A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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Ryszard J. Pawlak

ON DARBOUX POINTS AND THE PERFECTLY CLOSED CLASS OF FUNCTIONS

The present paper is intended to discuss problems connected with operations on Darboux functions at some point x_o.

In many papers has study operations on Darboux functions (see for example [2], [3], [5], [7], [10], [11] and [12]. The present paper is intended to discus problems connected with operations on Darboux functions at some point x_o (then we say also that x_o is Darboux point or that the function possesses Darboux property at x_o - see [3], [6], [8]). Precisely, in this paper is contained the answer to the following question: whether it is possible to form the perfectly closed family of functions relative to Darboux property at x_o. This paper end four open problems connected with operations on Darboux functions at point x_o.

We use the standard notions and notation.

By R we shall denote the set of all real numbers with the natural topology. Suppose that T is a topology in R different than the natural topology, then we shall write for example: T - neighbourhood or T - continuity, to make a distinction between two topologies under consideration (in the case of natural topology we omit the symbol of this topology). We say that a topology T is agree with the natural topology of line at a point x_o if there exists T - base $B(x_o)$ at x_o such that $B(x_o)$ is the base for R at x_o, too. The symbols Int A and $\bar{A}$ denote the interior and the closure of A (in natural topology of line), respectively. The closure of A in topology T we denote by $cl_T A$.

If A is a subset of R then by $\ell(A)$ we shall denote the set of all components of A (in the natural topology). We say that a set A is dense at x_o if there exists a neighbourhood U of x_o such that $U \subset \overline{A}$. In analogously way we can define a set T - dense at x_o.

Let f be an arbitrary function, then by $C_f(D_f)$ we denote the set of all continuity (discontinuity) points of f. If F is the family of functions then by C_F we denote the intersection $\bigcap_{f \in F} C_f$. By C(T, A), where T is some topology of real line and $A \subset R$, we denote the class of functions $f: (R, T) \to R$ such that the restriction $f|_A$ is T - continuous. If F is the family of functions, then by symbol $F|_A$ we denote the following class $\{f|_A : f \in F\}$.

The uniformly convergence of a sequence of functions $\{f_n\}$ to f we denote by $f_n \rightrightarrows f$. By B_1 we denote the family of all functions in Baire class one.

We say that a family of functions F is uniformly *-quasi--continuous at x_o, relative to some open set A if for every $\varepsilon > 0$ and $\eta > 0$, there exists a positive number $\delta \leq \eta$ such that for each $C \in \ell(A)$ for which $C \cap (x_o - \delta, x_o + \delta) \neq \emptyset$ there exists open interval $(a, b) \subset C \cap (x_o - \delta, x_o + \delta)$ such that $f((a, b)) \subset (f(x_o) - \varepsilon, f(x_o) + \varepsilon)$, for every $f \in F$. If the family $F = \{f\}$ is uniformly *-quasi-continuous at x_o, relative to A, then we say that f is *-quasi-continuous at x_o, relative to A.

We say that the family of function F is uniformly quasi-continuous at x_o, if for every $\varepsilon > 0$ and $\delta > 0$ there exists an open interval $(a, b) \subset (x_o - \delta, x_o + \delta)$ such that $f((a, b)) \subset (f(x_o) - \varepsilon, f(x_o) + \varepsilon)$, for every $f \in F$. If the family $F = \{f\}$ is uniformly quasi-continuous at x_o, then we say that f is quasi-continuous at x_o. In the case if f is quasi-continuous function at every point of its domain, we say short that f is quasi-continuous.

We say that a function $f: (R, T) \to R$ possesses T - Blumberg set (at x_o) if there exists a set B (containing x_o), T - dense (at x_o), such that $f|_B$ is T - continuous. We say that a function $f: R \to R$ possesses a strong set of Blumberg - B, if B is dense in R; the restriction $f|_B$ is continuous and $f(U) \subset \overline{f(U \cap B)}$, for every open set $U \subset R$.

THEOREM A [9]. For a function $f: [a, b] \to R$ the following conditions are equivalent:

a) f is quasi-continuous,

b) f possesses a strong set of Blumberg.

THEOREM B. Let X, Y be the topological spaces and $f: X \to Y$ be a quasi-continuous function. Then each Blumberg set of f is its strong set of Blumberg.

This theorem is contained in T. Šalats nonpublished papers "Some generalizations of the notion of continuity of Blumberg sets of functions".

DEFINITION 1. Let F be a family of function $f: R \to R$ and $A \subset \subset R$ be dense at $x_o \in R$. We say that a topology T is quasi-generated by (F, A, x_o) if

1^o T is finer than the natural topology of line and it is agree with the natural topology of line at each point of $A \setminus \{x_o\}$ and moreover A is T - dense set at x_o;

2^o if $f \in F$, then f possesses T - Blumberg set at every point of $\bar{A}$;

3^o if $f \in C(T, \bar{A})$ then $f|_{\bar{A}}$ possesses a strong set of Blumberg and it is *-quasi-continuous at x_o, relative to A.

DEFINITION 2. Let F be a class of function and $\mathcal{P}$ some property of functions. We say that F is perfectly closed relative to $\mathcal{P}$ if

1^o f possesses a property $\mathcal{P}$, for every $f \in F$;

2^o if $f, g \in F$, then $f + g$, $f \cdot g$, max (f, g), min (f, g) possesse the property $\mathcal{P}$;

3^o if $f_n \in F$ $n = 1, 2, \dots$ and $f_n \rightrightarrows f$, then f possesses the property $\mathcal{P}$.

The following proposition shows that the assumption of quasi--continuity of f at Darboux point x_o is every natural in the case if the set $f(D_f)$ is "small".

PROPOSITION. Let f be a function such that x_o is Darboux point of f and Int f $(D_f \cap [x_o, x_o+\delta)) = \emptyset$ or Int f $(D_f \cap (x_o+\delta, x_o]) = \emptyset$, for some $\delta > 0$. Then f is quasi-continuous at x_o.

The results connected with operations on Darboux functions suggest the following question: under what assumptions a family of functions F is perfectly closed relative to Darboux property at x_o. The partial answer to this question is contained in the following theorem.

THEOREM. Let F be the class of functions $f: R \to R$ such that the set $A = \text{Int } C_F$ is dense at x_o,

a) if there exists a topology T quasi-generated by (F, A, x_o), then $F \subset C(T, \bar{A})$ and moreover F and $C(T, \bar{A})$ are perfectly closed relative to Darboux property at x_o;

b) if $F_{|\bar{A}}$ is uniformly quasi-continuous and uniformly *-quasi-continuous at x_o, relative to A, then there exists a topology T quasi-generated by (F, A, x_o) and consequently, F is perfectly closed relative to Darboux property at x_o.

P r o o f a. First we shall show that $F \subset C(T, \bar{A})$. Let $f \in F$ and let $x \in \bar{A}$. Of course, if $x \in A$, then x is T - continuity point of $f_{|\bar{A}}$. Assume that $x \notin A$. Let $\varepsilon > 0$ and let B_x denotes T - Blumberg set at x and finally let U_x, be T - neighbourhood of x such that $U_x \subset cl_T B_x \cap cl_T A$ and $f(U_x \cap B_x) \subset (f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2})$. Remark that the set $B_x \cap A$ is T - dense in U_x (i.e. $U_x \subset cl_T (B_x \cap A)$). From T - continuity of f on A we deduce that there exists an open set C, T - dense in U_x and such that $f(C) \subset (f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2})$. Now let $y \in U_x \setminus B_x$ and let B_y be T - Blumberg set of f at y. Then the set $B_y \cap C$ is T - dense set at y, and consequently $f(y) \in [f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2}] \subset (f(x) - \varepsilon, f(x) + \varepsilon)$, which ends the proof of the inclusion $F \subset C(T, \bar{A})$.

Now we shall show that $C(T, \bar{A})$ is perfectly closed relative to Darboux property at x_o (according to the inclusion $F \subset C(T, \bar{A})$, this means that F is perfectly closed, too). Let δ be positive number such that $[x_o, x_o + \delta) \subset \bar{A}$. Suppose that $g \in C(T, \bar{A})$. Let

$x_o \notin C_g$. We shall prove that x_o is a right-hand Darboux point of g - see [8] (if $x_o \in C_g$ then, of course, x_o is Darboux point of g). Assume, to the contrary, that there exist a right-hand cluster number β of g at x_o ($-\infty \leqslant \beta \leqslant \infty$), different from $g(x_o)$ (let, for instance, $\beta > g(x_o)$) and a real number $\alpha \in (g(x_o), \beta)$ and $\delta_1 > 0$ such that

$$g^{-1}(\alpha) \cap [x_o, x_o + \delta_1) = \emptyset \tag{1}$$

Let $\delta_2 \leqslant \min(\delta_1, \delta)$ be a positive real number such that for every $C \in \ell(A)$ for which $C \cap (x_o - \delta_2, x_o + \delta_2) \neq \emptyset$ there exists an open interval $(a, b) \subset C \cap (x_o - \delta_2, x_o + \delta_2)$ such that

$$g((a, b)) \subset (-\infty, \alpha) \tag{2}$$

Let δ_o be an arbitrary positive real number less than δ_2. Since $g_{|C}$ possesses Darboux property, for each $C \in \ell(A)$, then, according to (1) and (2), we infer that

$$g(A \cap [x_o, x_o + \delta_o) \subset (-\infty, \alpha) \tag{3}$$

Let B^* be a strong Blumberg set for $g_{|\bar{A}}$. Let $z \in B^*$ be a number less than x_o such that $(z, x_o] \subset \bar{A}$ and let $t \in B^*$ be a number from the open interval $(x_o + \delta_o, x_o + \delta)$. It is easy to see, that $g_{|[z, t]}$ possesses a strong Blumberg set - $B^* \cap [z, t]$ and consequently, according to Theorem A, $g_{|[z, t]}$ is quasi-continuous, which means (according to (1) and (3)) that:

$$g([x_o, x_o + \delta_o]) \subset (-\infty, \alpha).$$

This contradicts the fact that β is right-hand cluster number of g at x_o. The obtained contradiction proved that x_o is a right-hand Darboux point of g. In the similar way, we can prove that x_o is a left-hand Darboux point of g and consequently x_o is Darboux point of g. Since sum, product, minimum and maximum of two functions from $C(T, \bar{A})$ is again a function from $C(T, \bar{A})$ and moreover $C(T, \bar{A})$ is closed relative to the uniformly convergence, then the proof of the fact that $C(T, \bar{A})$ is closed relative to Darboux property at x_o is finited.

P r o o f b. Let for every $n = 1, 2, \ldots$ $0 < \delta_n < \frac{1}{n}$ be a number such that for every $C \in \ell(A)$, for which $C \cap (x_o - \delta_n, x_o + \delta_n) \neq \emptyset$ there exists an open interval $(x_{n,C} - \varepsilon_{n,C}, x_{n,C} + \varepsilon_{n,C}) \subset C \cap (x_o - \delta_n, x_o + \delta_n)$ such that $f((x_{n,C} - \varepsilon_{n,C}, x_{n,C} + \varepsilon_{n,C})) \subset (f(x_o) - \frac{1}{n}, f(x_o) + \frac{1}{n})$ for every $f \in F$ (to the simplicity notation we assume that in the case if $C \cap (x_o - \delta_n, x_o + \delta_n) = \emptyset$, then $(x_{n,C} - \varepsilon_{n,C}, x_{n,C} + \varepsilon_{n,C}) = \emptyset$).

Put:

$$B(x_o) = \{\{x_o\} \cup \bigcup_{n=k}^{\infty} (\bigcup_{C \in \ell(A)} (x_{n,C} - \frac{1}{n}\varepsilon_{n,C}, x_{n,C} + \frac{1}{n}\varepsilon_{n,C})) :$$
$$: k = 1, 2, \ldots\}.$$

Let $x \in \bar{A} \setminus (A \cup \{x_o\})$. For every $n = 1, 2, \ldots$, by U_n^x we denote an open (in $\bar{A}$) set such that $U_n^x \subset (x - \frac{1}{n}, x + \frac{1}{n})$, $x_o \notin U_n^x$ and $f(U_n^x) \subset (f(x_o) - \frac{1}{n}, f(x_o) + \frac{1}{n})$. Then we put

$$B(x) = \{\{x\} \cup \bigcup_{n=k}^{\infty} U_n^x : k = 1, 2, \ldots\}.$$

If $x \in A \setminus \{x_o\}$, then we put

$$B(x) = \{(x - \frac{1}{n}, x + \frac{1}{n}) : n = k_x, k_x + 1, \ldots\},$$

where k_x denote the positive integer such that $(x - \frac{1}{k_x}, x + \frac{1}{k_x}) \subset A$.

In the case if $x \notin \bar{A}$, let $B(x) = \{\{x\}\}$.

It is easy to see that $\{B(x)\}_{x \in R}$ fulfils conditions of a local base (BP1), (BP2) and (BP3) from [4] p. 28. Let T be the topology generated by neighbourhoodsystem $\{B(x)\}_{x \in R}$ (see [4] Proposition 1.2.3, p. 39). Infer that T is finer than the natural topology of line and it is agree with the natural topology of line at each point of $A \setminus \{x_o\}$ and moreover A is T - dense set at x_o. From the construction of $\{B(x)\}_{x \in R}$ we deduce that if $f \in F$, then f is T - continuous and at the same time R is T - Blumberg set for every $f \in F$ at every point of R.

Now, let $f \in C(T, \bar{A})$. It is not hard to prove that $f|_{\bar{A}}$ is *-quasi-continuous at x_o, relative to A and $f|_{\bar{A}}$ is quasi-continuous. Moreover, it is easy to see that A is Blumberg set for $f|_{\bar{A}}$ and consequently, according to Theorem B, $f|_{\bar{A}}$ possesses the strong set of Blumberg. This ends the proof.

Respecting the above remarks, we can formulate the following problems:

Problem 1. Assume that x_o is Darboux point of some function $f(\in B_1)$. Characterize the class of functions g such that f + g possesses Darboux property at x_o.

Remark that there exists continuous function f and the function $g \in B_1$ such that 0 is Darboux point of g but f + g does not posses Darboux property at 0.

Problem 2. Let $x_o \in R$. Characterize the maximal additive class (see [1], Definition 3.1) for the family of function in Baire class one possessing Darboux property at x_o.

Problem 3. Under what hypothesis (different from the assumptions of our theorem) the uniformly limit of sequence of functions with Darboux property at x_o also possesses Darboux property at x_o.

Remark that there exists sequence $\{f_n\} \subset B_1$ such that 0 is Darboux point of $\{f_n\}$ $(n = 1, 2, ...)$ and $f_n \rightrightarrows f$ but f does not possess Darboux property at 0.

Problem 4. Under what hypothesis (different from the assumptions of our theorem) the maximum and minimum of two functions with Darboux property at x_o also possesse Darboux property at x_o

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Institute of Mathematics
University of Łódź

Ryszard J. Pawlak

O PUNKTACH DARBOUX I DOSKONALE ZAMKNIĘTYCH KLASACH FUNKCJI

W prezentowanym artykule rozważany jest problem związany z możliwością zachowania własności Darboux w ustalonym punkcie przy różnych operacjach wykonywanych na funkcjach posiadających tę własność w danym punkcie.