A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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A SIMPLE THREE-STATE DURATION MODEL

1. INTRODUCTION

Duration models have long been the subject of study by biometricians and zoologists but they have recently attracted the attention of criminologists and econometricians. In this new context we are concerned with determining the probability distribution of the first reoffence committed by an habitual criminal released from prison at time zero or with the probability distribution of the first period of employment enjoyed by an individual who became unemployed at time zero.

The conventional approach to these problems, as represented by A m e m i y a (1985, p. 433-455) in his survey of this area, uses a two-state model and assumes that every person released from prison will eventually reoffend and that every unemployed person will eventually find a job. This model is far too restrictive as it ignores the possibility that a criminal may be reformed or that an unemployed person may become unable to work. It also ignores the possibility that the individual may die before he is able to reoffend or find a job.

In this paper we will, therefore, generalise the conventional econometric model. We will retain the assumption that all individuals must eventually leave their initial state but will give them a choice of two states to enter. In the econometric example these three states may be labelled unemployed, employed and incapacitated respectively.

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2. CONSTANT HAZARD RATES

Suppose that N individuals, are released into state 1 at time 0 and that we are interested in studying the times of their first departures from this state. Then the conventional model with constant hazard rate λ asserts that the ith individual leaves state 1 for the first time before time t with probability

$$F^*(t) = 1 - \exp(-\lambda t) \tag{2.1}$$

where $F^*(\infty) = 1$ so that he cannot remain in this state for ever.

Now, suppose that a constant proportion κ/λ of these transfers out of state 1 represent entries into state 2 and that the remainder represent entries into state 3. Then the probability that the ith individual will leave state 1 between time t and time $t + \Delta t$ is given by

$$f^*(t)\Delta t = \lambda \exp(-\lambda t)\Delta t \tag{2.2}$$

and the probability that he will leave state 1 for state 2 in this interval by

$$f(t)\Delta t = \kappa \exp(-\lambda t)) \tag{2.3}$$

Finally, setting $\lambda = \kappa + \nu$ in equation (2.3) and integrating the result we have

$$F(t) = [\kappa/(\kappa + \nu)][1 - \exp(-(\kappa + \nu)t)] \tag{2.4}$$

which gives the probability that the ith individual will transfer from state 1 to state 2 before time t. Similarly, the probability that he will transfer from state 1 to state 3 before time t is given by

$$F^*(t) - F(t) = [\nu/(\kappa + \nu\ ][1 - \exp(-(\kappa + \nu)t)] \tag{2.5}$$

but we shall assume in this paper that such transfers to state 3 are not observed.

3. VARIABLE HAZARD RATES

In the last section we assumed that the instantaneous rates of transfer from state 1 to state 2 and from state 1 to state 3 were constant over time and common to all individuals. More generally we may suppose that these rates of transfer, $\kappa_i(t)$ and $\nu_i(t)$, are functions both of t and of $x_{i1}, x_{i2}, \ldots, x_{ip}$ which are a set of p variables which characterise the ith individual. In this context we have to replace equation (2.1) by

$$F_i^*(t) = 1 - \exp\left[-S_i(t)\right] \tag{3.1}$$

equation (2.3) by

$$f_i(t)\Delta t = \kappa_i(t)\left[\exp - S_i(t)\right]\Delta t \tag{3.2}$$

and equation (2.4) by

$$F_i(t) = \int_o^t \kappa_i(u) \exp\left[-S_i(u)\right]du \tag{3.3}$$

where $S_i(t) = \int_o^t \left[\kappa_i(u) + \nu_i(u)\right]du$ (3.4)

However, in the sequel we shall assume that the transfer rate functions are common to all individuals and shall, therefore, delete the subscript i from the functions $\kappa_i(t)$, $\nu_i(t)$, $S_i(t)$, $f_i(t)$ and $F_i(t)$.

4. INDIVIDUAL DATA

Suppose that N individuals were released into state 1 at time 0 and that n were observed to transfer to state 2 at known times $t_1, t_2, \ldots, t_n$ while the remaining N-n were not observed to make this transfer before time T when the experiment ended. Then the likelihood function for this problem is given by

$$L = m\left[1 - F(T)\right]^{N-n}\left[\prod_{i=1}^{n} f(t_i)\right] \tag{4.1}$$

where $m = N!/[n!(N - n)!]$

In particular, if $\kappa(t) = \kappa$ and $\nu(t)$ are constant then this expression becomes

$$L = m\{1 - \frac{\kappa}{\kappa + \nu}\left[1 - \exp(-(\kappa + \nu)T)\right]\}^{N-n}$$

$$x\kappa^n \exp\left[-(\kappa + \nu)\sum_{i=1}^{n} t_i\right] \tag{4.2}$$

5. GROUPED DATA

Now, suppose that the exact time at which the ith individual transferred from state 1 to state 2 was not observed but only the number N_t who transferred between time t and time t + 1 for t = 0, 1, ..., T - 1. Further, let $N_T = N - N_0 -, \ldots, -N_{T-1}$ then $N_0, N_1, \ldots, N_T$ have a multinomial distribution and the likelihood function for this model is given by

$$L^* = m^* \left[1 - F(T)\right]^{N-n} \prod_{t=0}^{T-1} \left[F(t + 1) - F(t)\right] \tag{5.1}$$

where $m^* = N!/[N_0!N_1! \ldots N_T!]$ (5.2)

Again specialising to the case when $\kappa(t) = \kappa$ and $\nu(t) = \nu$ are both constant, we have

$$F(t) = 1 - \frac{\kappa}{\kappa + \nu}\left[1 - \exp(-(\kappa + \nu)T)\right] \tag{5.3}$$

and

$$F(t+1) - F(t) = \frac{\kappa}{\kappa + \nu}\left[1 - \exp(-\kappa - \nu)\right] \exp(-(\kappa + \nu)t) \tag{5.4}$$

which may be substituted in expression (5.1).

6. CONCLUDING REMARKS

In this paper we have defined the likelihood function for two variants of a simple three-state duration model. These functions may readily be optimized to yield maximum likelihood estimates of their parameters, see Farebrother (1988).

REFERENCES

Amemiya J. (1985), *Advanced Econometrics*, Harvard University Press, Cambridge, Massachusetts.

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PROSTY TRÓJSTANOWY MODEL CZASU TRWANIA

W artykule przedstawiono uogólnienie dwustanowego modelu, który jest szeroko stosowany w ekonometrii do modelowania czasu trwania bezrobocia. Do zwykłych dwóch stanów "zatrudnienia" i "bezrobocia" dodaje się trzeci stan, reprezentujący te jednostki, które z różnych przyczyn stają się niezdolne do pracy.