ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 216, 2008

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ESTIMATION OF PARAMETERS OF TÖRNQUIST'S FUNCTIONS WITH NEWTON-RAPHSON'S METHOD

Abstract. In the work there has been discussed Newton-Raphson's iterative method of estimation of parameters of Törnquist's functions of general class and there are derived formulas of their partial derivatives of 1st and 2nd rank. It enabled presentation of uniform form of the vector of parameters of successive iterative approximations allowing their implementation in Excel calculation sheet.

Key words: Törnquist's functions, Newton-Raphson's method, estimation of parameters.

I. INTRODUCTION

In the analysis of economic phenomena there play an important role the functions describing forming of the size of the demand depending on the size of supply. Among them there are differentiated Törnquist's functions. In literature these functions were widely discussed from economic and analytical points of view by many authors, among from which, according to years, can be mentioned: Zając (1966), Pawłowski (1969), Kowalak (1973), Welfe (1978), Młynarski (1980), Stanisz (1986), Podolec (1995) oraz Rozmus i Wójcik (2003).

One of the important problems of Törnquist's functions is estimation of their parameters. For this purpose there are most often applied methods of linearization of functions, method of three points and combined methods.

In the work there has been discussed Newton-Raphson's iterative method of estimation of parameters of Törnquist's functions of general class and there are derived formulas of their partial derivatives of 1st and 2nd rank. It enabled presentation of uniform form of the vector of parameters of successive iterative approximations allowing their implementation in Excel calculation sheet. Application of Newton-Raphson's method has been illustrated on the numerical example.

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II. TÖRNQUIST'S FUNCTIONS

After Stanisz (1986) we present three types of Törnquist's functions together with meaning ranges of their parameters:

$$\text{T1: } y = \alpha \frac{x}{x+\beta}, \qquad \alpha > 0,\ \beta > 0,\ x \ge 0,$$

$$\text{T2: } y = \alpha \frac{x-\gamma}{x+\beta}, \quad \alpha > 0,\ \gamma > 0,\ \beta > -\gamma,\ x \ge \gamma,$$

$$\text{T3: } y = \alpha x \frac{x-\gamma}{x+\beta}, \quad \alpha > 0,\ \gamma > 0,\ \beta > -\gamma,\ x \ge \gamma,$$

at which we resigned from indexing of parameters of functions.

The mentioned functions can be written in general form

$$f(.) \equiv f(x;k,\alpha,\beta,\gamma) = \alpha x^k (x-\gamma)(x+\beta)^{-1}, \tag{1}$$

where the mentioned types are its particular cases: T1: $k = \gamma = 0$, T2: $k = 0$ and T3: $k = 1$. Derivation of derivatives for particular types are presented, among other things, in Stanisz's title (Stanisz 1986). They are all expressed by derivatives from (1) in the form

$$f'(x;k,\alpha,\beta,\gamma) = g(x;k,\alpha,\beta,\gamma) \cdot f(x;k,\alpha,\beta,\gamma),$$

where

$$g(x;\alpha,\beta,\gamma) = \frac{kx^2 + (1-k)\gamma x + (k+1)\beta x - k\beta\gamma}{x(x-\gamma)(x+\beta)}.$$

For further remarks there are necessary partial derivatives of functions (1) regarding parameters. Derivatives of 1st rank assume the form:

$$\frac{\partial f(.)}{\partial \alpha} = \frac{1}{\alpha} f(.), \quad \frac{\partial f(.)}{\partial \gamma} = \frac{-1}{x-\gamma} f(.), \quad \frac{\partial f(.)}{\partial \beta} = \frac{-1}{x+\beta} f(.),$$

that is, all are of the product type, and are formed by functions of a given parameter and initial Törnquist's function (1), however derivatives of 2nd rank are expressed by the formulas:

$$\frac{\partial^2 f(.)}{\partial\alpha^2}=\frac{\partial^2 f(.)}{\partial\gamma^2}=0,\quad \frac{\partial^2 f(.)}{\partial\beta^2}=\frac{2}{(x+\beta)^2}f(.),$$

$$\frac{\partial^2 f(.)}{\partial\alpha\partial\gamma}=\frac{-1}{\alpha(x-\gamma)}f(.),\quad \frac{\partial^2 f(.)}{\partial\alpha\partial\beta}=\frac{-1}{\alpha(x+\beta)}f(.),$$

$$\frac{\partial^2 f(.)}{\partial\gamma\partial\beta}=\frac{1}{(x-\gamma)(x+\beta)}f(.),$$

that is, they are expressed by the product of the function of two parameters and Törnquist's function (1). The presented derivatives will be used in chapter 4.

III. NEWTON-RAPHSON'S METHOD

For estimation of parameters of a function (1) we apply the least squares method (LSM). Denoting by $\boldsymbol{\beta}=(\alpha,\gamma,\beta)'$ the vector of parameters, the function (1) assumes the form $f(x;\boldsymbol{\beta})$. The form of vector $\boldsymbol{\beta}$, estimated from the two-dimensional sample $\{(x_i,y_i);\ i=1,2,...,n\}$, where n is the number of examined n units, is expressed by vector $\mathbf{b}=(b_0,b_1,b_2)$. We determine this vector from the model of non-linear regression $y=f(x;\boldsymbol{\beta})+\varepsilon$, where ε is a random component and we find it from LSM, at the assumption that function f of type (1) is multiple valued, continuous and double-differentiable.

We determine the minimum of criterion function $Q(\mathbf{b})=\sum_{i-1}^{n}(y_i-f(x_i;\mathbf{b}))^2$, calculating its partial derivatives regarding each component of vector **b,** i.e. $\frac{\partial Q(\mathbf{b})}{\partial b_j}$, $j=0,1,2$. After equating it to zero we receive the system of standard equations

$$\sum_{i=1}^{n} f(x_i;\mathbf{b})\frac{\partial f(x_i;\mathbf{b})}{\partial b_j}=\sum_{i=1}^{n} y_i\frac{\partial f(x_i;\mathbf{b})}{\partial b_j},\quad j=0,1,2. \tag{2}$$

For solution of the presented system there is proposed Newton-Raphson's method. It consists in execution of sequence of iterations which are to lead to finding vector **b**, in order to meet the criterion $\sum_{j=0}^{2} | b_j^{h+1} - b_j^h | < \rho$ for the given $\rho > 0$, where $h+1$ and h denote the numbers of iterations, at h = 0, 1, 2, ... Zero iteration $h = 0$ denotes assumed initial approximation of vector $\mathbf{b}^0$.

In this method there are differentiated the following steps:

a) we calculate partial derivatives of function $f(x, \mathbf{b})$ according to successive parameters, i.e. $\frac{\partial f(x;\mathbf{b})}{\partial b_j}$, at $j = 0,1,2,$

b) to each of equations of the system (2) we attribute one limiting function $f_j(\mathbf{b}) = \sum_{i=1}^{n} (y_i - f(x_i;\mathbf{b})) \frac{\partial f(x_i;\mathbf{b})}{\partial b_j}$, $j = 0,1,2,$

c) for limiting functions we calculate partial derivatives regarding parameters which are expressed by derivatives of 1st and 2nd rank of function $f(x, \mathbf{b})$, using formulas for determining them presented in chapter two.

$$\frac{\partial f_j(\mathbf{b})}{\partial b_k} = \sum_{i=1}^{n} y_i \frac{\partial^2 f(x_i;\mathbf{b})}{\partial b_j \partial b_k} - \sum_{i=1}^{n} \left[\frac{\partial f(x_i;\mathbf{b})}{\partial b_j} \cdot \frac{\partial f(x_i;\mathbf{b})}{\partial b_k} + f(x_i;\mathbf{b}) \frac{\partial^2 f(x_i;\mathbf{b})}{\partial b_j \partial b_k} \right],$$

$k = 0,1,2,$

d) we form square symmetrical Jacobian matrix $\mathbf{F} = \left(\frac{\partial f_j(\mathbf{b})}{\partial b_k} \right)$, $j, k = 0,1,2,$

e) for vector $\mathbf{b}^h$ in h-th iteration we calculate matrix $\mathbf{F}(\mathbf{b}^h)$ and its inverse matrix $\mathbf{F}^{-1}(\mathbf{b}^h)$,

f) let us write the vector of function $\mathbf{f}(\mathbf{b}) = (f_0(\mathbf{b}), f_1(\mathbf{b}), f_2(\mathbf{b}))'$,

g) from f) we determine the value of vector of function $\mathbf{f}(\mathbf{b}^h)$ for h-th iteration,

h) in iteration $h+1$ we calculate vector $\mathbf{b}^{h+1} = \mathbf{b}^h - \mathbf{F}^{-1}(\mathbf{b}^h)\mathbf{f}(\mathbf{b}^h)$, for h = 0, 1, 2, ...,

i) we substitute $\mathbf{b}^{h+1} \to \mathbf{b}^h$ and proceed to step e), and we end the process of iteration when there is fulfilled the condition $\sum_{j=0}^{2} | b_j^{h+1} - b_j^h | < \rho$.

IV. ESTIMATION OF PARAMETERS OF TÖRNQUIST'S FUNCTIONS

The general procedure of estimation of non-linear functions, presented in chapter three, in the case of Törnquist's functions requires determining vector $\mathbf{f}(\mathbf{b})=\begin{bmatrix} f_\alpha \\ f_\gamma \\ f_\beta \end{bmatrix}$ and matrix $\mathbf{F}=\begin{bmatrix} f_{\alpha\alpha} & f_{\alpha\gamma} & f_{\alpha\beta} \\ f_{\alpha\gamma} & f_{\gamma\gamma} & f_{\gamma\beta} \\ f_{\alpha\beta} & f_{\gamma\beta} & f_{\beta\beta} \end{bmatrix}$. Let us introduce auxiliary denotations $f_i = f(x_i;k,\alpha,\beta,\gamma)$, $g_i = (y_i - f_i)f_i$ and $h_i = (y_i - 2f_i)f_i$, which means that for given values of x-es and parameters there are determined values of presented functions.

Components of vector $\mathbf{f}(\mathbf{b})$ are expressed in the form:

$$f_\alpha = \frac{1}{\alpha}\sum_{i=1}^{n} g_i,\quad f_\gamma = -\sum_{i=1}^{n}\frac{1}{x_i-\gamma}g_i,\quad f_\beta = -\sum_{i=1}^{n}\frac{1}{x_i+\beta}g_i,$$

and elements of matrix $\mathbf{F}$ are:

$$f_{\alpha\alpha} = -\frac{1}{\alpha^2}\sum_{i=1}^{n} f_i^2,\quad f_{\alpha\gamma} = -\frac{1}{\alpha}\sum_{i=1}^{n}\frac{h_i}{x_i-\gamma},\quad f_{\alpha\beta} = -\frac{1}{\alpha}\sum_{i=1}^{n}\frac{h_i}{x_i+\beta},$$

$$f_{\gamma\gamma} = -\sum_{i=1}^{n}\frac{f_i^2}{(x_i-\gamma)^2},\quad f_{\gamma\beta} = \sum_{i=1}^{n}\frac{h_i}{(x_i-\gamma)(x_i+\beta)},\quad f_{\beta\beta} = \sum_{i=1}^{n}\frac{(2y_i-3f_i)f_i}{(x_i+\beta)^2}.$$

In the case of estimation of parameters for Törnquist's functions of type T1 one should apply substitutions $\mathbf{b}=\begin{bmatrix} b_0 \\ b_2 \end{bmatrix}$, $\mathbf{f}(\mathbf{b})=\begin{bmatrix} f_\alpha \\ f_\beta \end{bmatrix}$ and $\mathbf{F}=\begin{bmatrix} f_{\alpha\alpha} & f_{\alpha\beta} \\ f_{\alpha\beta} & f_{\beta\beta} \end{bmatrix}$.

V. DETERMINING INITIAL VALUES

Application of Newton-Raphson's iterative method requires giving initial values for parameters of Törnquist's functions. Within this scope there are several possibilities: (a) heuristic approach, (b) method of linearization and (c) combined methods. They will be presented in reference to the general form function (1).

(a) Heuristic approach:

(i) curves of type T1, T2 and T3, $\alpha = \max\{x_i\} - \varepsilon$, where ε is an positive constant,

(ii) curves of type T2 and T3, $\gamma = \min\{x_i\} - \eta$, where η is an optional positive constant,

(b) Methods of linearization:

Starting with the general form (1), we carry out a of transformations $(x+\beta)y = \alpha x^k (x-\gamma) \Leftrightarrow xy + \beta y = \alpha x^{k+1} - \alpha\gamma x^k$:

case A:

(i) $\beta y = -\alpha\gamma x^k + \alpha x^{k+1} - xy$,

(ii) $y = -\dfrac{\alpha\gamma}{\beta}x^k + \dfrac{\alpha}{\beta}x^{k+1} - \dfrac{1}{\beta}xy$,

(iii) substitutions:

$\beta_1 = -\dfrac{\alpha\gamma}{\beta},\ \beta_2 = \dfrac{\alpha}{\beta},\ \beta_3 = -\dfrac{1}{\beta},\ u_1 = x^k,\ u_2 = x^{k+1},\ u_3 = xy,$

(iv) $y = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3$ - model of multiple linear regression, for which we carry out estimation of regression coefficients with classical LSM.

For particular Törnquist's functions some substitutions are reduced to T1: $k=0,\ \gamma=0\ \Rightarrow \beta_1 = 0,\ u_1 = 1,\ u_2 = x;$ T2: $k = 0 \Rightarrow u_1 = 1,\ u_2 = x$ and T3: $k=1 \Rightarrow\ u_1 = x,\ u_2 = x^2$;

case B:

(i) $xy = \alpha x^{k+1} - \alpha\gamma x^k - \beta y$,

(i) $y = \alpha x^k - \alpha\gamma x^{k-1} - \beta \dfrac{y}{x}$,

(ii) substitutions:

$\beta_1 = \alpha,\ \beta_2 = -\alpha\gamma,\ \beta_3 = -\beta,\ u_1 = x^k,\ u_2 = x^{k-1},\ u_3 = \dfrac{y}{x},$

(iii) further as in (iv) at case A.

For particular Törnquist's functions some substitutions are reduced to: T1:

$$k=0,\ \gamma=0\ \Rightarrow \beta_2 = 0,\ u_1 = 1,\ u_2 = \frac{1}{x};\ \text{T2: } k=0 \Rightarrow u_1 = 1,\ u_2 = \frac{1}{x}$$

and T3: $k=1 \Rightarrow u_1 = x,\ u_2 = 1$.

We will not present combined methods for Törnquist's functions (1). For their particular types they are presented in Stanisz's work (1986).

VI. NUMERICAL EXAMPLE

In the example we will deal with Törnquist's function of type T2. For illustration of the procedure we used data from Kowalak's work (Kowalak 1973):

Annual incomes (10 thousand PLN)	1,6	1,7	1,8	1,9	2,0	2,1	2,4	2,6	2,9	3,1
Annual expenses (10 thousand PLN)	0,4	0,8	1,3	1,2	1,4	1,4	1,6	1,5	1,8	1,7

The mentioned data are presented in figure 1.

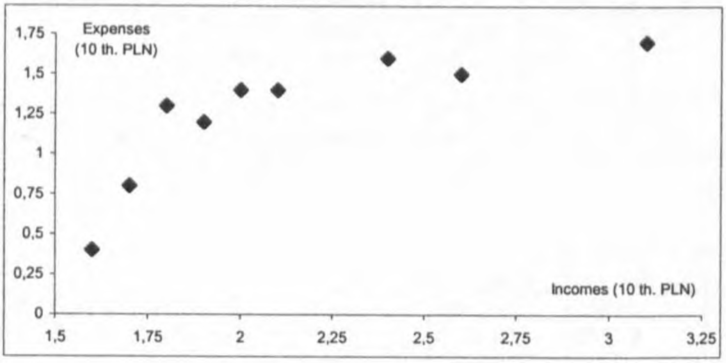

Fig. 1. Correlation plot
Source: own elaboration.

The lay-out of points shows the general tendency of changes of the characteristic Y at successive values of the characteristic X in accordance with the principle of monotonicity corresponding to Törnquista's curve of type I. For initial values for this function there were assumed magnitudes, whose way of determining was presented in chapter 5 in methods A and B and included in the setting-up in which *SSE* expresses the sum of squares for the error:

Methods	β_1	β_2	β_3	α	γ	β	SSE
A	1,9103	−1,6669	0,6466	1,8042	1,6374	−1,5465	2,8504
B	1,8528	−2,9294	1,4447	1,8528	1,5810	−1,4447	0,1061

From the presented initial estimations one should reject approximations received by method A, since the estimated value of parameter γ exceed minimum value for variable x, and it means that there are not fulfilled all boundary conditions for the considered Törnquist's function.

Next we will use initial approximations from method B for which there are fulfilled all boundary conditions of the function of type T2. We will show calculations connected with Newton-Raphson's method for 1st iteration, starting with approximation $\mathbf{b}^0 = \begin{bmatrix} 1,8528 \\ 1,5810 \\ -1,4447 \end{bmatrix}$:

a) calculation of values of derivatives of 1st and 2nd rank mentioned in chapter 5:

i	x_i	y_i	f_i	g_i	h_i	f_α	f_γ	f_β
1	1,6	0,4	0,2263	0,0393	–0,0119	0,0393	2,0733	0,2532
2	1,7	0,8	0,8634	–0,0548	–0,8003	–0,0548	–0,4605	–0,2146
...	...	...	...	...	...	...	...	...
9	2,9	1,8	1,6793	0,2027	–2,6172	0,2027	0,1537	0,1393
10	3,1	1,7	1,7002	–0,0004	–2,8912	–0,0004	–0,0003	–0,0002
					Total	–0,0600	1,8148	0,0900

i	$f_{\alpha\alpha}$	$f_{\alpha\gamma}$	$f_{\alpha\beta}$	$f_{\gamma\gamma}$	$f_{\gamma\beta}$	$f_{\beta\beta}$
1	0,0512	–0,6266	–0,0765	142,3940	–4,0357	1,1377
2	0,7455	–6,7277	–3,1352	52,6824	–26,3551	–13,1228
...	...	...	...	...	...	...
9	2,8199	–1,9843	–1,7984	1,6210	–1,3635	–1,1401
10	2,8908	–1,9034	–1,7467	1,2529	–1,1499	–1,0554
Total	18,7978	–36,9472	–25,9750	267,1746	–80,4955	–47,8597

b) vector and matrix of partial derivatives

$$\mathbf{f}(\mathbf{b}^0) = \begin{bmatrix} -0,0307 \\ -1,8148 \\ -0,0897 \end{bmatrix}, \quad \mathbf{F}(\mathbf{b}^0) = \begin{bmatrix} -5,4758 & 19,9412 & 14,0192 \\ 19,9412 & -267,1750 & -80,4955 \\ -14,0192 & -80,4955 & -47,8597 \end{bmatrix},$$

c) matrix inverse to matrix $\mathbf{F}(\mathbf{b}^0)$

$$\mathbf{F}^{-1}(\mathbf{b}^0) = \begin{bmatrix} -0,7882 & 0,0217 & -0,2675 \\ 0,0217 & -0,0082 & 0,0201 \\ -0,2675 & 0,0201 & -0,1331 \end{bmatrix},$$

d) new approximation after 1st iteration

e) $\mathbf{b}^1 = \mathbf{b}^0 - \mathbf{F}^{-1}(\mathbf{b}^0) = \begin{bmatrix} 1,8528 \\ 1,5810 \\ -1,4447 \end{bmatrix} - \begin{bmatrix} 0,0087 \\ 0,0124 \\ -0,0164 \end{bmatrix} = \begin{bmatrix} 1,8441 \\ 1,5686 \\ -1,4283 \end{bmatrix}$,

f) approximations for successive iterations

Number of iteration	0	1	2	3	4
alpha	1,8528	1,8441	1,8737	1,8836	1,8847
gamma	1,5810	1,5687	1,5604	1,5575	1,5571
beta	–1,4447	–1,4283	–1,4042	–1,3951	–1,3941
Sum SEE	0,1061	0,0813	0,0770	0,0767	0,0767
%	–	76,6727	72,6065	72,2821	72,2790

After 4th iteration there was obtained the final solution of determining the value of parameters of Törnquist's curve of type T2 in the form $y = 1,8847 \cdot \frac{x-1,5571}{x-1,3941}$. The received sum of squares of deviations (0,0767) constitutes 72,279% of its initial value from zero iteration (0,1061). The graphs of the examined curve matched with LSM and Newton-Raphson's method are presented in figure 2.

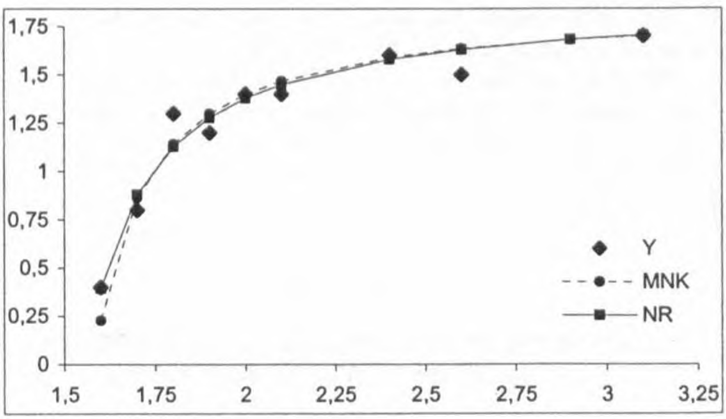

Fig. 2. Graph of empirical data and matched curves

Source: own elaboration.

SUMMARY

For determining parameters of Törnquist's functions there are proposed a lot of methods, among them methods of linearization. However, they are not much effective, although they constitute a good initial approximation of parameters for application of Newton-Raphson's method. In these situations optimum solutions are most often obtained already after several iteration steps.

Considerations carried out in the work concerned one general form for all three types of Törnquist's functions. It enabled presenting uniform formulas for partial derivatives regarding parameters of criterion function, and at the same time it allowed formulation of general form of vector of approximations in Newton-Raphson's method.

It is worth noticing that Törnquist's functions (1) can be written in more general form

$$y = \alpha\, x^{k_1} \left(\frac{x-\gamma}{x+\beta} \right)^{k_2},$$

where k_1, k_2 are given exponents being natural numbers.

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SZACOWANIE PARAMETRÓW FUNKCJI TÖRNQUISTA METODĄ NEWTONA-RAPHSONA

Dla wyznaczania parametrów funkcji Törnquista proponowanych jest wiele metod, wśród nich metody linearyzacji. Są one jednak mało efektywne, aczkolwiek stanowią dobre przybliżenia początkowe parametrów do zastosowania metody Newtona-

Raphsona. W tych sytuacjach rozwiązania optymalne najczęściej uzyskuje się już po kilku krokach iteracyjnych.

Rozważania przeprowadzone w pracy dotyczyły jednej ogólnej postaci dla wszystkich trzech typów funkcji Törnquista. Umożliwiło to podanie jednolitych wzorów na pochodne cząstkowe względem parametrów funkcji kryterialnej, a jednocześnie pozwoliło na sformułowanie ogólnej postaci wektora przybliżeń w metodzie Newtona-Raphsona.

Warto zauważyć, iż funkcje Törnquista (1) można zapisać w postaci ogólniejszej

$$y = \alpha\, x^{k_1} \left(\frac{x-\gamma}{x+\beta} \right)^{k_2},$$

gdzie k_1, k_2 są zadanymi wykładnikami potęg będącymi liczbami naturalnymi.