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Hamiltonian formalisms and symmetries of the Pais–Uhlenbeck oscillator

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Abstract

The study of the symmetry of Pais–Uhlenbeck oscillator initiated in Andrzejewski et al. (2014) [24] is continued with special emphasis put on the Hamiltonian formalism. The symmetry generators within the original Pais and Uhlenbeck Hamiltonian approach as well as the canonical transformation to the Ostrogradski Hamiltonian framework are derived. The resulting algebra of generators appears to be the central extension of the one obtained on the Lagrangian level; in particular, in the case of odd frequencies one obtains the centrally extended l-conformal Newton–Hooke algebra. In this important case the canonical transformation to an alternative Hamiltonian formalism (related to the free higher derivatives theory) is constructed. It is shown that all generators can be expressed in terms of the ones for the free theory and the result agrees with that obtained by the orbit method.

1. Introduction

The theories we are usually dealing with are Newtonian in the sense that the Lagrangian function depends on the first time derivatives only. There is, however, an important exception. It can happen that we are interested only in some selected degrees of freedom. By eliminating the remaining degrees one obtains what is called an effective theory. The elimination of a degree of

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freedom results in increasing the order of dynamical equations for remaining variables. Therefore, effective theories are described by Lagrangians containing higher order time derivatives [1]. Originally, these theories were proposed as a method for dealing with ultraviolet divergences [2]; this idea appeared to be quite successful in the case of gravity: the Einstein action supplied by the terms containing higher powers of curvature leads to a renormalizable theory [3]. Other examples of higher derivatives theories include the theory of the radiation reaction [4,5], the field theory on noncommutative spacetime [6,7], anyons [8,9] or string theories with the extrinsic curvature [10].

Of course, the appearance of terms with higher time derivatives leads to some problems. One of them is that the energy does not need to be bounded from below. To achieve a deeper insight into these problems and, possibly, to find a solution it is instructive to consider a quite simple, however nontrivial, higher derivatives theory. For example, it was shown in Ref. [11] (see also [12]) that the problem of the energy can be avoided (on the quantum level) in the case of the celebrated Pais–Uhlenbeck (PU) oscillator [13]. This model has been attracting considerable interest throughout the years (for the last few years, see, e.g., [11,12,14–24]). Recently, it has been shown (see [24]) that the properties of the PU oscillator, rather surprisingly, for some special values of frequencies change drastically and are related to nonrelativistic conformal symmetries. Namely, if the frequencies of oscillations are *odd* multiplicities of a basic one, i.e., they form an arithmetic sequences $\omega_k = (2k-1)\omega$, $\omega \neq 0$, for $k = 1, \dots, n$, then the maximal group of Noether symmetries of the PU Lagrangian is the l-conformal Newton–Hooke group with $l = \frac{2n-1}{2}$ (for more informations about these groups see, e.g., [25–28] and the references therein). Otherwise, the symmetry group is simpler (there are no counterparts of dilatation and conformal generators (see the algebra (2.5)).

Much attention has been also paid to Hamiltonian formulations of the PU oscillator. There exists a few approaches to Hamiltonian formalism of the PU model: decomposition into the set of the independent harmonic oscillators proposed by Pais and Uhlenbeck in their original paper [13], Ostrogradski approach based on the Ostrogradski method [29] of constructing Hamiltonian formalism for theories with higher time derivatives and the last one, applicable in the case of odd frequencies (mentioned above), which exhibits the l-conformal Newton–Hooke group structure of the model. Consequently, there arises a natural question about the relations between them as well as the realization of the symmetry on the Hamiltonian level? The aim of this work is to give the answer to this question.

The paper is organized as follows. After recalling, in Section 2, some informations concerning symmetry of the PU model on the Lagrangian level, we start with the harmonic decoupled approach. We find, on the Hamiltonian level, the form of generators (for both generic and odd frequencies) and we show that they, indeed, form the algebra which is central extension the one appearing on the Lagrangian level. Section 4 is devoted to the study of the relation between the above approach and the Ostrogradski one. Namely, we construct the canonical transformation which relates the Ostrogradski Hamiltonian to the one describing the decouple harmonic oscillator. This transformation enables us to find the remaining symmetry generators in terms of Ostrogradski variables. The next section is devoted to the case of odd frequencies where the additional natural approach can be constructed. In this framework the Hamiltonian is the sum of the one for the free higher derivatives theory and the conformal generator. We derive a canonical transformation which relates this new Hamiltonian to the one for the PU oscillator with odd frequencies. Moreover, we apply the method (see [30]) of constructing integrals of motion for the systems with symmetry to find all symmetry generators. Next, by direct calculations we show that they are related by the, above mentioned, canonical transformation to the ones of the PU model described in terms decoupled oscillators. We also express symmetry generators in terms of their counterparts in the free theory.

In concluding Section 6, we summarize our results and discuss possible further developments. Finally, Appendix A constitutes technical support for the mains results. We derive there some relations and identities which are crucial for our work.

2. PU oscillator and its symmetry

Let us consider the three-dimensional PU oscillator, i.e., the system which is described by the following Lagrangian [13]

$$L = -\frac{1}{2}\vec{x}\prod_{k=1}^{n}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\vec{x}, \tag{2.1}$$

where $0 < \omega_1 < \omega_2 < \ldots < \omega_n$ and $n = 1, 2, \ldots$. Lagrangian (2.1) implies the following equation of motion

$$\prod_{k=1}^{n}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\vec{x} = 0, \tag{2.2}$$

which possesses the general solution of the form

$$\vec{x}(t) = \sum_{k=1}^{n}(\vec{\alpha}_k \cos\omega_k t + \vec{\beta}_k \sin\omega_k t), \tag{2.3}$$

where $\vec{\alpha}$'s and $\vec{\beta}$'s are some arbitrary constants.

As it has been mentioned in the Introduction the structure of the maximal symmetry group of Lagrangian (2.1) depends on the values of ω's. If the frequencies of oscillation are odd, i.e., they form an arithmetic sequence $\omega_k = (2k-1)\omega$, $\omega \neq 0$, $k = 1, \ldots, n$, then the maximal group of Noether symmetries of the system (2.1) is the l-conformal Newton–Hooke group, with $l = \frac{2n-1}{2}$. It is the group which Lie algebra is spanned by $H, D, K, J^{\alpha\beta}$ and C_p^α, $\alpha, \beta = 1, 2, 3$, $p = 0, 1, \ldots, 2n-1$, satisfying the following commutation rules

$$\begin{aligned}
&[H, D] = H - 2\omega^2 K, \quad [H, K] = 2D, \quad [D, K] = K,\\
&[D, \vec{C}_p] = \left(p - \frac{2n-1}{2}\right)\vec{C}_p, \quad [K, \vec{C}_p] = (p-2n+1)\vec{C}_{p+1},\\
&[H, \vec{C}_p] = p\vec{C}_{p-1} + (p-2n+1)\omega^2\vec{C}_{p+1},\\
&\left[J^{\alpha\beta}, J^{\gamma\delta}\right] = \delta^{\alpha\delta}J^{\gamma\beta} + \delta^{\alpha\gamma}J^{\beta\delta} + \delta^{\beta\gamma}J^{\delta\alpha} + \delta^{\beta\alpha}J^{\alpha\gamma},\\
&\left[J^{\alpha\beta}, C_p^\gamma\right] = \delta^{\alpha\gamma}C_p^\beta - \delta^{\beta\gamma}C_p^\alpha.
\end{aligned} \tag{2.4}$$

Although this algebra is isomorphic to the l-conformal Galilei one (the latter can be obtained by a linear change of the basis $H \to H - \omega^2 K$, see [25,26,28] and [31–36] for more recent developments of this algebra) the use of the basis (2.4) implies the change of the Hamiltonian which alters the dynamics.

In the case of *generic* frequencies the maximal symmetry group is simpler. Its Lie algebra consists of H, $J^{\alpha\beta}$ and $\vec{C}_k^\pm$, $k = 1, \ldots, n$. The action of $J^{\alpha\beta}$ remains unchanged and only commutations rules between H and $\vec{C}$'s must be modified

$$\begin{aligned}&\left[H,\vec{C}_k^+\right]=-\omega_k\vec{C}_k^-,\\&\left[H,\vec{C}_k^-\right]=\omega_k\vec{C}_k^+.\end{aligned}\tag{2.5}$$

Both symmetry algebras posses central extension:

$$\left[C_p^\alpha,C_q^\beta\right]=(-1)^p p!q!\delta_{\alpha\beta}\delta_{2n-1,p+q},\tag{2.6}$$

in the odd case and

$$\left[C_k^{+\alpha}C_j^{-\beta}\right]=\frac{\omega_k}{\rho_k}\delta_{kj}\delta^{\alpha\beta},\tag{2.7}$$

in the generic case; which will turn out to be necessarily (see the next section) to construct the symmetry algebra on the Hamiltonian level.

3. Decoupled oscillators approach

An approach to the Hamiltonian formalism of the PU model was proposed in Ref. [13] where it was demonstrated that the Hamiltonian of the PU oscillator (in dimension one) turns into the sum of the harmonic Hamiltonians with alternating sign. To show this we follow the reasoning of Ref. [13] and introduce new variables

$$\vec{x}_k=\Pi_k\vec{x},\quad k=1,\dots,n;\tag{3.1}$$

where Π_k is the projective operator:

$$\Pi_k=\sqrt{|\rho_k|}\prod_{\substack{i=1\\i\neq k}}^n\left(\frac{d^2}{dt^2}+\omega_i^2\right),\tag{3.2}$$

and

$$\rho_k=\frac{1}{\prod_{\substack{i=1\\i\neq k}}^n(\omega_i^2-\omega_k^2)},\quad k=1,2,\dots,n.\tag{3.3}$$

Note that ρ_k are alternating in sign. Then one finds

$$\vec{x}=\sum_{k=1}^n(-1)^{k-1}\sqrt{|\rho_k|}\vec{x}_k,\tag{3.4}$$

as well as

$$L=-\frac{1}{2}\sum_{k=1}^n(-1)^{k-1}\vec{x}_k\left(\frac{d^2}{dt^2}+\omega_k^2\right)\vec{x}_k=\frac{1}{2}\sum_{k=1}^n(-1)^{k-1}\left(\dot{\vec{x}}_k-\omega_k^2\vec{x}_k^2\right)+t.d.\tag{3.5}$$

The corresponding Hamiltonian reads

$$H=\frac{1}{2}\sum_{k=1}^n(-1)^{k-1}\left(\vec{p}_k^2+\omega_k^2\vec{x}_k^2\right),\tag{3.6}$$

while the canonical equations of motion are of the form

$$\dot{\vec{x}}_k=(-1)^{k-1}\vec{p}_k,\quad\dot{\vec{p}}_k=(-1)^k\omega_k^2\vec{x}_k.\tag{3.7}$$

Taking into account the form of the general solution (2.3) we see that the dynamics of the new canonical variables is given by

$$\vec{x}_k = \frac{(-1)^{k-1}}{\sqrt{|\rho_k|}}\big(\vec{\alpha}_k \cos(\omega_k t) + \vec{\beta}_k \sin(\omega_k t)\big),$$
$$\vec{p}_k = \frac{\omega_k}{\sqrt{|\rho_k|}}\big(\vec{\beta}_k \cos(\omega_k t) - \vec{\alpha}_k \sin(\omega_k t)\big). \tag{3.8}$$

Therefore, we have a correspondence between the set of solutions of the Lagrange equation (2.2) and the set of solutions of the canonical equations (3.7). Consequently, we can translate the action of the group symmetry from the Lagrangian level to the Hamiltonian one and find all the symmetry generators in terms of oscillator canonical variables. We will show that the generators, obtained in this way, form the algebra which is the central extension of the symmetry algebra on the Lagrangian level.

In the generic case it is very easy to find the form of the remaining (the Hamiltonian is given by (3.6)) symmetry generators on the Hamiltonian level. First, let us note that the infinitesimal action of $\vec{\mu}_k \vec{C}_k^+$ and $\vec{\nu}_k \vec{C}_k^-$, $k = 1, \ldots, n$, on the Lagrangian level, takes the form

$$\vec{x}'(t) = \vec{x}(t) + \sum_{k=1}^{n} (\vec{\mu}_k \cos\omega_k t + \vec{\nu}_k \sin\omega_k t). \tag{3.9}$$

Acting with Π_k and applying Eq. (3.7) we find the infinitesimal action of $\vec{C}_k^{\pm}$ on the phase space; by virtue of

$$\delta(\cdot) = \epsilon\{\cdot, \text{Generator}\}, \tag{3.10}$$

we obtain the following generators:

$$\vec{C}_k^+ = \frac{(-1)^{k-1}}{\sqrt{|\rho_k|}} \cos(\omega_k t)\vec{p}_k + \frac{\omega_k}{\sqrt{|\rho_k|}} \sin(\omega_k t)\vec{x}_k,$$
$$\vec{C}_k^- = \frac{(-1)^{k-1}}{\sqrt{|\rho_k|}} \sin(\omega_k t)\vec{p}_k - \frac{\omega_k}{\sqrt{|\rho_k|}} \cos(\omega_k t)\vec{x}_k, \tag{3.11}$$

which commute to the central charge – according to (2.7). Similarly, the angular momentum generators read

$$J^{\alpha\beta} = \sum_{k=1}^{n} \big(x_k^\alpha p_k^\beta - p_k^\alpha x_k^\beta\big). \tag{3.12}$$

Consequently, we obtain the centrally extended algebra (2.5).

3.1. Odd frequencies

In the odd case the symmetry group is reacher and, therefore, this case is much more interesting. We assume now that the frequencies form the arithmetic sequence, i.e., $\omega_k = (2k-1)\omega$, $k = 1, \ldots, n$. In this case the main point is that the numbers ρ_k can be explicitly computed; the final result reads

$$\rho_k = \frac{(-1)^{k-1}(2k-1)}{(4\omega^2)^{n-1}(n-k)!(n+k-1)!}, \qquad k = 1, \ldots, n. \tag{3.13}$$

Consequently, one has useful relations

$$\frac{|\rho_k|}{|\rho_{k+1}|}=\frac{(2k-1)(n+k)}{(2k+1)(n-k)},\qquad k=1,\dots,n-1. \tag{3.14}$$

Next, let us note that the following Fourier expansion holds (see Appendix A)

$$\sin^p\omega t\cos^{2n-1-p}\omega t=\begin{cases}\sum_{k=1}^m\gamma^+_{kp}\cos(2k-1)\omega t, & p\text{-even};\\ \sum_{k=1}^m\gamma^-_{kp}\sin(2k-1)\omega t, & p\text{-odd};\end{cases} \tag{3.15}$$

where $\gamma^\pm_{kp}$ can be expressed in terms of sum of products of binomial coefficients; however, their explicit form is not very useful; for our purposes some properties of $\gamma^\pm_{kp}$ (see (A.2)–(A.6)) will turn out to be more fruitful. Now, using Eq. (3.15) we can rewrite the infinitesimal action (3.9), in the case of odd frequencies, in the equivalent form

$$\vec{x}'(t)=\vec{x}(t)+\frac{1}{\omega^p}\vec{\epsilon}_p\sin^p\omega t\cos^{2n-1-p}\omega t, \tag{3.16}$$

which gives suitable family of the generators $\vec{C}_p$, $p=0,1,2,\dots,2n-1$, on the Lagrangian level, i.e., satisfying commutation rules of the l-conformal Newton–Hooke algebra (cf. [24]).

In order to find the action of $\vec{C}_p$ in the Hamiltonian formalism, we use Eq. (3.15) together with (3.1) and (3.7), which yields

$$\vec{x}'_k=\vec{x}_k+\frac{(-1)^{k-1}\vec{\epsilon}_p}{\omega^p\sqrt{|\rho_k|}}\begin{cases}\gamma^+_{kp}\cos(2k-1)\omega t, & p\text{-even};\\ \gamma^-_{kp}\sin(2k-1)\omega t, & p\text{-odd};\end{cases} \tag{3.17}$$

$$\vec{p}'_k=\vec{p}_k+\frac{(2k-1)\omega\vec{\epsilon}_p}{\omega^p\sqrt{|\rho_k|}}\begin{cases}-\gamma^+_{kp}\sin(2k-1)\omega t, & p\text{-even};\\ \gamma^-_{kp}\cos(2k-1)\omega t, & p\text{-odd}.\end{cases} \tag{3.18}$$

Using Eq. (3.10) we derive the explicit expression for the generators $\vec{C}_p$ in terms of the canonical variables

$$\vec{C}_p=\sum_{k=1}^n\frac{\gamma^+_{kp}}{\omega^p\sqrt{|\rho_k|}}\big((-1)^{k-1}\cos\big((2k-1)\omega t\big)\vec{p}_k+(2k-1)\omega\sin\big((2k-1)\omega t\big)\vec{x}_k\big), \tag{3.19}$$

for p even, and

$$\vec{C}_p=\sum_{k=1}^n\frac{\gamma^-_{kp}}{\omega^p\sqrt{|\rho_k|}}\big((-1)^{k-1}\sin\big((2k-1)\omega t\big)\vec{p}_k-(2k-1)\omega\cos\big((2k-1)\omega t\big)\vec{x}_k\big), \tag{3.20}$$

for p odd. Eqs. (3.19) and (3.20) can be inverted to yield $\vec{x}_k$ and $\vec{p}_k$ in terms of the generators $\vec{C}_p$

$$\begin{aligned}\vec{p}_k={}&(-1)^{k-1}\sqrt{|\rho_k|}\cos\big((2k-1)\omega t\big)\sum_{p=0}^{2n-1}{}'' \beta^+_{pk}\omega^p\vec{C}_p\\ &+(-1)^{k-1}\sin\big((2k-1)\omega t\big)\sum_{p=0}^{2n-1}{}' \beta^-_{pk}\omega^p\vec{C}_p,\end{aligned} \tag{3.21}$$

$$\vec{x}_k = \frac{\sqrt{|\rho_k|}}{(2k-1)\omega}\sin\big((2k-1)\omega t\big)\sum_{p=0}^{2n-1}{}'' \beta^{+}_{pk}\omega^p\vec{C}_p - \frac{\sqrt{|\rho_k|}}{(2k-1)\omega}\cos\big((2k-1)\omega t\big)\sum_{p=0}^{2n-1}{}' \beta^{-}_{pk}\omega^p\vec{C}_p, \tag{3.22}$$

where β^+, β^- are the inverse matrices to γ^+, γ^- while one and two primes $'$, $''$ denote the sum over odd and even indices, respectively.[2]

Next, we find the action of the dilatation generator. To this end let us recall (cf. [24]) that the infinitesimal action of dilatation on coordinates is of the form

$$\vec{x}'(t) = \vec{x}(t) - \frac{\epsilon}{2\omega}\big((2n-1)\omega\cos(2\omega t)\vec{x}(t) - \sin(2\omega t)\dot{\vec{x}}(t)\big). \tag{3.23}$$

Substituting (3.4) and acting with the projectors Π_k we obtain, due to (3.1) and (3.7), the infinitesimal dilatation transformation on the phase space

$$\begin{aligned}\vec{x}'_k &= \vec{x}_k + \frac{\epsilon}{2\sqrt{|\rho_k|}}\cos(2\omega t)\big(\sqrt{|\rho_{k-1}|}(n-k+1)\vec{x}_{k-1} + \sqrt{|\rho_{k+1}|}(n+k)\vec{x}_{k+1}\big)\\ &\quad + \frac{\epsilon(-1)^k}{2\omega\sqrt{|\rho_k|}}\sin(2\omega t)\left(\frac{\sqrt{|\rho_{k-1}|}}{2k-3}(n-k+1)\vec{p}_{k-1} - \frac{\sqrt{|\rho_{k+1}|}}{2k+1}(n+k)\right)\vec{p}_{k+1},\\ \vec{x}'_1 &= \vec{x}_1 - \frac{\epsilon}{2\sqrt{|\rho_1|}}\left(\sqrt{|\rho_2|}(n+1)\cos(2\omega t)\vec{x}_2 + \sin(2\omega t)\sqrt{|\rho_2|}\frac{(n+1)}{3\omega}\vec{p}_2\right.\\ &\quad \left. - n\cos(2wt)\sqrt{|\rho_1|}\vec{x}_1 + \frac{n}{\omega}\sin(2\omega t)\sqrt{|\rho_1|}\vec{p}_1\right),\end{aligned} \tag{3.24}$$

$$\begin{aligned}\vec{p}'_k &= \vec{p}_k - \frac{\epsilon(2k-1)}{2\sqrt{|\rho_k|}}\cos(2\omega t)\left(\frac{\sqrt{|\rho_{k-1}|}}{2k-3}(n-k+1)\vec{p}_{k-1} + \frac{\sqrt{|\rho_{k+1}|}}{2k+1}(n+k)\vec{p}_{k+1}\right)\\ &\quad + \frac{\epsilon\omega(-1)^k(2k-1)}{2\sqrt{|\rho_k|}}\sin(2\omega t)\big(\sqrt{|\rho_{k-1}|}(n-k+1)\vec{x}_{k-1} - \sqrt{|\rho_{k+1}|}(n+k)\vec{x}_{k+1}\big),\\ \vec{p}'_1 &= \vec{p}_1 - \frac{\epsilon}{2\sqrt{|\rho_1|}}\cos(2\omega t)\left(-n\sqrt{|\rho_1|}\vec{p}_1 + \frac{\sqrt{|\rho_2|}}{3}(n+1)\vec{p}_2\right)\\ &\quad + \frac{\epsilon\omega}{2\sqrt{|\rho_1|}}\sin(2\omega t)\big(n\sqrt{|\rho_1|}\vec{x}_1 + \sqrt{|\rho_2|}(n+1)\vec{x}_2\big),\end{aligned} \tag{3.25}$$

where $k > 1$ and, by definition, we put $\vec{x}_{n+1} = \vec{p}_{n+1} = 0$. One can check, using Eq. (3.14), that (3.24) and (3.25) define the infinitesimal canonical transformation generated (according to (3.10)) by

$$D = \frac{-1}{2\omega}\big(\omega A\cos(2\omega t) + B\sin(2\omega t)\big), \tag{3.26}$$

where

[2] We will use this convention throughout the article.

$$A=-\sum_{k=1}^{n}\left(\sqrt{\left|\frac{\rho_{k-1}}{\rho_k}\right|}(n-k+1)\vec{x}_{k-1}+\sqrt{\left|\frac{\rho_{k+1}}{\rho_k}\right|}(n+k)\vec{x}_{k+1}\right)\vec{p}_k+n\vec{x}_1\vec{p}_1,$$

$$B=-\sum_{k=1}^{n}(-1)^k\frac{n-k+1}{2k-3}\sqrt{\left|\frac{\rho_{k-1}}{\rho_k}\right|}\left(\vec{p}_k\vec{p}_{k-1}-(2k-1)(2k-3)\omega^2\vec{x}_k\vec{x}_{k-1}\right)$$
$$+\frac{1}{2}n\left(\omega^2\vec{x}_1^2-\vec{p}_1^2\right), \tag{3.27}$$

and, by definition, $\vec{x}_0=\vec{p}_0=0$. The meaning of the components A and B will become more clear in Section 5 (see (5.5)).

Similar calculations can be done for the conformal generator K. Namely, the infinitesimal conformal transformation, on the Lagrangian level, reads

$$\vec{x}'(t)=\vec{x}(t)-\frac{\epsilon}{2\omega^2}\left((2n-1)\omega\sin(2\omega t)\vec{x}(t)+\left(\cos(2\omega t)-1\right)\dot{\vec{x}}(t)\right). \tag{3.28}$$

Substituting $\vec{x}$ and acting with the projector Π_k we obtain the infinitesimal conformal transformation on the phase space and consequently (due to (3.10)) the explicit form of the generator K

$$K=\frac{1}{2\omega^2}\left(B\cos(2\omega t)-\omega A\sin(2\omega t)+H\right). \tag{3.29}$$

Finally, the angular momentum takes the same form as in the generic case

$$J^{\alpha\beta}=\sum_{k=1}^{n}\left(x_k^\alpha p_k^\beta-p_k^\alpha x_k^\beta\right). \tag{3.30}$$

It remains to verify that obtained generators, indeed, yield integrals of motion and define the centrally extended l-conformal Newton–Hooke algebra. To this end we need a few identities which are proven in Appendix A. First, we compute the commutators of $\vec{C}$'s and check that they give the proper central extension. The only nontrivial case is $[C_p^\alpha, C_q^\beta]$ with p even and q odd (or conversely). We have

$$\begin{aligned}\left[C_p^\alpha, C_q^\beta\right]&=\frac{\omega(2\omega)^{2(n-1)}\delta^{\alpha\beta}}{\omega^{p+q}}\sum_{k=1}^{n}(-1)^{k-1}(n-k)!(n+k-1)!\gamma_{kp}^+\gamma_{kq}^-\\&=\frac{p!(2n-1-p)!\omega^{2n-1}\delta^{\alpha\beta}}{\omega^{p+q}}\sum_{k=1}^{n}(-1)^{k-1}\beta_{pk}^+\gamma_{kq}^-\\&=\frac{p!(2n-1-p)!\omega^{2n-1}\delta^{\alpha\beta}}{\omega^{p+q}}\sum_{k=1}^{n}\beta_{2n-1-p,k}^-\gamma_{kq}^-\\&=\frac{p!(2n-1-p)!\omega^{2n-1}\delta^{\alpha\beta}}{\omega^{p+q}}\delta_{2n-1,p+q}=p!q!\delta_{\alpha\beta}\delta_{2n-1,p+q},\end{aligned} \tag{3.31}$$

where we use consecutively Eqs. (3.19), (3.20), (3.13), (A.3) and (A.2). For p odd and q even we obtain the same result except the extra minus sign. Consequently, we obtain the central extension (2.6). In order to find the remaining commutators let us note that

$$
\begin{aligned}
&[A, B] = -2H, \\
&[B, H] = 2\omega^2 A, \\
&[A, H] = -2B.
\end{aligned} \tag{3.32}
$$

The proof of the above relations is straightforward although tedious and involve the use of (3.14). Now, by virtue of Eq. (3.32), it is easy to check that the generators H, D, K satisfy the first line of Eqs. (2.4).

Now, we find the adjoint action of $H, D, K, J^{\alpha\beta}$ on $\vec{C}_p$. Since the calculations are rather wearisome and lengthy we sketch only the main points. To show that $[H, \vec{C}_p]$ gives proper rule we use the identity (A.4). The case $[D, \vec{C}_p]$ is more involved; however, using repeatedly Eqs. (3.14) and (A.5) we arrive at the desired result. Similarly to obtain $[H, \vec{C}_k]$, first, we use Eq. (3.14) and then Eq. (A.6). Finally, it is easy to compute the commutators involving angular momentum.

Having all the commutation rules and (A.4) it is not hard to check that the obtained generators are constants of motion. This concludes the proof that, on the Hamiltonian level, they are symmetry generators and form the centrally extended l-conformal Newton–Hooke algebra.

4. Ostrogradski approach

Since the PU oscillator is an example of higher derivatives theory, it is natural to use the Hamiltonian formalism proposed by Ostrogradski [29]. To this end let us expand Lagrangian (2.1) in the sum of higher derivatives terms (here $\vec{Q} = \vec{x}$)

$$
L = -\frac{1}{2}\vec{Q}\prod_{k=1}^{n}\left(\frac{d^2}{dt^2} + \omega_k^2\right)\vec{Q} = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k-1}\sigma_k\big(\vec{Q}^{(k)}\big)^2, \tag{4.1}
$$

where

$$
\sigma_k = \sum_{i_1<\ldots<i_{n-k}} \omega_{i_1}^2 \cdots \omega_{i_{n-k}}^2, \qquad k = 0, \ldots, n, \qquad \sigma_n = 1. \tag{4.2}
$$

It can be shown (by standard reasoning) that the following identities hold

$$
\sum_{k=1}^{n} \rho_k \omega_k^{2m} = 0, \qquad m = 0, \ldots, n-2, \tag{4.3}
$$

$$
\sum_{k=1}^{n} \rho_k \omega_k^{2(n-1)} = (-1)^{n+1}, \tag{4.4}
$$

$$
\sum_{m=0}^{n} \sigma_m (-1)^m \sum_{k=1}^{n} \rho_k \omega_k^{2(r-n+m-1)} = 0, \qquad r \geq n, \tag{4.5}
$$

where ρ_k is given by Eq. (3.3). Now, we introduce the Ostrogradski variables

$$
\begin{aligned}
&\vec{Q}_k = \vec{Q}^{(k-1)}, \\
&\vec{P}_k = \sum_{j=0}^{n-k}\left(-\frac{d}{dt}\right)^j \frac{\partial L}{\partial \vec{Q}^{(k+j)}} = (-1)^{k-1}\sum_{j=k}^{n}\sigma_j \vec{Q}^{(2j-k)},
\end{aligned} \tag{4.6}
$$

for $k = 1, \dots, n$. Then the Ostrogradski Hamiltonian takes the form

$$H = \frac{(-1)^{n-1}}{2}\vec{P}_n^2 + \sum_{k=2}^{n}\vec{P}_{k-1}\vec{Q}_k - \frac{1}{2}\sum_{k=1}^{n}(-1)^k\sigma_{k-1}\vec{Q}_k^2. \tag{4.7}$$

By virtue of Eqs. (3.7) and (4.6), for $k = 1, \dots, n$, we find

$$\begin{aligned}
\vec{Q}_k &= (-1)^{\frac{k-1}{2}}\sum_{j=1}^{n}\sqrt{|\rho_j|}(-1)^{j-1}\omega_j^{k-1}\vec{x}_j, \quad k\text{-odd}; \\
\vec{Q}_k &= (-1)^{\frac{k}{2}-1}\sum_{j=1}^{n}\sqrt{|\rho_j|}\omega_j^{k-2}\vec{p}_j, \quad k\text{-even};
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\vec{P}_k &= (-1)^{\frac{k}{2}-1}\sum_{i=1}^{n}(-1)^{i-1}\sqrt{|\rho_i|}\left(\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k}\right)\vec{x}_i, \quad k\text{-even}; \\
\vec{P}_k &= (-1)^{\frac{k-3}{2}}\sum_{i=1}^{n}\sqrt{|\rho_i|}\left(\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k-1}\right)\vec{p}_i, \quad k\text{-odd}.
\end{aligned} \tag{4.9}$$

One can show that Eqs. (4.8) and (4.9) define a canonical transformation; to compute the Poisson brackets $\{\vec{Q}_k, \vec{Q}_j\}$ and $\{\vec{Q}_k, \vec{P}_j\}$ we use (4.3) and (4.3)–(4.5), respectively; computing $\{\vec{P}_k, \vec{P}_j\}$ is the most complicated one and involves considering two cases $k - j \lesseqgtr 1$ as well as applying Eqs. (4.3) and (4.5).

Next, let us note that the inverse transformation is of the form

$$\begin{aligned}
\vec{x}_i &= {\sum_{k=1}^{n}}' (-1)^{\frac{k-3}{2}}\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k-1}\sqrt{|\rho_i|}\vec{Q}_k + {\sum_{k=1}^{n}}'' (-1)^{\frac{k}{2}}\sqrt{|\rho_i|}\omega_i^{k-2}\vec{P}_k, \\
\vec{p}_i &= {\sum_{k=1}^{n}}'' (-1)^{\frac{k}{2}+i-1}\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k}\sqrt{|\rho_i|}\vec{Q}_k + {\sum_{k=1}^{n}}' (-1)^{\frac{k+1}{2}+i}\sqrt{|\rho_i|}\omega_i^{k-1}\vec{P}_k.
\end{aligned} \tag{4.10}$$

No, we can try to find the symmetry generators (both in the odd and generic cases) in terms of the Ostrogradski variables. Of course, we expect that the Hamiltonian (3.6) should be transformed into the Ostrogradski one. Indeed, using (4.3)–(4.5) repeatedly we arrive, after straightforward but rather arduous computations (considering two cases: n-odd, even), at the Ostrogradski Hamiltonian (4.7).

Similarly, applying Eqs. (4.3)–(4.5), we check that the angular momentum (in both cases (3.12) and (3.30)) transforms under (4.10) into Ostrogradski angular momentum

$$J^{\alpha\beta} = \sum_{k=1}^{n}\left(Q_k^\alpha P_k^\beta - Q_k^\beta P_k^\alpha\right). \tag{4.11}$$

As far as the generators $\vec{C}$'s are concerned (again using (4.3)–(4.5)) we obtain the following expressions:

$$\vec{C}_k^- = \sum_{k=1}^{n} (\cos\omega_i t)^{(k-1)} \vec{P}_k - \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{j=k}^{n} \sigma_j (\cos\omega_i t)^{2j-k} \right) \vec{Q}_k,$$
$$\vec{C}_k^- = \sum_{k=1}^{n} (\sin\omega_i t)^{(k-1)} \vec{P}_k - \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{j=k}^{n} \sigma_j (\sin\omega_i t)^{2j-k} \right) \vec{Q}_k, \tag{4.12}$$

in the case of generic frequencies, and

$$\vec{C}_p = \frac{1}{\omega^p} \sum_{k=1}^{n} \Bigg(\vec{P}_k \left(\sin^p \omega t \cos^{2n-1-p} \omega t\right)^{(k-1)} + (-1)^k \vec{Q}_k \sum_{j=k}^{n} \sigma_j \left(\sin^p \omega t \cos^{2n-1-p} \omega t\right)^{(2j-k)} \Bigg), \tag{4.13}$$

in the odd case; which perfectly agrees with the definitions of the Ostrogradski canonical variables (4.6) and the action of $\vec{C}$'s on Q (Eqs. (3.9) and (3.16)). Similar reasoning can be done for the remaining two generators D and K in the odd case. Then, they become bilinear forms in the Ostrogradski variables; however the explicit form of coefficients is difficult to simplify and not transparent thus we skip it here.

5. Algebraic approach to odd case

Since the l-conformal Newton–Hooke algebra is related to the l-conformal Galilei one by the change of Hamiltonian

$$H = \tilde{H} + \omega^2 \tilde{K}, \tag{5.1}$$

where tilde refers to generators of the free theory (which possesses the l-conformal Galilei symmetry); therefore, it would be instructive to construct an alternative Hamiltonian formalism for the PU-model (in the case of *odd* frequencies) with the help of the one for the *free* higher derivatives theory.

Denoting by $\vec{q}_m, \vec{\pi}_m$, $m = 0, \dots, n-1$ the phase space coordinates of the free theory and adapting the results of Ref. [37] to our conventions we obtain the following form of the generators of the free theory (at time $t = 0$)

$$\tilde{H} = \frac{(-1)^{n+1}}{2} \pi_{n-1}^2 - \sum_{m=1}^{n-1} \vec{q}_m \vec{\pi}_{m-1},$$
$$\tilde{D} = \sum_{m=0}^{n-1} \left(m - \frac{2n-1}{2} \right) \vec{q}_m \vec{\pi}_m,$$
$$\tilde{K} = (-1)^{n+1} \frac{n^2}{2} \vec{q}_{n-1}^2 + \sum_{m=0}^{n-2} (2n-1-m)(m+1) \vec{q}_m \vec{\pi}_{m+1},$$
$$\tilde{J}^{\alpha\beta} = \sum_{m=0}^{n-1} \left(q_m^\alpha \pi_m^\beta - q_m^\beta \pi_m^\alpha \right),$$
$$\tilde{\vec{C}}_m = (-1)^{m+1} m! \vec{\pi}_m, \quad m = 0, \dots, n-1,$$
$$\tilde{\vec{C}}_{2n-1-m} = (2n-1-m)! \vec{q}_m, \quad m = 0, \dots, n-1. \tag{5.2}$$

Of course, the change of the algebra basis given by (5.1) induces the corresponding one for the coordinates in dual space of the algebra (denoted in the same way); consequently we define the new Hamiltonian as follows

$$H = \tilde{H} + \omega^2 \tilde{K} = \frac{(-1)^{n+1}}{2}\pi_{n-1}^2 - \sum_{m=1}^{n-1} \vec{q}_m \vec{\pi}_{m-1} + (-1)^{n+1}\frac{n^2\omega^2}{2}\vec{q}_{n-1}^2 + \sum_{m=0}^{n-2}(2n-1-m)(m+1)\omega^2 \vec{q}_m \vec{\pi}_{m+1}. \tag{5.3}$$

We will show that (5.3) is indeed the PU Hamiltonian in $\vec{q}_m, \vec{\pi}_m$ coordinates and we will find the remaining generators in terms of them. To this end let us define the following transformation

$$\vec{x}_k = (-1)^k \left(\sum_{m=0}^{n-1}{}'' \frac{\omega^{-m}}{m!\sqrt{|\rho_k|}} \gamma_{km}^+ \vec{q}_m + \sum_{m=0}^{n-1}{}' \frac{m!\omega^m\sqrt{|\rho_k|}}{(2k-1)\omega} \beta_{2n-1-m,k}^+ \vec{\pi}_m \right),$$
$$\vec{p}_k = (-1)^k \left(\sum_{m=0}^{n-1}{}' \frac{\omega^{-m}(2k-1)\omega}{m!\sqrt{|\rho_k|}} \gamma_{k,2n-1-m}^+ \vec{q}_m + \sum_{m=0}^{n-1}{}'' m!\omega^m\sqrt{|\rho_k|}\beta_{mk}^+ \vec{\pi}_m \right), \tag{5.4}$$

for $k = 1, \ldots, n$. Using (3.14) and (A.3) we check that (5.4) define a canonical transformation. Moreover, by applying Eqs. (3.14) and (A.2)–(A.4) we check that the PU Hamiltonian (3.6) (with odd frequencies) transforms into (5.3). The remaining generators can be also transformed. First, using (3.14), (A.2), (A.3), (A.5) and (A.6), after troublesome computations, we find that

$$A = -2\tilde{D},$$
$$B = -\tilde{H} + \omega^2 \tilde{K}, \tag{5.5}$$

and, consequently, we obtain a nice interpretation of A and B. Using Eqs. (5.5), one checks that H, D, K take the form

$$H = \tilde{H} + \omega^2 \tilde{K},$$
$$D = \tilde{D}\cos 2\omega t + \frac{1}{2\omega}\left(\tilde{H} - \omega^2 \tilde{K}\right)\sin 2\omega t,$$
$$K = \frac{1}{2}(1 + \cos 2\omega t)\tilde{K} + \frac{1}{2\omega^2}(1 - \cos 2\omega t)\tilde{H} + \frac{\sin 2\omega t}{\omega}\tilde{D}. \tag{5.6}$$

Finally, the angular momentum reads

$$J^{\alpha\beta} = \sum_{m=0}^{n-1}\left(q_m^\alpha \pi_m^\beta - q_m^\beta \pi_m^\alpha\right), \tag{5.7}$$

i.e., takes the same form as the one for the free theory (according to it commutes with H). The generators $\vec{C}_k$ are obtained by plugging (5.4) into (3.19) and (3.20), see also (5.17).

Summarizing, we expressed all PU symmetry generators in terms of the ones for free theory (and consequently in terms of $\vec{q}_m$ and $\vec{\pi}_m$) and we see that the both sets of generators (except Hamiltonian) agree at time $t = 0$. This result becomes even more evident if we apply the algorithm of constructing integrals of motion for Hamiltonian system with symmetry presented in Ref. [30]. Namely, for the Lie algebra spanned by X_i, $i = 1, \dots, n$, $[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$, with the adjoint action

$$Ad_g(X_i) = gX_ig^{-1} = \sum_{j=1}^{n} D_i^j(g)X_j, \tag{5.8}$$

the integrals of motion $X_i(\xi, t)$ corresponding to the generators X_i are of the form

$$X_i(\xi, t) = \sum_{j=1}^{n} D_i^j\left(e^{tH}\right)\xi_j, \tag{5.9}$$

where ξ's are the coordinates of the dual space to the Lie algebra (more precisely, their restriction to the orbits of the coadjoint action in the dual space).

Let us apply this approach to our case. One can check that for H, D, K, $J^{\alpha\beta}$ Eq. (5.9) gives (5.6) and (5.7). For $\vec{C}_p$ we have

$$\vec{C}_p = e^{tH}\tilde{\vec{C}}_p e^{-tH} = \sum_{r=0}^{2n-1} a_{pr}(t)\tilde{\vec{C}}_r, \quad p = 0, \dots, 2n-1, \tag{5.10}$$

where the functions a_{pr} satisfy the set of equations

$$\dot{a}_{pr}(t) = (r+1)a_{p,r+1}(t) + (r-2n)\omega^2 a_{p,r-1}(t), \tag{5.11}$$

with $a_{k,-1} = a_{k,2n} = 0$ and the initial conditions $a_{pr}(0) = \delta_{pr}$. Substituting $a_{pr}(t) = \hat{a}_{pr}(t\omega)\omega^r$ we obtain

$$\dot{\hat{a}}_{pr}(t) = (r+1)\hat{a}_{p,r+1}(t) + (r-2n)\hat{a}_{p,r-1}(t), \tag{5.12}$$

with appropriate initial conditions. It turns out that for fixed p Eq. (5.12) is strongly related to the evolution of $\vec{q}$'s and $\vec{\pi}$'s in the PU model with odd frequencies. More precisely, the canonical equations of motion for the Hamiltonian (5.3) are equivalent to Eq. (5.12) for fixed p (cf. [38]). Consequently, the solution can be written in terms of combinations of harmonics with odd frequencies:

$$\hat{a}_{pr}(t) = \sum_{a=-(2n-1)}^{2n-1}{}' i^r \beta_{ra} e^{iat} s_a^p, \tag{5.13}$$

where β_{pa} is given by (A.24) and s_a^p are some constants (see [38])). Taking into account the initial conditions, we obtain

$$a_{pr}(t) = \omega^{r-p} \sum_{a=-(2n-1)}^{2n-1}{}' i^r \beta_{ra}\gamma_{ap} e^{iat\omega}. \tag{5.14}$$

By virtue of Eqs. (A.25) and (A.29), we have

$$a_{pr}(t) = \omega^{r-p} \sum_{k=1}^{n} \beta_{rk}^{\pm}\gamma_{kp}^{\pm} \cos(2k-1)\omega t, \tag{5.15}$$

where upper (lower) sign corresponds to p, r even (odd); and

$$a_{pr}(t) = \mp\omega^{r-p} \sum_{k=1}^{n} \beta_{rk}^{\mp}\gamma_{kp}^{\pm} \sin(2k-1)\omega t, \tag{5.16}$$

where upper (lower) sign corresponds to p even and r odd (p odd and r even). Having the explicit form of $a_{pr}(t)$, and using Eqs. (5.2) and (5.10) we obtain $\vec{C}$'s in terms of $\vec{q}$'s and $\vec{\pi}$'s:

$$\vec{C}_p = \sum_{r=0}^{n-1} \big((-1)^{r-1} r! a_{pr}(t) \vec{\pi}_r + (2n-1-r)! a_{p,2n-1-r}(t) \vec{q}_r \big). \tag{5.17}$$

As we have mentioned above (5.17) is related by canonical transformation (5.4) to (3.19) and (3.20).

6. Discussion

Let us summarize. In the present paper we focused on the Hamiltonian approaches to the PU model and its symmetries. First, we derived the form of the symmetry generators, in the original Pais and Uhlenbeck approach (for both generic and odd frequencies). We have shown that the resulting algebra is the central extension of the one obtained on the Lagrangian level, i.e., the centrally extended l-conformal Newton–Hooke algebra in the case of odd frequencies and the algebra defined by Eqs. (2.5) and (2.7), in the generic case. Next, we considered the Ostrogradski method of constructing Hamiltonian formalism for theories with higher derivatives. We derived the canonical transformation (Eqs. (4.8)–(4.9)) leading the Ostrogradski Hamiltonian to the one in decoupled oscillators approach.

Let us note that the both approaches, mentioned above, do not distinguish the odd frequencies and in that case do not uncover the richer symmetry. A deeper insight is attained by nothing that for odd frequencies an alternative Hamiltonian formalism can be constructed. It is based on the Hamiltonian formalism for the free higher derivatives theory exhibiting the l-conformal Galilei symmetry. More precisely, we add to the Hamiltonian of the free theory the conformal generator. As a result, we obtain the new Hamiltonian, which turns out to be related, by canonical transformation (5.4), to the PU one. This construction can be better understood from the orbit method point of view, where the construction of dynamical realizations of a given symmetry algebra is related to a choice of one element of the dual space of the algebra as the Hamiltonian (see [30] and the references therein). In our case, both algebras (l-Galilei and l-Newton–Hooke) are isomorphic to each other; only the one generator, corresponding to the Hamiltonian, differ by adding the conformal generator of the free theory. This gives the suitable change in the dual space and consequently the definition (5.3).

The change of the Hamiltonian alters the dynamics, which implies different time dependence of the symmetry generators (which do not commute with H); however, all PU generators should be expressed in terms of the generators of the free theory (for $t = 0$). This fact was confirmed by applying the method presented in Ref. [30] as well as, directly, by the canonical transformation (5.4) to the decoupled oscillators approach for the PU model.

Turning to possible further developments, let us recall that in the classical case ($l = \frac{1}{2}$) the dynamics of harmonic oscillator (on the half-period) is related to the dynamics of free particle by well known Niederer's transformation [39] (this fact has also counterpart on the quantum level). It turns out that this relation can be generalized to an arbitrary half-integer l [24] on the Lagrangian level; on the Hamiltonian one, we encounter some difficulties since there is no straightforward transition to the Hamiltonian formalism for a theory with higher derivatives. However, in the recent paper [40] the canonical transformation which relates the Hamiltonian (5.3) to the one for free theory (the first line of (5.2)) has been constructed; it provides a counterpart of classical Niederer's transformation for the Hamiltonian formalism developed in Section 5. Using our results one can obtain similar transformation for both remaining Hamiltonian approaches. We also believe that the results presented here can help in constructing quantum counterpart of the Niederer's transformation for higher l as well as to study of the symmetry of the quantum version of PU oscillator.

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Appendix A

In this appendix we prove the following Fourier expansion

$$\sin^p t \cos^{2n-1-p} t = \begin{cases} \sum_{k=1}^n \gamma_{kp}^+ \cos(2k-1)t, & p\text{-even}; \\ \sum_{k=1}^n \gamma_{kp}^- \sin(2k-1)t, & p\text{-odd}; \end{cases} \tag{A.1}$$

and derive some, crucial for the main part of the paper, properties of the expansion coefficients; namely

$$\gamma_{kp}^+ = (-1)^{k-1}\gamma_{k,2n-1-p}^-, \qquad \beta_{pk}^+ = (-1)^{k-1}\beta_{2n-1-p,k}^-, \tag{A.2}$$

$$2p!(2n-1-p)!\beta_{pk}^\pm = 2^{2n-1}(n-k)!(n+k-1)!\gamma_{kp}^\pm, \tag{A.3}$$

$$(2k-1)\gamma_{kp}^\pm = \mp p\gamma_{k,p-1}^\mp \pm (2n-1-p)\gamma_{k,p+1}^\mp, \tag{A.4}$$

$$(n+k)\gamma_{k+1,p}^\pm + (n-k+1)\gamma_{k-1,p}^\pm \pm n\gamma_{kp}^\pm\delta_{k1} = (2n-1-2p)\gamma_{kp}^\pm, \tag{A.5}$$

$$(n+k)\gamma_{k+1,p}^\pm - (n-k+1)\gamma_{k-1,p}^\pm \mp n\gamma_{kp}^\pm\delta_{k1} = \mp p\gamma_{k,p-1}^\mp \mp (2n-1-p)\gamma_{k,p+1}^\mp, \tag{A.6}$$

where $\beta^\pm$ is the inverse matrix of $\gamma^\pm$ and by definition $\gamma_{kp}^\pm = 0$ whenever $p < 0$, $p > 2n-1$, $k < 1$, $k > n$. Let us stress that $\beta_{pk}^+, \gamma_{kp}^+$ $(\beta_{pk}^-, \gamma_{kp}^-)$ are defined only for p even (odd).

Let us consider, for fixed n, $n = 1, 2, \ldots$, the set of functions

$$P_k^+(\tau) = \frac{\sqrt{2}\cos(2k-1)t}{\cos^{2n-1} t}\bigg|_{t=\arctan\tau},$$
$$P_k^-(\tau) = \frac{\sqrt{2}\sin(2k-1)t}{\cos^{2n-1} t}\bigg|_{t=\arctan\tau}, \tag{A.7}$$

where k is, a priori, an integer. One can check that functions (A.7) satisfy the orthonormality relations

$$\int_{-\infty}^{\infty} \frac{P_k^+(\tau)P_j^+(\tau)}{\pi(1+\tau^2)^{2n}} d\tau = \int_{-\infty}^{\infty} \frac{P_k^-(\tau)P_j^-(\tau)}{\pi(1+\tau^2)^{2n}} d\tau = \delta_{kj},$$
$$\int_{-\infty}^{\infty} \frac{P_k^\pm(\tau)P_j^\mp(\tau)}{\pi(1+\tau^2)^{2n}} d\tau = 0, \tag{A.8}$$

and the following identities

$$P_0^{\pm} = \pm P_1^{\pm}, \tag{A.9}$$

$$\left(1+\tau^2\right)\left(P_k^{\pm}\right)' = \mp(2k-1)P_k^{\mp} + (2n-1)\tau P_k^{\pm}, \tag{A.10}$$

$$\left(1+\tau^2\right)P_{k+1}^{\pm} = P_k^{\pm}\left(1-\tau^2\right) \mp 2\tau P_k^{\mp}, \tag{A.11}$$

$$\left(1+\tau^2\right)P_{k-1}^{\pm} = P_k^{\pm}\left(1-\tau^2\right) \pm 2\tau P_k^{\mp}, \tag{A.12}$$

$$(n-k)P_{k+1}^{\pm} + (n+k-1)P_{k-1}^{\pm} = (2n-1)P_k^{\pm} - 2\tau\left(P_k^{\pm}\right)', \tag{A.13}$$

$$(n-k)P_{k+1}^{\pm} - (n+k-1)P_{k-1}^{\pm} = (2k-1)P_k^{\pm} \mp 2\left(P_k^{\mp}\right)'. \tag{A.14}$$

Let X denote the operator

$$X = \left(1+\tau^2\right)\frac{d}{d\tau} - (2n-1)\tau. \tag{A.15}$$

Then

$$XP_k^{\pm} = \mp(2k-1)P_k^{\mp}; \tag{A.16}$$

consequently the action of the operator $Y = X^2$ is as follows

$$YP_k^{\pm} = -(2k-1)^2 P_k^{\pm}, \tag{A.17}$$

i.e., P's are eigenvectors of the operator Y.

Now, the point is that for $k = 1, \dots, n$ the functions $P_k^{\pm}$ are *polynomials* of degree less than or equal to $2n-1$ (this can be seen by expanding $\sin(2k-1)t$ and $\cos(2k-1)t$). Due to (3.1) they form the orthonormal basis in the space $W^{2n-1}(\tau)$ of all polynomials degree less than $2n$ with the scalar product

$$(f,g) = \int_{-\infty}^{\infty} \frac{f(\tau)g(\tau)}{\pi(1+\tau^2)^{2n}} d\tau. \tag{A.18}$$

Since $P_k^+, (P_k^-)$ are even (odd) functions the expansion with respect to the standard basis $\{\tau^p\}_{p=0}^{2n-1}$ of $W^{2n-1}(\tau)$ is of the form

$$P_k^+(\tau) = \sqrt{2}\sum_{p=0}^{2n-1}{}'' \beta_{pk}^+ \tau^p, \quad k = 1, \dots, n;$$

$$P_k^-(\tau) = \sqrt{2}\sum_{p=0}^{2n-1}{}' \beta_{pk}^- \tau^p, \quad k = 1, \dots, n. \tag{A.19}$$

Moreover, since $P_0^+ = P_1^+$ and $P_0^- = -P_1^-$ we have $\beta_{p0}^+ = \beta_{p1}^+$ and $\beta_{p0}^- = -\beta_{p1}^-$. Denoting by $\gamma^{\pm}$ the inverse matrix of $\beta^{\pm}$ we get the following relations

$$\tau^p = \frac{1}{\sqrt{2}}\sum_{k=1}^{n} \gamma_{kp}^+ P_k^+(\tau), \quad p\text{-even};$$

$$\tau^p = \frac{1}{\sqrt{2}}\sum_{k=1}^{n} \gamma_{kp}^- P_k^-(\tau), \quad p\text{-odd}. \tag{A.20}$$

Substituting $\tau = \tan t$ in Eqs. (A.20) we obtain the expansions

$$\tan^p t = \sum_{k=1}^{n} \gamma^+_{kp} \frac{\cos(2k-1)t}{\cos^{2n-1} t}, \quad p\text{-even};$$
$$\tan^p t = \sum_{k=1}^{n} \gamma^-_{kp} \frac{\sin(2k-1)t}{\cos^{2n-1} t}, \quad p\text{-odd}; \tag{A.21}$$

which are equivalent to the Fourier expansion (A.1).

Now, we prove the identities (A.2)–(A.6). First, let us note that the operator X was considered in Ref. [38][3] as acting on the space $W^{2n-1}_{\mathbb{C}}$ (the space of complex values polynomials of degree less than $2n$). It was shown there that the polynomials

$$P_a(\tau) = (1+i\tau)^{\frac{2n-1+a}{2}}(1-i\tau)^{\frac{2n-1-a}{2}}, \tag{A.22}$$

where the index a is an odd integer belonging to the set $\{-(2n-1), \dots, (2n-1)\}$, form an orthonormal basis of $W^{2n-1}_{\mathbb{C}}$ and are the eigenvectors of X, i.e.,

$$XP_a = iaP_a. \tag{A.23}$$

Moreover, it was proved that the coefficients of the expansion

$$P_a(\tau) = \sum_{p=0}^{2n-1} i^p \beta_{pa} \tau^p, \tag{A.24}$$

satisfy the relations

$$\beta_{p,-a} = (-1)^p \beta_{pa}, \quad \beta_{2n-1-p,a} = (-1)^{\frac{2n-1-a}{2}} \beta_{pa}. \tag{A.25}$$

Furthermore, with (γ_{ap}) being the inverse matrix to (β_{pa}) the following important relation holds

$$p!(2n-1-p)!\beta_{pa} = G(n,a)\gamma_{ap} i^p, \tag{A.26}$$

where

$$G(n,a) = 2^{2n-1}\left(\frac{2n-1+a}{2}\right)!\left(\frac{2n-1-a}{2}\right)!. \tag{A.27}$$

We can use this information to obtain some relations for $\beta^\pm$ and $\gamma^\pm$. To this end let us note that we have

$$P^+_k = \sqrt{2}\,\mathrm{Re}(P_{2k-1}) = \sqrt{2}\,\mathrm{Re}(P_{-(2k-1)}),$$
$$P^-_k = \sqrt{2}\,\mathrm{Im}(P_{2k-1}) = -\sqrt{2}\,\mathrm{Im}(P_{-(2k-1)}), \tag{A.28}$$

which implies

$$\beta^+_{pk} = (-1)^{\frac{p}{2}} \beta_{p,2k-1}, \quad \beta^-_{pk} = (-1)^{\frac{p-1}{2}} \beta_{p,2k-1},$$
$$\gamma^+_{kp} = 2\gamma_{2k-1,p}, \quad \gamma^-_{kp} = 2i\gamma_{2k-1,p}, \tag{A.29}$$

where p is even (odd) for the $+(-)$ case, respectively.

[3] For our convention n must be replaced there with $n-1$.

Now, we are ready to prove the relations (A.2)–(A.6). First, using (A.25), (A.26) and (A.29) we get (A.2) and (A.3). Recursion relation (A.4) is obtained by differentiating (A.1). Substituting (A.19) to (A.13) and using (A.3), (A.9) we arrive at (A.5). Similarly, inserting (A.19) into (A.14) and applying (A.2), (A.3), (A.9) we get (A.6).

References

[1] E.S. Fradkin, A.A. Tseytlin, Nucl. Phys. B 201 (1982) 469.
[2] W. Thiring, Phys. Rev. 77 (1950) 570.
[3] K.S. Stelle, Phys. Rev. D 16 (1977) 953.
[4] A. Mironov, A. Morozov, Int. J. Mod. Phys. A 23 (2008) 4677.
[5] D. Galakhov, JETP Lett. 87 (2008) 452.
[6] M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. 73 (2001) 977.
[7] R.J. Szabo, Phys. Rep. 378 (2003) 207.
[8] M.S. Plyushchay, Phys. Lett. B 243 (1990) 383.
[9] M.S. Plyushchay, Phys. Lett. B 262 (1991) 71.
[10] A.M. Polyakov, Nucl. Phys. B 268 (1986) 406.
[11] K. Andrzejewski, J. Gonera, P. Maślanka, Prog. Theor. Phys. 125 (2011) 247.
[12] S.W. Hawking, T. Hertog, Phys. Rev. D 65 (2002) 103515.
[13] A. Pais, G.E. Uhlenbeck, Phys. Rev. 79 (1950) 145.
[14] O. Sarioglu, B. Tekin, Class. Quantum Gravity 23 (2006) 7541.
[15] V.V. Nesterenko, Phys. Rev. D 75 (2007) 087703.
[16] A.V. Smilga, SIGMA 5 (2009) 017.
[17] K. Andrzejewski, J. Gonera, P. Machalski, K. Bolonek-Lasoń, Phys. Lett. B 706 (2012) 427.
[18] B. Bagchi, A.G. Choudhury, P. Guha, Mod. Phys. Lett. A 28 (2013) 1375001.
[19] J.B. Jiménez, E. Di Dio, R. Durrer, J. High Energy Phys. 1304 (2013) 030.
[20] M. Pavšič, Phys. Rev. D 87 (2013) 107502.
[21] I. Masterov, J. Math. Phys. 55 (2014) 102901.
[22] D.S. Kaparulin, S.L. Lyakhovich, A.A. Sharapov, Classical and quantum stability of higher-derivative dynamics, arXiv:1407.8481, 2014.
[23] G. Pulgar, J. Saavedra, G. Leon, Y. Leyva, Higher order Lagrangians inspired in the Pais–Uhlenbeck oscillator and their cosmological applications, arXiv:1408.5885, 2014.
[24] K. Andrzejewski, A. Galajinsky, J. Gonera, I. Masterov, Nucl. Phys. B 885 (2014) 150.
[25] J. Negro, M.A. del Olmo, A. Rodriguez-Marco, J. Math. Phys. 38 (1997) 3810.
[26] A. Galajinsky, I. Masterov, Phys. Lett. B 702 (2011) 265.
[27] C. Duval, P. Horvathy, J. Phys. A 44 (2011) 335203.
[28] A. Galajinsky, I. Masterov, Phys. Lett. B 723 (2013) 1960.
[29] M. Ostrogradski, Mem. Acad. St. Petersburg 4 (1850) 385.
[30] J. Gonera, J. Math. Phys. 54 (2013) 113507.
[31] C. Duval, P. Horvathy, J. Phys. A 42 (2009) 465206.
[32] S. Fedoruk, E. Ivanov, J. Lukierski, Phys. Rev. D 83 (2011) 085013.
[33] J. Gomis, K. Kamimura, Phys. Rev. D 85 (2012) 045023.
[34] N. Aizawa, Y. Kimura, J. Segar, J. Phys. A 46 (2013) 405204.
[35] N. Aizawa, Z. Kuznetsova, F. Toppan, J. Math. Phys. 54 (2013) 093506.
[36] M. Henkel, A. Hosseiny, S. Rouhani, Nucl. Phys. B 879 (2014) 292.
[37] K. Andrzejewski, J. Gonera, P. Maślanka, Phys. Rev. D 86 (2012) 065009.
[38] K. Andrzejewski, J. Gonera, Phys. Rev. D 88 (2013) 065011.
[39] U. Niederer, Helv. Phys. Acta 46 (1973) 191.
[40] K. Andrzejewski, Phys. Lett. B 738 (2014) 405.