_o - t)| < \frac{\gamma}{2}\})$. Let $x_1 \in (x_o + c, x_o + d)$ be a point of the symmetric τ-continuity of the function f, and let $\delta > 0$. We shall choose $\alpha > 0$ such that $\alpha < \min(\delta, x_1 - x_o)$ and $(x_1 - \alpha, x_1 + \alpha) \subset (x_o + c, x_o + d)$. By our assumption about the point x_1, we know that

$$\mathrm{int}(\{t > 0: |f(x_1 + t) - f(x_1 - t)| < \tfrac{\gamma}{2}\}) \cap (0, \alpha) \neq \emptyset.$$

Let $(c_1, d_1) \subset \mathrm{int}\{t > 0: |f(x_1 + t) - f(x_1 - t)| < \frac{\gamma}{2}\}) \cap (0, \alpha)$ and $(a, b) = x_o + (c_1, d_1)$. Then, for each $y \in (a, b)$, we have $y - x_o \in (c_1, d_1)$ and $x_1 - y \in (c, d)$. Therefore

$$|f(x_1 + (y - x_o)) - f(x_1 - (y - x_o))| < \frac{\gamma}{2},$$

$$|f(x_o + (x_1 - y)) - f(x_o - (x_1 - y))| < \frac{\gamma}{2}$$

and, by the above,

$$|f(y + (x_1 - x_o)) - f(y - (x_1 - x_o)| < \gamma.$$

Now, let x_o be a point of the symmetric τ-continuity of the function f, and let $\delta > 0$, $\gamma > 0$. Then there exists a set $F = \bigcup_{n=1}^{\infty} [a_n, b_n]$ such that $0 \in \phi^+(F)$ and $F \subset \mathrm{int}(\{t > 0: |f(x_o + t) - f(x_o - t)| < \frac{\gamma}{2}\})$. By the assumption, we know that, for each $n \in N$, the set $\{x \in x_o + [a_n, b_n]$: f is symmetrically τ-continuous at $x\}$ is a residual subset of $x_o + [a_n, b_n]$ and, therefore, the proof of the lemma is completed.

LEMMA 2. Let $f: R \to R$, $\alpha \in R$, $\alpha > 0$ and

$$B = \{x \in R: \text{if } F = \mathrm{cl}(F) \text{ and } o \in \phi^+(F) \text{ then } \mathrm{int}\, F \not\subset \{t > 0: |f(x + t) - f(x)| < \alpha\}\}.$$

If (a, b) is an interval such that $\mathrm{cl}(B) \supset [a, b]$, then for any sets F and A such that $A \subset F$ and:

1) $F = \bigcup_{n=1}^{\infty} [a_n, b_n]$,

2) $0 \in \phi^{+}(F)$,

3) for each $n \in N$, $[a_n, b_n] \setminus A \in \mathbf{I}$,

and, for each $(c, d) \subset (a, b)$ there exist two points $x_1, x_2 \in (c, d)$ such that $x_2 - x_1 \in A$ and $|f(x_1) - f(x_2)| \geq \frac{\alpha}{2}$.

P r o o f. Let $y_1 \in (c, d) \cap B$ and $C = \{t > 0: |f(y_1 + t) - f(y_1)| \geq \frac{\alpha}{2}\}$. We assume that n_o is a natural number such that $C \cap (a_{n_o}, b_{n_o})$ is a subset of the second category of (a_{n_o}, b_{n_o}). Then $A \cap (a_{n_o}, b_{n_o}) \cap C \neq \emptyset$. Let $t_o \in A \cap (a_{n_o}, b_{n_o}) \cap C$. Then $|f(y_1 + t_o) - f(y_1)| \geq \frac{\alpha}{2}$ and $y_1 + t_o - y_1 \in A$. Thus, we put $x_1 = y_1$, $x_2 = y_1 + t_o$.

Now, we assume that, for each $n \in N$, $C \cap [a_n, b_n]$ is a subset of the first category of $[a_n, b_n]$. We denote, for each $n \in N$, $D_n = \{t \in [a_n, b_n]: |f(y_1 + t) - f(y_1)| < \frac{\alpha}{2}\}$. Then $[a_n, b_n] \setminus D_n \in \mathbf{I}$. Let $F_1 = (F \cap (0, d - y_1)) \cup \{0\}$. Then $0 \in \phi^{+}(F_1)$. We know that $y_1 \in B$ and, therefore, $\operatorname{int}(F_1) \not\subset \{t > 0: |f(y_1 + t) - f(y_1)| < \alpha\}$. Let t_1 be a point such that

$$t_1 \in \operatorname{int} F_1 \cap \{t > 0: |f(y_1 + t) - f(y_1)| \geq \alpha\} \neq \emptyset.$$

Let k be a natural number such that $t_1 \in (a_k, b_k)$. Then, for each $t \in D_k$, we have

$$\begin{aligned} &|f(y_1 + t_1) - f(y_1 + t)| \\ &\geq |f(y_1 + t_1) - f(y_1)| - |f(y_1) - f(y_1 + t)| \\ &\geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}. \end{aligned}$$

Let $\gamma > 0$ be such that $t_1 + \gamma < \min(b_k, d - y_1)$. The set $A \cap (0, \gamma)$ is a subset of the second category of $(0, \gamma)$ and, therefore, $t_1 + (A \cap (0, \gamma))$ is a subset of the second category of $(t_1, t_1 + \gamma) \subset (a_k, b_k)$. Thus there exists a point $t_2 \in (t_1 + (A \cap (0, \gamma)) \cap D_k$, and $t_2 > t_1$.

Then

$$c < y_1 < y_1 + t_1 < y_1 + d - y_1 = d,$$

$$c < y_1 < y_1 + t_2 < y_1 + \gamma < y_1 + d - y_1 = d,$$

and

$$|f(y_1 + t_1) - f(y_1 + t_2)| \geq \frac{\alpha}{2}.$$

Now, we put $x_1 = y_1 + t_1$, $x_2 = y_1 + t_2$ Then $x_2 - x_1 = t_2 - t_1 \in A$ and the proof of the lemma is completed.

LEMMA 3. If $\{x \in R$: f is right-hand continuous at x with respect to the topology $\tau\}$ is a dense subset of R, then $f \in C_{I\text{-a.e.}}$

P r o o f. Suppose that there exist a natural number n and an open interval (a, b) such that, for each $(c, d) \subset (a, b)$ there exist $x_1, x_2 \in (c, d)$ such that $|f(x_1) - f(x_2)| \geq \frac{1}{n}$. Let $x_o \in (a, b)$ be a point of right-hand continuity with respect to the topology τ of the function f. Then int $\{x > x_o: |f(x) - f(x_o)| < \frac{1}{2n}\} \cap (x_o, b) \neq \emptyset$. Therefore, there exists an interval (c, d) such that $(c, d) \subset \text{int}\, \{x > x_o: |f(x) - f(x_o)| < \frac{1}{2n}\} \cap (a, b)$. Then, for any $x_1, x_2 \in (c, d)$, $|f(x_1) - f(x_2)| < \frac{1}{n}$, which gives a contradiction with our assumption. Thus $f \in C_{I\text{-a.e.}}$.

THEOREM 5. Let $f: R \to R$ and $f \in \tau\text{-}\mathbf{SC}_{I\text{-a.e.}}$, then $f \in \mathbf{C}_{I\text{-a.e.}}$.

P r o o f. By Lemma 3, we may reduce our consideration to the case if $\{x \in R$: f is right-hand continuous with respect to the topology τ at $x\}$ is not a dense subset of R. Thus, if

$$A = \bigcup_{n=1}^{\infty} \{x \in R: \text{if } F = \text{cl}\,(F) \text{ and } 0 \in \phi^+(F), \text{ then}$$

$$\text{int}\, F \not\subset \{t > 0: |f(x + t) - f(x)| < \tfrac{1}{n}\} = \bigcup_{n=1}^{\infty} A_n,$$

there exist an open interval (a, b) and a natural number n such that $\text{cl}\,(A_n \cap (a, b)) = [a, b]$.

Let, for any natural k, p, for each $h < \frac{1}{p}$ and for each $i \in \{1, \ldots, n\}$,

$$B_{kphi} = \{x \in (a, b): r \in [\tfrac{i-1}{k}h, \tfrac{i}{k}h]$$

$$\Longrightarrow |f(x + r) - f(x - r)| < \tfrac{1}{6n}\}.$$

By our assumption, we have

$$\{x \in (a, b): f \text{ is symmetrically } \tau\text{-continuous at } x\}$$
$$= \bigcup_{k=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{h<\frac{1}{p}} \bigcup_{i=1}^{k} B_{kphi}$$

(see lemma [4]). Therefore, there exist natural numbers k, p such that $S = \bigcap_{h<\frac{1}{p}} \bigcup_{i=1}^{k} B_{kphi}$ is a subset of the second category of (a, b). Let $a_1 = a$, $b_1 < b$ be such that S is a subset of the second category of (a_1, b_1) and let $h<\frac{1}{p}$ be such that $b_1 + h < b$. Then there exist $i \in \{1, ..., k\}$ and an open interval $(a_2, b_2) \subset$ $\subset (a_1, b_1)$ such that B_{kphi} is a dense subset of (a_2, b_2). We may assume that $b_2 - a_2 < \frac{1}{2k} \cdot h$. Then $\frac{i}{k}h + a_2 - \frac{i-1}{k}h - b_2 > 0$.

Let $(c, d) \subset [\frac{i-1}{k}h + b_2, \frac{i}{k}h + a_2]$ be such that $d - c <$ $< 2(b_2 - a_2)$ and let $x \in (a_2, b_2) \cap B_{kphi}$ and $x_1 \in (c, d)$. Then $x_1 - x \in [\frac{i-1}{k}h, \frac{i}{k}h]$ and $|f(x + (x_1 - x)) - f(x - (x_1 - x))| <$ $< \frac{1}{6n}$. Now, we put $c' = 2a_2 - c$, $d' = 2b_2 - c$, $c'' = 2a_2 - d$, $d'' = 2b_2 - d$. Then $d'' - c' > 0$.

Let $x_o \in (c', d'')$ be a point of the symmetric τ-continuity of the function f. By Lemma 1, there exists a set $F = \bigcup_{n=1}^{\infty}[a_n, b_n]$ such that $0 \in \phi^+(F)$ and, for any natural n and a positive real δ,

$$D_n = \{x \in x_o + [a_n, b_n]: \operatorname{int}(\{t: |f(t + (x - x_o))$$
$$- f(t - (x - x_o))| < \tfrac{1}{6n}\} \cap (x_o, x_o + \delta) \neq \emptyset\}$$

is a residual subset of $[a_n, b_n] + x_o$. Let $D = \bigcup_{n=1}^{\infty} D_n$, $F_1 = 2 \cdot F$, $D_1 = 2(D - x_o)$. Then F_1 and D_1 satisfy the assumptions of Lemma 2. We may assume that $F \subset (x_o, d'') - x_o$. Therefore, there exist $x_2, x_3 \in (c, d)$ such that $x_3 - x_2 \in D_1$ and $|f(x_2) - f(x_3)| \geq \frac{1}{2n}$. Then $\frac{1}{2}(x_3 - x_2) \in D - x_o \subset F$ and $x_o + \frac{1}{2}(x_3 - x_2) \in D \subset F + x_o \subset$ $\subset (x_o, d'')$. We put $x_1 = x_o + \frac{1}{2}(x_3 - x_2) < d''$. There exists an open interval $(c_1, d_1) \subset \operatorname{int}\{t \in R: |f(t + (x_1 - x_o)) - f(t - (x_1 - x_o))| < \frac{1}{6n}\} \cap (x_o, x_1 - x_o)$, which means that, for each $t \in (c_1, d_1)$, $|f(t + (x_1 - x_o)) - f(t - (x_1 - x_o))| < \frac{1}{6n}$.

Let $\xi_1 = \frac{1}{2}(x_3 + x_2)$, $x' \in (\frac{\xi_1 + c_1}{2}, \frac{\xi_1 + d_1}{2}) \cap B_{kphi} \subset (a_2, b_2) \cap B_{kphi}$ and $x^* = 2x' - \xi_1 \in (c_1, d_1)$. Then

$$|f(x' + (x_2 - x')) - f(x' - (x_2 - x'))| < \frac{1}{6n},$$
$$|f(x' + (x_3 - x')) - f(x' - (x_3 - x'))| < \frac{1}{6n}$$

and

$$x' - (x_2 - x') = x^* + (x_1 - x_o),$$
$$x' - (x_3 - x') = x^* - (x_1 - x_o).$$

Since $x^* \in (c_1, d_1)$ therefore

$$|f(x' - (x_2 - x')) - f(x' - (x_3 - x'))|$$
$$= |f(x^* + (x_1 - x_o)) - f(x^* - (x_1 - x_o))| < \frac{1}{6n}.$$

Thus

$$|f(x_2) - f(x_3)| \le |f(x' + (x_2 - x')) - f(x' - (x_2 - x'))|$$
$$+ |f(x' - (x_2 - x')) - f(x' - (x_3 - x'))|$$
$$+ |f(x' - (x_3 - x')) - f(x' + (x_3 - x'))| < \frac{1}{2n},$$

which gives a contradiction because $|f(x_2) - f(x_3)| \ge \frac{1}{2n}$. Therefore the proof of the theorem is completed.

PROPOSITION 4. If a function f is symmetrically continuous at $x \in R$, then f is symmetrically τ-continuous at x. There exists a function $f \in \tau\text{-}\mathbf{SC}\setminus\mathbf{SC}$.

P r o o f. The Dirichlet function satisfies the required condition.

By Theorems 5 and 3 and by the above proposition, we have $\tau\text{-}\mathbf{SC} \subset \tau\text{-}\mathbf{SC}_{I\text{-a.e.}} \subset \mathbf{C}_{I\text{-a.e.}} = \mathbf{SC}_{I\text{-a.e.}} \subset \tau\text{-}\mathbf{SC}_{I\text{-a.e.}}$ and therefore, $\tau\text{-}\mathbf{SC}_{I\text{-a.e.}} = \mathbf{SC}_{I\text{-a.e.}} = \mathbf{C}_{I\text{-a.e.}}$.

Thus, we have:

COROLLARY 4. There exists a function f nonmeasurable in the sense of Lebesgue, such that $f \in \tau\text{-}SC_{I\text{-a.e.}}$.

By [1] and by the above, we have:

COROLLARY 5. There exists a function f nonmeasurable in the sense of Borel such that $f \in \tau\text{-}SC$.

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O PEWNYCH UOGÓLNIENIACH SYMETRYCZNEJ CIĄGŁOŚCI

W pracach [7] i [5] zostały wprowadzone pojęcia I-ciągłości i τ-ciągłości. W tej pracy podane są definicje uogólnień symetrycznej ciągłości funkcji f: R → R, a mianowicie symetrycznej I-ciągłości oraz symetrycznej τ-ciągłości. Udowodnione są również pewne własności tych uogólnień oraz inkluzje zachodzące pomiędzy klasami funkcji ciągłych, symetrycznie ciągłych, symetrycznie I-ciągłych oraz symetrycznie τ-ciągłych.