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THE OSCILLATORY BEHAVIOUR OF $\mathcal{J}$-APPROXIMATE DERIVATIVES

The derivatives considered are the $\mathcal{J}$-approximate derivatives. We shall prove that if $f'_{\mathcal{J}\text{-ap}}$ exists everywhere on [a, b] and is bounded above or below on (a, b), then $f'_{\mathcal{J}\text{-ap}} = f'$ on [a, b] (one-sided at a and b).

Since the $\mathcal{J}$-approximate derivative possesses the Darboux property, the above forces $f'_{\mathcal{J}\text{-ap}}$ to attain every value indeed infinitely often on any interval where $f'_{\mathcal{J}\text{-ap}}$ is not f'. Thus $f'_{\mathcal{J}\text{-ap}}$ must oscillate between positive and negative values whose absolute value may be as large as desired.

On the other hand, since $\mathcal{J}$-approximate derivative is a function of Baire class one, the above implies the existence of an open dense subset V of I_o on which $f'_{\mathcal{J}\text{-ap}}$ is f'. So the question arises whether the oscillation mentioned in the above paragraph occurs on the component intervals of this set V. In what follows, an affirmative answer is furnished to this question.

Let R be the real line, N the set of all natural numbers, $\mathcal{B}$ the σ-algebra of subsets of R having the Baire property, $\mathcal{J}$ the σ-ideal of sets of the first category. If $A \subset R$ and $x_o \in R$, denote $x_o \cdot A = \{x_o \cdot x: x \in A\}$ and $A - x_o = \{x - x_o: x \in A\}$; χ_A will mean the characteristic function of the set A.

Recall that 0 is an I-density point of a set $A \in \mathcal{B}$ if and only if, for every increasing sequence $\{n_m\}_{m\in N}$ of natural numbers, there exists a subsequence $\{n_{m_p}\}_{p\in N}$ such that $\chi_{n_{m_p}} \cdot A \cap [-1, 1] \xrightarrow[p\to\infty]{} 1$ except on a set belonging to $\mathcal{J}$. Further, x_o is an $\mathcal{J}$-density point of $A \in \mathcal{B}$ (denoted by $d_{\mathcal{J}}(A, x_o) = 1$) if and only if 0 is an $\mathcal{J}$-density point of $A - x_o \cdot A$ point x_o is an $\mathcal{J}$-dispersion point of $A \in \mathcal{B}$ (denoted by $d_{\mathcal{J}}(A, x_o) = 0$) if and only if $d_{\mathcal{J}}(R \setminus A, x_o) = 1$ (see [4]).

Throughout this paper, all functions are real-valued functions of one variable. The notations cl(E) and int(E) will denote, respectively, the closure and the interior of E in the natural topology.

DEFINITION 1. Let f be any function defined in some neighbourhood of x_o and having there the Baire property.

$$\mathcal{J}\text{-}\liminf_{x \to x_o} f(x) = \sup \{\alpha: d_{\mathcal{J}}(\{x: f(x) < \alpha\}, x_o) = 0\},$$

$$\mathcal{J}\text{-}\limsup_{x \to x_o} f(x) = \inf \{\alpha: d_{\mathcal{J}}(\{x: f(x) > \alpha\}, x_o) = 0\}.$$

We shall say that f is $\mathcal{J}$-approximately continuous at x_o if and only if

$$\mathcal{J}\text{-}\liminf_{x \to x_o} f(x) = \mathcal{J}\text{-}\limsup_{x \to x_o} f(x) = f(x_o).$$

DEFINITION 2. Let f be any function defined in some neighbourhood of x_o and having there the Baire property, and let

$$C(x, x_o) = \frac{f(x) - f(x_o)}{x - x_o} \quad \text{for} \quad x \neq x_o.$$

We shall define the $\mathcal{J}$-approximate upper derivative as

$$\overline{f}'_{\mathcal{J}\text{-ap}}(x_o) = \mathcal{J}\text{-}\limsup_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o}.$$

The $\mathcal{J}$-approximate lower derivative $\underline{f}'_{\mathcal{J}\text{-ap}}(x_o)$ is defined similarly. If $\overline{f}_{\mathcal{J}\text{-ap}}(x_o) = \underline{f}_{\mathcal{J}\text{-ap}}(x_o)$, their common value is called the $\mathcal{J}$-approximate derivative of f at x_o, $f'_{\mathcal{J}\text{-ap}}(x_o)$.

To prove the above-mentioned results, we need a preliminary lemmas and some theorems:

THEOREM 1 [1]. Let G be an open subset of R. A point 0 is an $\mathcal{J}$-dispersion point of G if and only if, for each $n \in N$, there exist $k \in N$ and a real number $\delta > 0$ such that, for each $h \in (0, \delta)$ and for each $i \in \{1, ..., n\}$, there exist $j_r, j_l \in \{1, ..., k\}$ such that

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j_r - 1}{nk}\right)h,\ \left(\frac{i-1}{n} + \frac{j_r}{nk}\right)h\right) = \emptyset$$

and

$$G \cap \left(-\left(\frac{i-1}{n} + \frac{j_1}{nk}\right)h,\ -\left(\frac{i-1}{n} + \frac{j_1 - 1}{nk}\right)h\right) = \emptyset.$$

We shall use the above theorem for $x \in R$ by translating the set, if necessary. It is easily seen that if G in theorem 1 is replaced by an arbitrary set $A \in \mathfrak{B}$, then in the above conditions we should write

$$A \cap \left(\left(\frac{i-1}{n} + \frac{j_r - 1}{nk}\right)h,\ \left(\frac{i-1}{n} + \frac{j_r}{nk}\right)h\right) \in \mathcal{J}$$

and

$$A \cap \left(-\left(\frac{i-1}{n} + \frac{j_1}{nk}\right)h,\ -\left(\frac{i-1}{n} + \frac{j_1^{-1}}{nk}\right)h\right) \in \mathcal{J}.$$

THEOREM 2 [2]. Let $f\colon [0, 1] \to R$ have a finite $\mathcal{J}$-approximate derivative $f'_{\mathcal{J}\text{-ap}}(x)$ for all $x \in [0, 1]$. Then:

(a) the function f is a function of Baire class one;

(b) the function f has the Darboux property;

(c) the function $f'_{\mathcal{J}\text{-ap}}$ has the Darboux property.

THEOREM 3 [1]. If $f\colon [0, 1] \to R$ has a finite $\mathcal{J}$-approximate derivative $f'_{\mathcal{J}\text{-ap}}(x)$ at all $x \in [0, 1]$, then $f'_{\mathcal{J}\text{-ap}}$ is of Baire class one.

THEOREM 4 [2]. If f is $\mathcal{J}$-approximately differentiable on $[0, 1]$ and $f'_{\mathcal{J}\text{-ap}}(x) \geq 0$ at each $x \in [0, 1]$, then f is non-decreasing on $[0, 1]$.

THEOREM 5 [2]. Let f be an increasing function defined on $[0, 1]$. For each $x_o \in (0, 1)$, $D_+f(x_o) = D_{+\mathcal{J}\text{-ap}}f(x_o)$. The corresponding equalities for the other extrema derivates and extrema $\mathcal{J}$-approximate derivates are also valid.

THEOREM 6. If $f'_{\mathcal{J}\text{-ap}}$ exists everywhere on $[a, b]$ and is bounded above or below on (a, b), then $f'_{\mathcal{J}\text{-ap}} = f'$ on $[a, b]$ (one-sided at a and b).

P r o o f. We shall assume that there exists a real number M such that, for each $x \in (a, b)$, $f'_{\mathcal{J}\text{-ap}}(x) > M$ and let $h(x) =$

$= f(x) - Mx$ for each $x \in [a, b]$. For each $[c, d] \subset (a, b)$ and for each $x \in [c, d]$, $h'_{\mathcal{I}\text{-ap}}(x) > 0$. Then, by Theorem 4, the function h is nondecreasing on $[c, d]$. It is easy to see that h is increasing on $[c, d]$. Then, by the Darboux property, the function h is increasing on $[a, b]$. By Theorem 5, we have that $h'(x) = h'_{\mathcal{I}\text{-ap}}(x)$ at all $x \in (a, b)$. In the similar way as in Theorem 5, we can prove that $h'^{+}(a) = h'^{+}_{\mathcal{I}\text{-ap}}(a)$ and $h'^{-}(b) = h'^{-}_{\mathcal{I}\text{-ap}}(b)$. Therefore $f' = f'_{\mathcal{I}\text{-ap}}$ on $[a, b]$ (one-sided at a and b), and the proof of Theorem 6 is completed.

LEMMA 1 ([3]). Let f be a function, x a point in the domain of f, λ a real number and K a positive number. If, for each $\varepsilon > 0$, there is a $\delta > 0$ such that $0 < |y - x| < \delta$ and y in the domain of f imply there are numbers y_1 and y_2 with $y_1 < y < y_2$, satisfying:

(1) $|f(y_i) - f(x) - \lambda(y_i - x)| < \varepsilon|y_i - x|$ for $i = 1, 2$,

(2) $|y_i - y| < \varepsilon|y - x|$ for $i = 1, 2$ and

(3) a) $f(z) + Kz$ is increasing on $[y_1, y_2]$ or

b) $f(z) - Kz$ is decreasing on $[y_1, y_2]$ or

c) $|f(y_i) - f(y)| \le K|y_i - y|$ for $i = 1$ or $i = 2$,

then $f'(x)$ exists and equals λ.

LEMMA 2. Let f be a function and x a point at which f has an $\mathcal{I}$-approximate derivative $f'_{\mathcal{I}\text{-ap}}(x) = \lambda$. Let $\varepsilon > 0$ be given. There is a $\delta > 0$ such that $0 < |y - x| < \delta$ implies there are numbers y_1 and y_2 with $y_1 < y < y_2$, satisfying:

(1) $|f(y_i) - f(x) - \lambda(y_i - x)| < \varepsilon|y_i - x|$ for $x = 1, 2$ and

(2) $|y_i - y| < \varepsilon|y - x|$ for $i = 1, 2$.

P r o o f. It suffices to consider just $0 < \varepsilon < 1$. Let $A = \{t: |f(t) - f(x) - \lambda(t - x)| < \varepsilon|t - x|\}$. Then $d_{\mathcal{I}}(A, x) = 1$ and, by Theorem 1, for $n \in N$ such that $n \ge 3$ and $\frac{1}{n - 1} < \varepsilon$, there exists a $\delta_1 > 0$ such that, for each $h \in (0, \delta_1)$ and for each $i \in \{1, \ldots, n\}$,

$$A \cap (x + \frac{i - 1}{n}h,\ x + \frac{i}{n}h) \notin \mathcal{I},$$

$$A \cap (x - \frac{i}{n}h,\ x - \frac{i - 1}{n}h) \notin \mathcal{I}.$$

Now, let $\delta = \frac{n-1}{n}\delta_1$ and y be fixed with $0 < |x - y| < \delta$. It may be assumed without loss of generality that $y > x$. Let $h = \frac{n}{n-1}(y - x)$. Then $h < \delta_1$ and $y = x + \frac{n-1}{n}h$. Therefore $(x + \frac{n-2}{n}h, y) \cap A \neq \emptyset$ and $(y, x + h) \cap A \neq \emptyset$, which implies the existence of two points $y_1 < y < y_2$ such that $y_1, y_2 \in A$, $|y - y_1| < \frac{1}{n}h < \frac{1}{n-1}|y - x| < \varepsilon|y - x|$ and $|y - y_2| < \frac{1}{n}h < \varepsilon|y - x|$, which completes the proof of the lemma.

LEMMA 3. Suppose of is $\mathcal{T}$-approximately continuous on an interval I_o. Let $K > 0$ be given and let $A(x) = \{y: |f(y) - f(x)| < K|y - x|\}$. Let $n, m, p \in N$ and $H_{nmp} = \{x$: for each $h \in (0, \frac{1}{p})$, there exist $i_l(x), i_p(x) \in \{1, \ldots, n\}$ such that

$$(x - \frac{i_l(x)}{n}h,\ x - \frac{i_l(x) - 1}{n}h)\setminus A(x) \in \mathcal{T}$$

and

$$(x + \frac{i_r(x) - 1}{n}h,\ x + \frac{i_r(x)}{n}h)\setminus A(x) \in \mathcal{T},$$

and, for each $i \in \{1, \ldots, n\}$, there exist $j_l(x, i), j_r(x, i) \in \{1, \ldots, m\}$ such that

$$(x - (\frac{i-1}{n} + \frac{j_l(x, i)}{nm})h,\ x - (\frac{i-1}{n} + \frac{j_l(x,i)-1}{nm})h)\setminus A(x) \in \mathcal{T}$$

and

$$(x + (\frac{i-1}{n} + \frac{j_r(x, i)-1}{nm})h,\ x - (\frac{i-1}{n} + \frac{j_r(x,i)}{nm})h)\setminus A(x) \in \mathcal{T}.$$

Then:

(a) if $x, y \in cl(H_{nmp})$ and $|x - y| < \frac{1}{p}$, then

$$|f(x) - f(y)| \leq K|x - y|,$$

(b) if $x \in cl(H_{nmp})$ and $h < \frac{1}{p}$, then for each $i \in \{1,\ldots,n\}$,

$$(x + \frac{i-1}{n}h,\ x + \frac{i}{n}h) \cap \{y: |f(y) - f(x)| \leq K|y - x|\} \notin \mathcal{T}$$

and

$$(x - \frac{i}{n}h,\ x - \frac{i-1}{n}h) \cap \{y: |f(y) - f(x)| \leq K|y - x|\} \notin \mathcal{T}.$$

P r o o f. Let $x, y \in cl(H_{nmp})$ and $|x - y| < \frac{1}{p}$. It may be assumed without loss of generality that $y > x$. Since f is an $\mathcal{T}$-approximately continuous function at x and y, thus, by Theorem 1, for each $s \in N$, there exists $\delta > 0$ such that, for each $h \in (0, \delta)$ and for each $j \in \{1, ..., n\}$,

$$(x + \frac{j-1}{n}h, x + \frac{j}{n}h) \cap \{t: |f(t) - f(x)| < \frac{1}{s}\} \notin \mathcal{T},$$

$$(x - \frac{j}{n}h, x + \frac{j-1}{n}h) \cap \{t: |f(t) - f(x)| < \frac{1}{s}\} \notin \mathcal{T},$$

$$(y + \frac{j-1}{n}h, y + \frac{j}{n}h) \cap \{t: |f(t) - f(y)| < \frac{1}{s}\} \notin \mathcal{T},$$

$$(y - \frac{j}{n}h, y - \frac{j-1}{n}h) \cap \{t: |f(t) - f(y)| < \frac{1}{s}\} \notin \mathcal{T}.$$

Let $\delta_o > 0$ be such that $\delta_o < \min(\frac{1}{sK}, \frac{1}{p}, \delta)$ and $|x - y| + 2\delta_o < \frac{1}{p}$. We choose $x_1 \in (x - \delta_o, x + \delta_o) \cap H_{nmp}$ and $y_1 \in (y - \delta_o, y + + \delta_o) \cap H_{nmp}$. We may assume that $x_1 < x < y_1 < y$. Then $x - x_1 < \delta$ and, for each $j \in \{1, ..., n\}$,

$$(x - \frac{j}{n}(x - x_1), x - \frac{j-1}{n}(x - x_1) \cap \{t: |f(t) - f(x)| < \frac{1}{s}\} \in \mathcal{T}$$

and there exists $i_p(x_1) \in \{1, ..., n\}$ such that

$$(x_1 + \frac{i_p(x_1) - 1}{n}(x - x_1), x_1 + \frac{i_p(x_1)}{n}(x - x_1))\setminus A(x_1) \in \mathcal{T}.$$

Thus there exists $x_1' \cap (x_1, x) \cap A(x_1) \cap \{t: |f(t) - f(x)| < \frac{1}{s}\}$. Analogously, we can choose $y_1' \in (y_1, y) \cap A(y_1) \cap \{t: |f(t) - - f(y)) y \frac{1}{s}\}$. Since $x_1, y_1 \in H_{nmp}$ and $|y_1 - x_1| \le |y_1 - y| + +|x - y|+|x - x_1| < 2\delta_o+|x - y| < \frac{1}{p}$ there exists $i_r(x_1) \in \{1, ..., n\}$ such that

$$(x_1 + \frac{i_r(x_1) - 1}{n}(y_1 - x_1), x_1 + \frac{i_r(x_1)}{n}(y_1 - x_1))\setminus A(x_1) \in \mathcal{T},$$

and, for each $j \in \{1, ..., n\}$,

$$(y_1 - \frac{j}{n}(y_1 - x_1), y_1 - \frac{j-1}{n}(y_1 - x_1)) \cap A(y_1) \notin \mathcal{T}.$$

So, there exists $x_o \in (x_1, y_1) \cap A(x_1) \cap A(y_1)$, and $|f(x) - f(y) \le$

$\leq |f(x) - f(x_1')| + |f(x_1') - f(x_1)| + |f(x_1) - f(x_o)| + |f(x_o) - f(y_1)| + |f(y_1) - f(y_1')| + |f(y_1') - f(y)| < \frac{1}{s} + K|x_1' - x_1| + K|x_1 - x_o| + K|x_o - y_1| + K|y_1 - y_1'| + \frac{1}{s} < \frac{4}{s} + K|x_1 - y_1| \leq \frac{4}{s} + K|x_1 - x_1'| + K|x_1' - x| + K|x - y| + K|y - y_1'| + K|y_1' - y_1| < \frac{8}{s} + K|x - y|$. Thus $|f(x) - f(y)| \leq \lim_{s\to\infty} (\frac{8}{s} + K|x - y|) = K|x - y|$.

Now, let $x \in cl(H_{nmp})$, $0 < h < \frac{1}{p}$ and $i \in \{1, ..., n\}$. Let $0 < \delta < \frac{1}{4nmp} \cdot h$ and $\{x_s\}_{s\in N} \subset H_{nmp}$ such that $x = \lim_{s\to\infty} x_s$ and, for each $s \in N$, $x_s \in (x - \delta, x + \delta)$. Then, for each $s \in N$, there exists $j(x_s, i) \in \{1, ..., m\}$ such that

$$(x_s + \frac{(i-1)m + j(x_s, i) - 1}{mn}h,\ x_s + \frac{(i-1)m + j(x_s, i)}{mn}h)\setminus A(x_s) \in \mathcal{T}.$$

Let $\{x_{s_r}\}_{r\in N} \subset \{x_s\}_{s\in N}$ such that, for all $r \in N$, $j(x_{s_r}, i)$ is cammon (for example, for each $s \in N$, $j(x_{s_r}, i) = j_o$). Then, for each $r \in N$,

$$(\alpha, \beta) = (x + \delta + \frac{(i-1)m + j_o - 1}{nm}h,\ x - \delta + \frac{(i-1)m + j_o}{nm}h) \subset (x_{s_r} + \frac{(i-1)m + j_o - 1}{nm}h,\ x_{s_r} + \frac{(i-1)m + j_o}{nm}h)$$

and $(\alpha, \beta)\setminus A(x_{s_r}) \in \mathcal{T}$. Thus $(\alpha, \beta)\setminus \bigcap_{r\in N} A(x_{s_r}) \in \mathcal{T}$ and, moreover, $(\alpha, \beta) \subset (x + \frac{i-1}{n}h,\ x + \frac{i}{n}h)$. Let $y \in \bigcap_{r\in N} A(x_{s_r}) \cap (\alpha, \beta)$. Then, for each $r \in N$,

$$|f(x) - f(y)| \leq |f(x) - f(x_{s_r})| + |f(x_{s_r}) - f(y)| < K|x - x_{s_r}| + K|x_{s_r} - y|$$

and

$$|f(x) - f(y)| \le \lim_{r\to\infty} (K|y - x_{s_r}| + K|x_{s_r} - x|) = K|x - y|.$$

Thus we have shown that

$$(x + \frac{i-1}{n}h,\ x + \frac{i}{n}h) \cap \{t\colon |f(t) - f(x)| \le K|t - x|\} \notin \mathcal{T}.$$

The proof of the second condition is analogous.

By the above lemmas, we shall prove the main result of this paper. Its proof is similar to that of [3, Theorem 4.1]. We shall denote by I and I_o arbitrary intervals.

THEOREM 7. Suppose f has a finite $\mathcal{T}$-approximate derivative $f'_{\mathcal{T}\text{-ap}}(x)$ at each $x \in I_o$ and let $M \ge 0$. If $f'_{\mathcal{T}\text{-ap}}$ attains both M and -M on I_o, then there is a subinterval I of I_o on which $f'_{\mathcal{T}\text{-ap}} = f'$ and f' attains both M and -M on I.

P r o o f. Suppose no such interval I exists. Then, for each interval $I \subset I_o$ on which $f'_{\mathcal{T}\text{-ap}} = f'$, we have $f'(y) > -M$ for all $y \in I$ or $f'(y) < M$ for all $y \in I$, for otherwise the Darboux property of f' would imply that f' attains M and -M on I. Let $V = \{x \in I_o\colon$ there is an open interval $I \subset I_o$ such that $x \in I$ and $f'_{\mathcal{T}\text{-ap}}(y) = f'(y)$ for all $y \in I\}$. By Theorems 2 and 6, V is an open dense subset of I_o. Since $f' > -M$ or $f' < M$ on each component (a, b) of V, it follows from Theorem 6 that f has a right-sided derivative at b and f has a left-sided derivative at a. Thus the set $I_o \setminus V = P$ is a perfect nowhere dense set.

Since the function $f'_{\mathcal{T}\text{-ap}}$ is Baire 1, P contains points at which $f'_{\mathcal{T}\text{-ap}}$ is continuous relative to P. At any such point x_o, $|f'_{\mathcal{T}\text{-ap}}(x_o)| \le M$. Suppose that $f'_{\mathcal{T}\text{-ap}}(x_o) > M$. (A similar argument holds if $f_{\mathcal{T}\text{-ap}} < -M$. Then there is an open interval I containing x_o for which $f'_{\mathcal{T}\text{-ap}}(x) > M$ for $x \in I \cap P$. For any component (a, b) of V with $(a, b) \subset I$, a is in $I \cap P$, and thus, $f'_{\mathcal{T}\text{-ap}}(a) > M$. Hence $f'_{\mathcal{T}\text{-ap}}(x) > -M$ for $x \in (a, b)$. By combining these two facts, it follows that $f'_{\mathcal{T}\text{-ap}} > -M$ on I. Therefore, by Theorem 6, $I \subset V$, which contradicts $x_o \in P$.

Now, by selecting any point x_o of P at which $f'_{\mathcal{T}\text{-ap}}$ is continuous at x_o relative to P, we can choose an open interval (c,

d) with c and d in V, $c < x_o < d$ and $|f'_{\mathcal{T}\text{-ap}}(x)| < M + 1$ on $(c, d) \cap P$. Then, for $K = M + 1$, the sets H_{nmp} defined as in Lemma 3 have the property that $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{p=1}^{\infty} H_{nmp} \cap P_1 = P_1$ where $P_1 = P \cap [c_1, d_1]$ and $[c_1, d_1] \cap (c, d)$. Indeed, let $x \in P_1$. Then $|f'_{\mathcal{T}\text{-ap}}(x)| < K$ and $d_{\mathcal{T}}(\{t: |f(t) - f(x)| < K|t - x|\}, x) = 1$. Therefore, by Theorem 1, there exist $q, n \in N$ such that, for each $0 < h < \frac{1}{q}$, there exist $i_r(x), i_l(x) = \{1, ..., n\}$ such that

$$(x + \frac{i_r(x) - 1}{n}h,\ x + \frac{i_r(x)}{n}h)\setminus\{t: |f(t) - f(x)| < K|t - x|\} \in \mathcal{T}$$

and

$$(x - \frac{i_l(x)}{n}h,\ x - \frac{i_l(x) - 1}{n}h)\ \{t: |f(t) - f(x)| < K|t - x|\} \in \mathcal{T}$$

Again by Theorem 1, there exist $m, r \in N$ for n, such that, for each $0 < h < \frac{1}{r}$ and for each $i \in \{1, ..., n\}$, there exist $j_l(x,i)$, $j_r(x, i) \in \{1, ..., m\}$ such that

$$(x + (\frac{i - 1}{n} + \frac{j_r(x, i) - 1}{nm})h,\quad x + (\frac{i - 1}{n} + \frac{j_r(x, i)}{nm})h)\setminus$$
$$\setminus\{t: |f(t) - f(x)| < K|t - x|\} \in \mathcal{T}.$$

and

$$(x - (\frac{i - 1}{n} + \frac{j_l(x, i)}{nm})h,\quad x - (\frac{i - 1}{n} + \frac{j_l(x, i) - 1}{nm})h)\setminus$$
$$\setminus\{t: |f(t) - f(x)| < K|t - x|\} \in \mathcal{T}.$$

Now, let $p \ge \max\{q, s\}$. Then $x \in H_{nmp}$ and $P_1 = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{p=1}^{\infty} H_{nmp} \cap P_1$. The Baire category theorem quarantees the existence of integers n_o, m_o, p_o and an open interval $J \subset (c, d)$ with $J \cap P \neq 0$ and $J \cap P \subset J \cap cl(H_{n_o}, m_o, p_o)$. It may be assumed that $n_o > 1$, $\mu(\mathcal{T}) < \frac{1}{p_o}$, and that the endpoints of J are in V. However, as will be shown below, under these conditions f is differentiable on J, which contradicts $J \cap P \neq \emptyset$.

Now, we shall show that, for each $x \in V$, if (a, b) is a component of V such that $(a, b) \subset J$ and $x \in (a, b)$, then

$$|f(x) - f(a)| \le 3(M + 1)|x - a|$$

and

$$|f(x) - f(b)| \le 3(M + 1)|x - b|.$$

It will suffice to prove only the first of these inequalities in the case where $f' < M$ on (a, b). The other inequality and the case where $f' > -M$ on (a, b) have parallel proofs.

By the assumption that $f' < M$ on (a, b) and by the Darboux property of f, we have that, for all $x, y \in [a\ b]$ auch that $x \le y$,

(*) $f(y) - f(x) \le M(y - x)$.

Therefore, it need only be established that $f(x) - f(a) \ge -3(M + 1)(x - a)$. First, let $(a, b)/2 \le x \le b$. Since $a, b \in J \cap P$, it follows that

$$f(b) \ge f(a) - (M + 1)(b - a)$$

and, by (*), we have

$$f(x) \ge f(b) - M(b - x).$$

Thus

$$|f(x) \ge f(a) - (M + 1)(b - a) - M(b - x) =$$
$$= f(a) - (M + 1)(b - x) - (M + 1)(x - a) - M(b - x)$$

and $0 \le b - x \le x - a$. So, we have

$$f(x) \ge f(a) - 3(M + 1)(x - a).$$

Let $a < x < \frac{a + b}{2}$. Let x_o be such that $x = a + \frac{n_o - 1}{n_o}(x_o - a)$.

Then $x_o = x + \frac{1}{n_o}(x_o - a) \le x + \frac{1}{2}(x_o - a)$ and $x_o \le 2x - a < b$.

Since $a \in \mathrm{cl}(H_{n_o}, m_o, p_o)$, it follows that

$$\{t: |f(t) - f(a)| \le (M + 1)|t - a|\} \cap (x, x_o) \ne \emptyset.$$

Thus there exists $y \in (x, x_o)$ such that

$$|f(y) - f(a)| \le (M + 1)|y - a|,$$

and hence,

$$f(y) \ge f(a) - (M + 1)(y - a).$$

Again by (*), we have

$$f(x) \ge f(y) - M(y - x).$$

Finally, $0 < y - x < \frac{1}{n_o}(x_o - a) \le \frac{n_o - 1}{n_o}(x_o - a) = x - a$ and

$$f(x) \ge f(a) - 3(M + 1)(x - a).$$

It is further shown that, for any two points $x, y \in J$ which are not in the same component of V,

$$|f(x) - f(y)| \le 3(M+1)\,|x - y|.$$

This is clear if x and y both belong to $P \cap J$. Then $x, y \in$ $\in cl(H_{n_o}, m_o, p_o) \cap J$ and $|f(x) - f(y)| \le (M+1)\,|x - y| < 3(M+$ $+1)|x - y|$. We assume that $x \in V$ and $y \in P \cap J$. We may assume that $x < y$ and let (a, b) be a component of V such that $a < x <$ $< b \le y$. Then, by the above.

$$|f(x) - f(y)| \le |f(x) - f(b)| + |f(b) - f(y)|$$
$$\le 3(M+1)\,|x - b| + (M+1)\,|b - y| <$$
$$< 3(M+1)\,|x - y|.$$

Now, we assume that $x < y$, $x \in (a, b)$, $y \in (c, d)$, $(a, b) \cap$ $\cap (c, d) = \emptyset$ and (a, b), (c, d) are components of V. Then, by the above,

$$|f(x) - f(y)| < |f(x) - f(b)| + |f(b) - f(c)| +$$
$$+ |f(c) - f(y)| < 3(M+1)\,|x - y|.$$

Finally, we shall apply Lemma 1. Let $x_o \in J$, $\lambda = f'_{\mathcal{P}\text{-ap}}(x_o)$, $L = 3(M+1)$ and $0 < \varepsilon < 1$. Then, by Lemma 2, there is $\delta > 0$ such that $0 < |y - x_o| < \delta$ implies there are numbers y_1 and y_2 with $y_1 < y < y_2$, satisfying $|f(y_i) - f(x_o) - \lambda(y_i - x_o)| <$ $< \varepsilon|y_i - x|$ for $i = 1, 2$ and $|y_i - y| < \varepsilon|y - x_o|$ for $i = 1, 2$. Now, let $\delta_o > 0$ be such that $(x_o - 2\delta_o, x_o + 2\delta_o) \subset J$ and $\delta_o < \delta$. Then, by the above, for $0 < |y - x_o| < \delta_o$, there are y_1, y_2 such that $y_1 < y < y_2$, $|y_i - x_o| < 2\delta_o$ for $i = 1, 2$ and y_1, y_2 satisfy conditions (1), (2) of Lemma 1. We shall show that y_1, y_2 satisfy condition (3) of Lemma 1. If there exists (a, b) such that it is a component of V, and $y_1, y_2 \in (a, b)$, then $f' < M$ on $[y_1, y_2]$ or $f' > -M$ on $[y_1, y_2]$. Therefore $f(x) - Mx$ is decreasing on $[y_1, y_2]$ or $f(x) + Mx$ is increasing on $[y_1, y_2]$. If y_1, y_2 are not in the same component of V, then y_1, y are not on the same component of V or y, y_2 are not in the same component of V. Therefore, by the above,

$$|f(y) - f(y_1)| \le L|y - y_1| \text{ or } |f(y) - f(y_2)| \le L|y - y_2|.$$

So, all conditions of Lemma 1 are satisfied and f is differentiable at x_o. Since x_o was an arbitrary point of J, we know that f is a differentiable function on J, which contradicts $J \cap P \neq \emptyset$. Thus the proof of Theorem 7 is completed.

To finish with, we shall give applications of Theorem 7.

THEOREM 8. Let f have a finite $\mathcal{T}$-approximate derivative $f'_{\mathcal{T}\text{-ap}}(x)$ for each $x \in I_o$ and let α be a real number. If $\{x: f'_{\mathcal{T}\text{-ap}}(x) = \alpha\} \neq \emptyset$, then there is $x_o \in \operatorname{int}(\{x: f'(x) \text{ exists}\})$ such that $f'(x_o) = \alpha$.

P r o o f. It may be assumed that $\operatorname{int}(\{x: f'(x) \text{ exists}\}) \neq$ $\neq I_o$, for otherwise the conclusion is obvious. Let M be any number with $M > |\alpha|$. Theorem 7 quarantees the existence of a component (a, b) of int ({x: f'(x) exists}) on which f' takes the values M and -M. Since f' has the Darboux property on (a, b), f' also attains α on (a, b).

COROLLARY 1. Let f have a finite $\mathcal{T}$-approximate derivative $f'_{\mathcal{T}\text{-ap}}(x)$ for each x in I_o. If $\{x: f(x) = 0\}$ is dense in I_o, then f is identically zero on I_o.

COROLLARY 2. Let f and g have finite $\mathcal{T}$-approximate derivatives $f'_{\mathcal{T}\text{-ap}}(x)$ and $g'_{\mathcal{T}\text{-ap}}$ x, respectively, for each x in I_o. If $\{x: f(x) = g(x)\}$ is dense on I_o, then f = g on I_o.

COROLLARY 3. Let f have a finite $\mathcal{T}$-approximate derivative $f'_{\mathcal{T}\text{-ap}}(x)$ and g a finite derivative g'(x) for each x in I_o. If f' = g' on int ({x: f'(x) exists}), then f' = g' on I_o.

P r o o f. Let h = f - g. Then h has a finite $\mathcal{T}$-approximate derivative on I_o and int ({x: h'(x) exists}) = int ({x: f'(x) exists}). Moreover, h' = 0 on int ({x: h'(x) exists}). Theorem 8 quarantees that $h'_{\mathcal{T}\text{-ap}} = 0$ on I_o and the conclusion follows.

THEOREM 9. Let $\mathcal{P}$ be a property of functions saying that any function which is differentiable and possesses $\mathcal{P}$ on an interval I is monotone on I. If f has a finite $\mathcal{T}$-approximate derivative $f'_{\mathcal{T}\text{-ap}}(x)$ at each x in I_o and if f has property $\mathcal{P}$ on I_o, then f is monotone on I_o.

P r o o f. It suffices to show that $f'_{\mathcal{T}\text{-ap}}$ is unsigned on I_o (see Theorem 4). Suppose the contrary. It follows from Theorem 7 that there is a subinterval I of I_o on which $f'_{\mathcal{T}\text{-ap}} = f'$ and f' attains both positive and negative values. Then f is not monotone on I, which contradicts the assumption.

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OSCYLACYJNE ZACHOWANIE 𝒯-APROKSYMATYWNEJ POCHODNEJ

W pracy rozważano 𝒯-aproksymatywną pochodną. Udowodniono w niej dwa twierdzenia.

Twierdzenie. Jeżeli 𝒯-aproksymatywna pochodna $f'_{\mathcal{T}\text{-ap}}$ funkcji f istnieje w każdym punkcie przedziału [a, b] i jest ograniczona z góry lub z dołu w przedziale (a, b), to dla każdego $x \in [a, b]$ $f'_{\mathcal{T}\text{-ap}}(x) = f'(x)$.

Twierdzenie. Niech $M \geq 0$ oraz niech f będzie funkcją posiadającą skończoną 𝒯-aproksymatywną pochodną $f'_{\mathcal{T}\text{-ap}}$ w każdym punkcie pewnego przedziału I_o. Jeżeli $f'_{\mathcal{T}\text{-ap}}$ osiąga M i -M na I_o, to istnieje podprzedział $I \subset I_o$, na którym $f'_{\mathcal{T}\text{-ap}} = f'$ oraz f' osiąga M i -M na I_o.