Analytic and Algebraic Geometry
Łódź University Press 2013, 11 – 32
http://dx.doi.org/10.18778/7969-017-6.02

GEOMETRIC DESINGULARIZATION OF CURVES IN MANIFOLDS *) **)

STANISŁAW ŁOJASIEWICZ

1. Introduction

The article does not pretend to any originality. In the literature there exists a number of descriptions of desingularizations in the case of curves. Deciding for this description the author think it is worth looking in details into this fascinating topic in an easily accessible case, namely – in the effects of multi blowings-up for curves in manifolds and for coherent sheaves on 2-dimensional manifolds.

All the needed facts from analytic geometry can be find in the author's books [L1], [L2].

2. The canonical blowing-up of $\mathbb{C}^n$ at 0

The *blow-up of* $\mathbb{C}^n$ *at* 0 is

$$\Pi = \Pi_n = \{(z, \lambda) : z \in \lambda\} \subset \mathbb{C}^n \times \mathbb{P}, \quad \mathbb{P} = \mathbb{P}_{n-1}.$$

Taking the inverse atlas for $\mathbb{C}^n \times \mathbb{P}$

$$\gamma_k : \mathbb{C}^n \times \mathbb{C}^{n-1} \ni (z, w_{(k)}) \mapsto$$
$$(z, \mathbb{C}(w_1, ..., \underset{(k)}{1}, ..., w_n)) \in \mathbb{C}^n \times \{\mathbb{P} \setminus \mathbb{P}(\{z_k = 0\}))= G_k,\ k = 1, ..., n,$$

2010 *Mathematics Subject Classification.* Primary 32Sxx, Secondary 14Hxx.

Key words and phrases. Resolution of singularities, curve, blowing-up, coherent analytic sheaf.

*) This article was published (in Polish) in the proceedings of Xth Workshop on Theory of Extremal Problems (1989) and has never appeared in translation elsewhere. To honor this outstanding mathematician (who passed away in 2002) this article was translated into English (by T. Krasiński) in order to make it accesible to the mathematical community.

**) The translator thanks Dinko Pervan (an Erasmus student from Croatia) for preparing the article in TeX and W. Kucharz, A. Płoski and Sz. Brzostowski for improving the English text.

(that is $\gamma_k = (\mathrm{id}\,\mathbb{C}^n) \times$ (inverse mapping to the k-th canonical map on $\mathbb{P}$)), we have the inverse images of Π

$$\Gamma_k = \gamma_k^{-1}(\Pi) = \{(z, w_{(k)}) : z \in \mathbb{C}(w_1, ..., 1, ..., w_n)\} = \big\{(z, w_{(k)}) : z_{(k)} = z_k w_{(k)}\big\}\,;$$

they are graphs of the polynomial mappings $(z_k, w_{(k)}) \to z_k w_{(k)}$, whence $\Pi \subset \mathbb{C}^n \times \mathbb{P}$ is an n-dimensional closed submanifold, $(\gamma_k)_{\Gamma_k} : \Gamma_k \to \Pi \cap G_k$ – its inverse maps (they give an inverse atlas on Π); composing them with biholomorphisms: $(z_k, w_{(k)}) \to (z_k w_1, ..., z_k, ..., z_k w_n, w_{(k)})$ (domains onto the graphs of the preceding polynomial mappings) we obtain an inverse atlas on Π

$$(*) \qquad \chi_k : \mathbb{C}^n \ni (z_k, w_{(k)}) \to (z_k w_1, ..., z_k, ..., z_k w_n, \mathbb{C}(w_1, ..., 1, ..., w_n)) \in \Pi \cap G_k.$$

The canonical projection $p : \Pi \to \mathbb{C}^n$ is called the canonical blowing-up. The fiber $S_0 = p^{-1}(0) = 0 \times \mathbb{P}$ (biholomorphic to $\mathbb{P}$) is called the exceptional set (the exceptional submanifold); $\Pi_{\mathbb{C}^n \setminus 0}$ is the graph of the holomorphic mapping $\mathbb{C}^n \setminus 0 \ni z \to \mathbb{C}z \in \mathbb{P}$, whence $p^{\mathbb{C}^n \setminus 0} : \Pi_{\mathbb{C}^n \setminus 0} \to \mathbb{C}^n \setminus 0$ is a biholomorphism. Hence the blowing-up $p : \Pi \to \mathbb{C}^n$ is a modification of $\mathbb{C}^n$ at 0. The inverse image $p^{-1}(E)$ of a set $E \subset \mathbb{C}^n$ in the k-th coordinate system $(*)$ can be expressed by

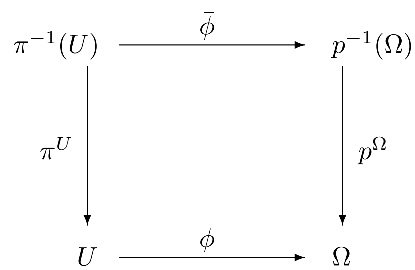
$$(**) \qquad \begin{cases} \chi_k^{-1}(p^{-1}(E)) = (p \circ \chi_k)^{-1}(E) \text{ where} \\ p \circ \chi_k \ni (z_k, w_{(k)}) \to (z_k w_1, ..., z_k, ..., z_k w_n) \in \mathbb{C}^n. \end{cases}$$

In particular $\chi_k^{-1}(S_0) = \{z_k = 0\}$.

The restrictions $p^{\Omega} : \Pi_{\Omega} \to \Omega$, where Ω is an open neighbourhood of 0 at $\mathbb{C}^n$, are called the local canonical blowings-up.

3. The blowing-up of a manifold at a point

Let M be an n-dimensional manifold and $a \in M$. A blowing-up of M at the point a is a holomorphic mapping of manifolds $\pi : \bar{M} \to M$ such that $\pi^{M \setminus a} : \bar{M} \setminus \pi^{-1}(a) \to M \setminus a$ is a biholomorphism and for an open neighbourhood U of a, the mapping π^U is isomorphic to a local canonical blowing-up p^{Ω} i.e. we have a commutative diagram

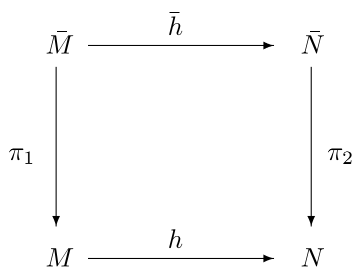


for some biholomorphisms $\phi : U \to \Omega$, $\phi(a) = 0$ and $\bar{\phi} : \pi^{-1}(U) \to p^{-1}(\Omega)$. (Notice that U and Ω can be abitrarily diminished). π is a proper mapping (because $\pi^{M \setminus a}$ and π^U are proper). The fiber $S = \pi^{-1}(a)$, biholomorphic to $\mathbb{P}$, is called the exceptional set (the exceptional submanifold) of the blowing-up. Thus π is a modification of M at a.

The existence of blowing-up. We take a chart (a coordinate system) at a: $\phi : U \to \Omega$, $\phi(a) = 0$, and define $\bar{M}$ as a gluing-up of π_Ω with $M \backslash a$ by the biholomorphism $(\phi_{U\backslash a})^{-1} \circ p^{\Omega\backslash 0} : \Pi_{\Omega\backslash 0} \to U\backslash a$. (Its graph is closed in $\Pi_\Omega \times (M\backslash a)$ because $\phi^{-1} \circ p^\Omega$ is a closed set in $\Pi_\Omega \times M$ and $\left(\phi^{-1} \circ p^\Omega\right) \cap (\Pi_\Omega \times M\backslash a) = \phi^{-1}_{U\backslash a} \circ p^{\Omega\backslash 0})$. So we have the identifying biholomorphisms $h_0 : \Pi_\Omega \to D_0$, $h_1 : M\backslash a \to D_1$, where $D_i \subset \bar{M}$, $i = 0, 1$, are open sets, $\bar{M} = D_0 \cup D_1$ and $h_1^{-1} \circ h_0 = \phi^{-1}_{U\backslash a} \circ p^{\Omega\backslash 0}$. Hence $h_1^{-1}(D_0) = U\backslash a$ (the domains of both sides) which implies $h_1(U\backslash a) \subset D_0$. Next $g = \phi^{-1} \circ p \circ h_0^{-1} : D_0 \to M$ contains $(h_1^{-1})_{D_0}$, and hence $\pi = h_1^{-1} \cup g : \bar{M} \to M$ is a holomorphic maping. Then $\pi^{M\backslash a} = h_1^{-1}$ (because $h^{-1} \supset \phi^{-1} \circ p^{\Omega\backslash 0} \circ h_0^{-1} = g^{M\backslash a}$) is a biholomorphism on the image. At last, $\phi \circ \pi^U \supset \phi \circ g \supset p^\Omega \circ h_0^{-1}$ which implies the equality, because the domains are equal ($\pi^{-1}(U) = h_1^{-1}(U\backslash a) \cup D_0 = D_0$), whence the above diagram is commutative with $\bar{\phi} := h_0^{-1}$.

Remark 1. *Obviously, if G is an open neighbourhood of a at M then $\pi : \bar{M} \to M$ is a blowing-up at a if and only if $\pi^{M\backslash a}$ is a biholomorphism and π^G is a blowing-up at a.*

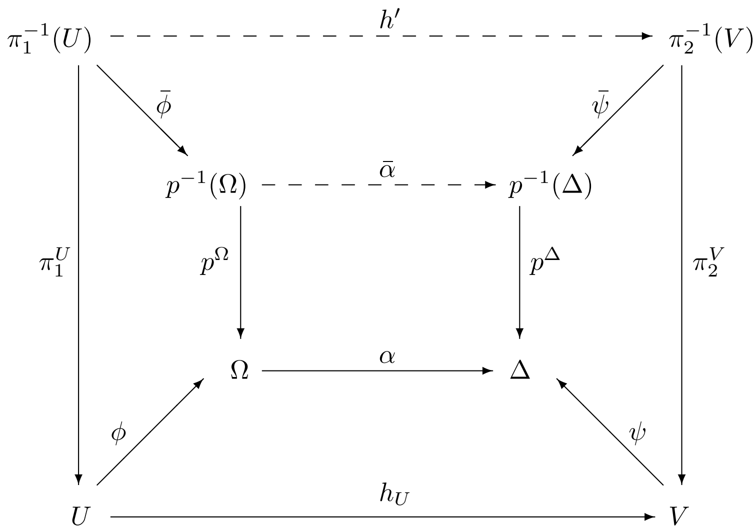
Proposition 1. *If $h : M \to N$ is a biholomorphism of manifolds, $h(a) = b$, $\pi_1 : \bar{M} \to M$ is a blowing-up at a, $\pi_2 : \bar{N} \to N$ a blowing-up at b, then there exists a biholomorphism $\bar{h} : \bar{M} \to \bar{N}$ such that the diagram*



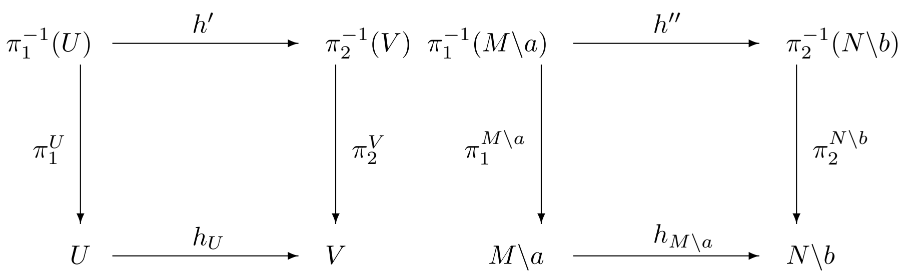
(#)

is commutative

Dowód. Choosing by definition: $\phi : U \to \Omega$ and $\bar{\phi}$ - for π_1, and $\psi : V \to \Delta$ and $\bar{\psi}$ - for π_2, such that $h(U) = V$, we have a commutative diagram



where $\alpha := \psi \circ h_U \circ \phi^{-1}$, and it suffices to complement it by biholomorphisms: $\bar{\alpha} : p^{-1}(\Omega) \to p^{-1}(\Delta)$ and $h' := \bar{\psi}^{-1} \circ \bar{\alpha} \circ \bar{\phi}$. Then in the commutative diagrams



where the biholomorphism h'' is defined by the remaining arrows (which are biholomorphisms), the biholomorphisms h' and h'' give rise to a biholomorphism $\bar{h} = h' \cup h'' : \bar{M} \to \bar{N}$. In fact, it suffices to find a holomorphic mapping $\bar{\alpha} : p^{-1}(\Omega) \to p^{-1}(\Delta)$ such that $p^\Delta \circ \bar{\alpha} = \alpha \circ p^\Omega$ (i.e. the commutativity of the inner rectangle) and a similar holomorphic mapping $\bar{\beta} : p^{-1}(\Delta) \to p^{-1}(\Omega)$ for α^{-1}, since then we obtain the commutative triangle

$$\begin{array}{ccc} p^{-1}(\Omega) & \xrightarrow{\bar\beta\circ\bar\alpha} & p^{-1}(\Omega) \\ & {}_{p^{\Omega}}\searrow \quad \swarrow_{p^{\Delta}} & \\ & \Omega & \end{array}$$

which implies $\bar\beta \circ \bar\alpha = \mathrm{id}_{p^{-1}(\Omega)}$ (because we have the equality on the dense set $p^{-1}(\Omega)\backslash S_0$), and similarly $\bar\alpha \circ \bar\beta = \mathrm{id}_{p^{-1}(\Delta)}$. Obviously it suffices to find $\bar\alpha$ (because the construction of $\bar\beta$ is analogous) for sufficiently small Ω and Δ.

According to the Hadamard Lemma (since $\alpha(0) = 0$) one can choose neighbourhoods Ω, Δ such that $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i(z) = \sum_{j=1}^{n} a_{ij}(z)z_j$ and $\det a_{ij}(z) \neq 0$ in Ω. Define $a(z,w) = (\sum_{j=1}^{n} a_{1j}(z)w_j, ..., \sum_{j=1}^{n} a_{nj}(z)w_j)$ in $\Omega \times \mathbb{C}^n$; then $a(z,z) = \alpha(z)$ and $a(z,w) \neq 0$ for $w \neq 0$. Hence we may define a holomorphic mapping $\bar a : \Omega \times \mathbb{P} \ni (z, \mathbb{C}w) \to (\alpha(z), \mathbb{C}a(z,w)) \in \Delta \times \mathbb{P}$. Since $\bar a(z, \mathbb{C}z) = (\alpha(z), \mathbb{C}\alpha(z))$ for $z \in \Omega\backslash 0$ and $\bar a(0 \times \mathbb{P}) \subset 0 \times \mathbb{P}$, then we have the holomorphic restriction $\bar\alpha = \bar a_{\Pi_\Omega} : \Pi_\Omega \to \Pi_\Delta$, and hence $p^{\Delta}(\bar\alpha(z, \mathbb{C}z)) = \alpha(z) = \alpha(p^{\Omega}(z, \mathbb{C}z))$ for $z \in \Omega\backslash 0$, that is $p^{\Delta} \circ \bar\alpha = \alpha \circ p^{\Omega}$ by density of $\Pi_{\Omega\backslash 0}$ in Π_Ω. □

4. The proper inverse image

Let $\pi : \bar M \to M$ be a blowing-up at a point $a \in M$. The proper inverse image (by π) of a set $V \subset M$ closed in a neighbourhood of a (i.e. $V \cap U$ is a closed set in U for some neighbourhood U of a) is defined by

$$\bar V = \text{the closure of the set } \pi^{-1}(V\backslash a) = \pi^{-1}(V)\backslash S \text{ in } \pi^{-1}(V).$$

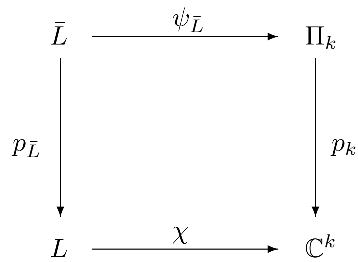
(It is obtained from the set $\pi^{-1}(V)\backslash S$ by adding to it its accumulation points belonging to S). If V is analytic in a neighbourhood of a then $\bar V$ is analytic in a neighbourhood of the exceptional set S (since $\pi^{-1}(V)$ and S are analytic in a neighbourhood of S). Obviously

$$\pi^{-1}(V) = \bar V \cup S.$$

If U is an open neighbourhood of a, then the proper inverse image of the set $V \cap U$ is $\bar V \cap \pi^{-1}(U)$. If $W \subset V$ then $\bar W \subset \bar V$, and if $V = \bigcup_{i=1}^{k} Z_i$, then $\bar V = \bigcup_{i=1}^{k} \bar Z_i$, (provided W, Z_i are closed in a neighbourhood of a). If $D \supset V$ is an open neighbourhood of a then $\bar V$ is the proper inverse image of V if and only if it is the same by the blowing-up π^D.

In Proposition 1 the biholomorphism $\bar{h}$ sends the exceptional submanifold $\pi_1^{-1}(a)$ onto the exceptional submanifold $\pi_2^{-1}(b)$, and the proper inverse image of V onto the proper inverse image of $h(V)$.

The proper inverse image of a linear subspace $L \subset \mathbb{C}^n$ of dimension k by the canonical blowing-up is $\bar{L} = \{(z, \lambda) \in L \times \mathbb{P}(L) : z \in \lambda\}$; it is a submanifold of dimension k and $p_{\bar{L}} : \bar{L} \to L$ is a blowing-up at 0. (For taking an isomorphism $\chi : L \to \mathbb{C}^k$ we have the commutative diagram



where $\psi = \chi \times \chi' : L \times \mathbb{P}(L) \to \mathbb{C}^k \times \mathbb{P}_k$, $\chi' : \mathbb{P}(L) \ni \lambda \to \chi'(\lambda) \in \mathbb{P}_k$ are biholomorphisms and $\psi(\overline{L}) = \Pi_k$).

5. The transversality

Proposition 2. *If M is a linear space of dimension n then linear subspaces $L_1, ..., L_r \subset M$ intersect transversally (in M) if and only if in some linear coordinate system in M it is*

$$L_i = \{z_\upsilon = 0, \upsilon \in I_i\}, \quad \textit{where } I_1, ..., I_r \subset \{1, ..., n\} \textit{ are disjoint.}$$

Dowód. The sufficiency is obvious because $\operatorname{codim} L_i = \# I_i$. Conversely, if L_i intersect transversally, then the sum $\sum L_i^\perp = (\bigcap L_i)^\perp$ is direct because $\dim \sum L_i^\perp = \operatorname{codim} \bigcap L_i = \sum \operatorname{codim} L_i = \sum \dim L_i^\perp$. Hence there exists a basis $\phi_1, ..., \phi_n$ of the dual space M^* such that $\{\phi_\upsilon : \upsilon \in I_i\}$ generate $L_i^\perp$ where $I_i \subset \{1, ..., n\}$ are disjoint. Then $L_i = \{\phi_\upsilon = 0, \upsilon \in I_i\}$, that is $L_i = \{z_\upsilon = 0, \upsilon \in I_i\}$ in the coordinate system $\phi = (\phi_1, ..., \phi_n)$ (because $\phi^{-1}(\{z_\upsilon = 0, \upsilon \in I_i\}) = L_i$). □

Corollary 1. *If L_i, $i \in I$, intersect transversally and $J \subset I$, then also L_i, $i \in J$, intersect transversally. If $I \cap J = \emptyset$ and L_i, $i \in I \cup J$, intersect transversally then so do $\bigcap_I L_i$ and $\bigcap_J L_i$. If $L_1, ..., L_r, T$ intersect transversally then so do $L_1 \cap T, ..., L_r \cap T$ in T.*

Proposition 3. *If M is a manifold of dimension n, then submanifolds $N_1, ..., N_r$ intersect transversally at a point $a \in \bigcap N_i$ if and only if there exists a chart (a coordinate system at a) $\phi : U \to \Omega$, $\phi(a) = 0$, such that $\phi(N_i \cap U) = T_i \cap \Omega$, where*

$T_i \subset \mathbb{C}^n$ are subspaces that intersect transversally, so it may be

$$T_i = \{u_i = 0\}, \text{ where } z = (u_1, ..., u_r, v) \in \mathbb{C}^n = \mathbb{C}^{I_1} \times ... \times \mathbb{C}^{I_r} \times \mathbb{C}^J.$$

Dowód. The sufficiency is clear. For the necessity we may assume $M = \mathbb{C}^n$, $a = 0$ and $T_0 N_i = T_i$ as above. Then there exists an open neighbourhood $U = \Omega_1 \times ... \times \Omega_r \times \Delta$ of the origin in $\mathbb{C}^n$ and functions $\varepsilon_i(u_{(i)}, v)$ with values in $\mathbb{C}^{I_i}$, holomorphic in $U_i = \Omega_1 \times ..._{(i)}... \times \Omega_r \times \Delta$, such that $d_0\varepsilon_i = 0$ and $N_i \cap U = \{u_i = \varepsilon_i(u_{(i)}, v), (u_{(i)}, v) \in U_i\}$. After shrinking U the mapping $\phi : U \ni z \to (u_1 - \varepsilon_1(u_{(1)}, v), ..., u_r - \varepsilon_r(u_{(r)}, v), v) \in \Omega$ is a biholomorphism onto a neighbourhood Ω of the origin and hence $N_i \cap U = \phi^{-1}(T_i)$ which implies $\phi(N_i \cap U) = T_i \cap \Omega$. □

Corollary 2. *If submanifolds N_i, $i \in I$, intersect transversally at a point a and $J \subset I$, then so do the submanifolds N_i, $i \in J$. If $I \cap J = \emptyset$ and submanifolds $N_i, i \in I \cup J$, intersect transversally at a then so do the submanifolds $\bigcap_I N_i$ and $\bigcap_J N_i$.*

Corollary 3. *If submanifolds N_i intersect transversally then $N = \bigcap N_i$ is a submanifold and* $\operatorname{codim} N = \sum \operatorname{codim} N_i$.

We say submanifolds N_i of a manifold M are mutually transversal in an open set $G \subset M$, if $N_i \cap G$ are closed and for each $a \in G$ submanifolds N_i containing a intersect transversally at a. Notice that if subspaces of a linear space intersect transversally then they are mutually transversal in this space (by Corollary 1 and from the fact that if subspaces intersect transversally, then they intersect transversally at each point of their intersection). Hence (by Proposition 3)

Corollary 4. *If submanifolds N_i intersect transversally at $a \in \bigcap N_i$, then they are mutually transversal in a neighbourhood of the point a.*

6. The effect of blowing-up

Let M be a manifold of dimension n and let $\pi : \bar{M} \to M$ be a blowing-up at point $a \in M$, and $S = \pi^{-1}(a) \subset \bar{M}$ – the exceptional set.

Proposition 4. *If $\Gamma \subset M$, $\Gamma \ni a$, is a submanifold of dimension s then its proper inverse image $\bar{\Gamma} \subset \bar{M}$ is a submanifold of dimension s which intersects S transversally and the submanifold $\bar{\Gamma} \cap S$ is biholomorphic to $\mathbb{P}_{s-1}$. Then $\pi_{\bar{\Gamma}} : \bar{\Gamma} \to \Gamma$ is a blowing-up at a with the exceptional set $\bar{\Gamma} \cap S$.*

Dowód. The set $\bar{\Gamma} \backslash S = \pi^{-1}(\Gamma \backslash a)$ is a submanifold of dimension s and $(\pi_{\bar{\Gamma}})^{\Gamma \backslash a} : \bar{\Gamma} \backslash S \to \Gamma \backslash a$ is a biholomorphism. Let us take a chart $\phi : U \to \Omega$, $\phi(a) = 0$, such that $\phi(\Gamma \cap U) = L \cap \Omega$, where $L = \{z_1 = ... = z_r = 0\}$ $(r = n - s)$. It suffices to show the proposition for π^U and $\Gamma \cap U$ because then the proper inverse image of $\Gamma \cap U$, that is $\bar{\Gamma} \cap \pi^{-1}(U)$, will be a submanifold (of dimension s) and $(\pi^U)_{\bar{\Gamma} \cap \pi^{-1}(U)} = (\pi_{\bar{\Gamma}})^{\Gamma \cap U}$ will be a blowing-up at a, whence $\bar{\Gamma}$ will be a submanifold and $\pi_{\bar{\Gamma}}$ a blowing-up at

a (see Remark 1). According to Proposition 1, it suffices to prove the proposition for p^Ω, $L\cap\Omega$ and 0. Since the proper inverse image of $L\cap\Omega$ is $\bar{L}\cap p^{-1}(\Omega)$, where $\bar{L}$ is the proper inverse image of L by p, and $(p^\Omega)_{\bar{L}\cap p^{-1}(\Omega)} = (p_{\bar{L}})^{L\cap\Omega}$, then it suffices to prove the proposition for p, L and 0. But $\bar{L}$ is a submanifold of dimension s, $p_{\bar{L}} : \bar{L} \to L$ is a blowing-up at 0 and $\bar{L}\cap S_0 = 0\times\mathbb{P}(L)$ (see Section 4). It remains to prove the transversality. We have (see (**) in Section 2)

$$\chi_k^{-1}(p^{-1}(L) = \begin{cases} \{z_k = 0\} \text{ if } \ k \leqslant r \\ \{z_k = 0\} \cup \{w_1 = ... = w_r = 0\} \text{ if } \ k > r, \end{cases}$$

so by $\chi_k^{-1}(S_0) = \{z_k = 0\}$ it is

$$\chi_k^{-1}(\bar{L}) = \begin{cases} \emptyset \text{ if } \ k \leqslant r \\ \{w_1 = ... = w_r = 0\} \text{ if } \ k > r, \end{cases}$$

whence (Proposition 2) the transversality of the intersection of $\bar{L}$ and S_0 follows. □

Proposition 5. *If submanifolds $\Gamma_1, ..., \Gamma_r \subset M$ intersect transversally at a and $\bar{\Gamma}_1, ..., \bar{\Gamma}_r$ are their proper inverse images then $\bar{\Gamma}_1, ..., \bar{\Gamma}_r, S$ are mutually transversal in a neighbourhood of S. If additionally Γ_i intersect transversally then the proper inverse image of $\Gamma = \bigcap \Gamma_i$ is $\bar{\Gamma} = \bigcap \bar{\Gamma}_i$.*

Dowód. If U is an open neighbourhood of a then the proper inverse image of $\Gamma_i\cap U$ $(\Gamma\cap U)$ is $\bar{\Gamma}_i\cap\pi^{-1}(U)$ $(\bar{\Gamma}\cap\pi^{-1}(U))$. By Propositions 3 and 1 it suffices to consider the canonical blowing-up p and $\Gamma_i = T_i = \{z_\upsilon = 0, \upsilon\in I_i\}$, I_i disjoint (by the fact $\bar{\Gamma}\backslash S = \bigcap(\bar{\Gamma}_i\backslash S)$). Let $\bar{T}_i$ denote the proper inverse image of T_i. We have (see (**) in Section 2)

$$\chi_k^{-1}(p^{-1}(T_i) = \begin{cases} \{z_k = 0\} \text{ if } \ k \in I_i \\ \{z_k = 0\} \cup \{w_\upsilon = 0, \upsilon \in I_i\} \text{ if } \ k \notin I_i, \end{cases}$$

so

$$\chi_k^{-1}(\bar{T}_i) = \begin{cases} \emptyset \text{ if } k \in I_i \\ \{w_\upsilon = 0, \upsilon \in I_i\} \text{ if } k \notin I_i, \end{cases}$$

which implies (Proposition 2) that $\bar{T}_i, ..., \bar{T}_r, S$ are mutually tranversal in Π. If $\bar{T}$ is the proper inverse image of $T = \bigcap T_i$ then $T = \{z_\upsilon = 0, \upsilon\in I\}$, where $I = \bigcup I_i$, and in the same way

$$\chi_k^{-1}(\bar{T}) = \begin{cases} \emptyset \text{ if } k \in I \\ \{w_\upsilon = 0, \upsilon \in I\} \text{ if } k \notin I, \end{cases}$$

so $\chi_k^{-1}(\bar{T}) = \bigcap \chi_k^{-1}(\bar{T}_i)$, whence $\bar{T} = \bigcap \bar{T}_i$. □

Let $\mathcal{C}(a) = \mathcal{C}(a, M)$ denote the set of curves $\Gamma \subset M$ (i.e. local analytic subsets of constant dimension 1) such that $a \in \Gamma$ and the germ Γ_a is irreducible. Then

$$\mathcal{C}(a) = \bigcup_{p=1}^{\infty} \mathcal{C}_p(a), \tag{6.1}$$

where $\mathcal{C}_p(a) = \mathcal{C}_p(a, M)$ denotes the set of curves Γ in $\mathcal{C}(a)$ having, in some coordinate system ϕ in a (i.e. ϕ is a chart such that $\phi(a) = 0$), the form (that is $\phi(\Gamma)$ is a set of the form)

$$\text{(6.2)} \qquad \begin{cases} z_1 = t^p \\ v = c(t)t^q \end{cases} \qquad |t| < \sigma,$$

where $v = (z_2, ...z_n)$, $q \geqslant p$, and c is a holomorphic function in $\{|t| < \sigma\}$ $(\sigma > 0)$. (For it is of the form $\{f(t) : |t| < \sigma\}$, where f is a holomorphic mapping, a homeomorphism onto its image, $f(0) = 0$; it is $f(t) = g(t)t^p$, $p \geqslant 1$, $g(0) \neq 0$, and after changing the system of coordinates one may have $g_1(0) \neq 0$; then $g_1 = \gamma^p$ with γ holomorphic in a neighbourhood of the origin, $\gamma(0) \neq 0$, and it suffices to change the parameter putting $\tau = \gamma(t)t$ in a neighbourhood of the origin). In particular, $\mathcal{C}_1(a)$ is the set of all curves $\Gamma \ni a$ smooth at a.

A set Γ_0 of the form (6.2) (without any restriction on q) is always a curve in $\mathbb{C}^n$ having its germ irreducible at 0. (For the mapping $\{|t| < \sigma\} \ni t \to (t^p, c(t)t^q) \in \{|z_1| < \sigma^p\} \subset \mathbb{C}^n$ is proper). Let us notice that replacing σ by $0 < \bar{\sigma} < \sigma$ we obtain an open neighbourhood of 0 in Γ_0 (precisely $\Gamma_0 \cap \{|z_1| < \bar{\sigma}^p\}$). If $0 < q < p$ and $c(0) \neq 0$ then $\Gamma_0 \in \mathcal{C}_q$. In fact, if for example $c_2(0) \neq 0$ then (changing the parameter to $\tau = t\gamma(t)$, where $\gamma^q = c_2$) for sufficiently small ε, a neighbourhood U_ε of the origin and holomorphic b_i, the sets $\Gamma_\varepsilon = \{z_1 = t^p, v = c(t)t^q, t \in U_\varepsilon\} = \{z_2 = \tau^q, z_i = b_i(\tau)\tau^q,\ i \neq 2,\ |\tau| < \varepsilon\}$ are neighbourhoods of 0 in Γ_0. But $\Gamma_{\varepsilon_0} \subset \Gamma_0 \cap \{|z_1| < \sigma_0\} \subset \Gamma_\varepsilon$ for some $\sigma_0, \varepsilon_0 > 0$, hence Γ_{ε_0} is an open set in Γ_ε and so in Γ_0.

It is

$$\text{(6.3)} \qquad \mathcal{C}_p(a) = \mathcal{C}_1(a) \cup \bigcup \mathcal{C}_{p,q}(a),$$

where $\mathcal{C}_{p,q}(a), q > p$ is not divisible by p, is the set of all the curves in $\mathcal{C}(a)$ that have the form (6.2) in some coordinate system at a, where $c(0) \neq 0$. In fact, if in (6.2) we have $v = \sum c_{p\nu}t^{p\nu}$ then the curve (6.2) is smooth (it suffices to change the parameter to $\tau = t^p$). In the remaining cases $v = a_pt^p + ... + a_{kp}t^{kp} + c(t)t^q$, where $c(0) \neq 0$ and $pk < q < p(k+1)$, and it suffices to replace the coordinates to $z_1' = z_1, v' = v - a_pz_1 - ... - a_{kp}z_1^k$ (it is a biholomorphism of $\mathbb{C}^n$ onto $\mathbb{C}^n$).

Let us notice that if a curve $\Gamma \ni a$ is smooth at a, then its proper inverse image $\bar{\Gamma}$ intersects S at a unique point: $\bar{\Gamma} \cap S = \{\bar{a}\}$ and in a transversall way.

Proposition 6. *Let Γ be a curve in $\mathcal{C}_{p,q}$, $p > 1$. Then its proper inverse image $\bar{\Gamma}$ is a curve and $\bar{\Gamma} \cap S = \{\bar{a}\}$; if $q > 2p$ then $\bar{\Gamma} \in \mathcal{C}_{p,q-p}(\bar{a})$, and if $q < 2p$ then $\bar{\Gamma} \in \mathcal{C}_{q-p}(\bar{a})$.*

Dowód. We may restrict considerations to the canonical blowing-up ($a = 0$) and Γ of form (2), where $c(0) \neq 0$ and $|c(t)| \leqslant M$. Then (see $(**)$ in Section 2)

$$\chi_1^{-1}(p^{-1}(\Gamma)) = \{z_1 = t^p, z_1 w_{(1)} = c(t)t^q, |t| < \sigma\}$$
$$= \{z_1 = 0\} \cup \{z_1 = t^p, w_{(1)} = c(t)t^{q-p},\ |t| < \sigma\},$$

and for $k > 1$

$$\chi_k^{-1}(p^{-1}(\Gamma)) = \{z_k w_1 = t^p, ..., z_k = c_k(t)t^q, ...,\quad |t| < \sigma\}$$
$$\subset \{z_k = 0\} \cup \{|z_k|^{q-p}|w_1|^q \geqslant M^{-p}\}.$$

Hence

$$\chi_1^{-1}(\bar\Gamma) = \{z_1 = t^p, w_{(1)} = c(t)t^{q-p},\ |t| < \sigma\} \in \begin{cases} \mathcal{C}_{p,q-p}(0) \text{ if } q > 2p \\ \mathcal{C}_{q-p}(0) \text{ if } q < 2p \end{cases}$$

and $\chi_k^{-1}(\bar\Gamma) \cap \chi_k^{-1}(S) = \emptyset$ for $k > 1$. Then $\bar\Gamma \cap S = \{\bar a\}$, where $\bar a = \chi_1(0)$, and $\bar\Gamma \in \mathcal{C}_{p,q-p}(\bar a)$ if $q > 2p$, and $\bar\Gamma \in \mathcal{C}_{q-p}(\bar a)$ if $q < 2p$. □

Smooth curves $\Gamma_1, \Gamma_2 \ni a$ are tangent of order p at a if in some (and then in each) coordinate system ϕ at a in which they are topographic: $\phi(\Gamma_i) = \{v = g_i(z_1), z_1 \in U_i\}$, the function $g_2 - g_1$ has a zero of order p at 0.

Proposition 7. *Let smooth curves $\Gamma_1, \Gamma_2 \ni a$ be tangent of order p at a, and let $\bar\Gamma_1, \bar\Gamma_2$ be their proper inverse images. If $p > 1$ then $\bar\Gamma_1 \cap S = \bar\Gamma_2 \cap S = \{\bar a\}$ and $\bar\Gamma_1, \bar\Gamma_2$ are tangent of order $p - 1$ at a; if $p = 1$ then $\bar\Gamma_1 \cap S \neq \bar\Gamma_2 \cap S$.*

Dowód. We may restrict considerations to the canonical blowing-up ($a = 0$) and $\Gamma_1 = \{v = 0, |z_1| < \sigma\}$, $\Gamma_2 = \{v = c(z_1)z_1^p, |z_1| < \sigma\}$, c is a holomorphic mapping, $c(0) \neq 0$, $|c(z_1)| \leqslant M$. Then (see $(**)$ in Section 2) $\chi_1^{-1}(p^{-1}(\Gamma_1)) = \{z_1 = 0\} \cup \{w_{(1)} = 0, |z_1| < \sigma\}$, $\chi_1^{-1}(p^{-1}(\Gamma_2)) = \{z_1 = 0\} \cup \{w_{(1)} = c(z_1)z_1^{p-1}, |z_1| < \sigma\}$ and $\chi_k^{-1}(p^{-1}(\Gamma_i)) \subset \{|z_k| \leqslant M|z_k w_1|^p\} \subset \{z_k = 0\} \cup \{|z_k|^p|w_1|^{p-1} \geqslant 1/M\}$ for $k > 1$. Hence $\chi_k^{-1}(\bar\Gamma_i) \cap \chi_k^{-1}(S) = \emptyset$ for $k > 1$ and $\chi_1^{-1}(\bar\Gamma_1) = \{w_{(1)} = 0, |z_1| < \sigma\}$ and $\chi_1^{-1}(\bar\Gamma_2) = \{w_{(1)} = c(z_1)z_1^{p-1}, |z_1| < \sigma\}$. So if $p > 1$ then $\bar\Gamma_1 \cap S = \bar\Gamma_2 \cap S = \{\bar a\}$, where $\bar a = \chi_1(0)$, and $\bar\Gamma_1, \bar\Gamma_2$ are tangent of order $p - 1$ at $\bar a$. If in turn $p = 1$ then $\bar\Gamma_1 \cap S = \{\chi_1(0)\}$ and $\bar\Gamma_2 \cap S = \{\chi_1(0, c(0))\}$. □

A smooth curve $\Gamma \ni a$ is tangent of order p at a to a submanifold $N \ni a$ if it is tangent of order p at a to a smooth curve $\Gamma_0 = N \cap L$, where L is a submanifold of dimension $\operatorname{codim} N + 1$ transversal to N and containing a neighbourhood of a at Γ.

Proposition 8. *Let a smooth curve $\Gamma \ni a$ be tangent of order p at a to a submanifold $N \ni a$; let $\bar\Gamma, \bar N$ be their proper inverse images and let $\bar\Gamma \cap S = \{\bar a\}$. If $p > 1$ then $\bar a \in \bar N$ and $\bar\Gamma$ is tangent of order $p - 1$ at $\bar a$ to $\bar N$; if $p = 1$ then $\bar a \notin \bar N$.*

Dowód. One can assume that the submanifold L contains Γ, is transversal to N and the smooth curve $\Gamma_0 = N \cap L$ is tangent of order p at a to Γ. So, we have $\bar L \supset \bar\Gamma$, $\bar L$ is transversal to $\bar N$ and $\bar\Gamma_0 = \bar N \cap \bar L$ is a smooth curve (Proposition 5). According to Proposition 7: if $p > 1$ then $\bar{\Gamma}$ and $\bar{\Gamma}_0$ are tangent of order $p-1$ at $\bar{a}$, so $\bar{a} \in \bar{N}$ and $\bar{\Gamma}$ is tangent of order $p-1$ at $\bar{a}$ to $\bar{N}$; if $p = 1$ then $\bar{N} \cap \bar{L} \cap S = \bar{\Gamma}_0 \cap S = \{\bar{c}\}$, $\bar{c} \neq \bar{a}$, but $\bar{a} \in \bar{L}$, so $\bar{a} \notin \bar{N}$. □

7. Geometric desingularization of a curve in a manifold

Let M be a manifold. We say an analytic subset $V \subset M$ is a normal crossing subset if irreducible components of its germs V_a, $a \in V$, are germs of smooth hypersurfaces intersecting transversally at a. In particular such sets are:

Sets of type τ: they are unions of smooth compact hypersurfaces which are mutually transversal. By Propositions 4 and 5:

(1) The inverse-image of a set of type τ (with irreducible components $N_1, ..., N_r$ if $r > 0$) by a blowing-up is a set of type τ (with irreducible components $\bar{N}_1, ..., \bar{N}_r, S$ if $r > 0$, where S is the exceptional set).

A set of type τ' is one of type τ or one-point set. Obviously, the inverse image of a set of type τ' by a blowing-up is a set of type τ. Let $Z \subset M$ be of type τ'. We say a curve $\Gamma \subset M$ is crosswise to Z (at $c \in Z$) if it is closed, $\Gamma \cap Z = c$, Γ_c is irreducible and $\Gamma - c$ is smooth. In particular Γ is crosswise to c.

We say sets E_i are separated by a set F if $E_i \backslash F$ are disjoint. This property is preserved by the operation of taking inverse images.

(2) Let $\pi : \bar{M} \to M$ be a blowing-up at $a \in Z$, Z of type τ'. Then: Γ is crosswise to Z implies $\bar{\Gamma}$ is crosswise to $\pi^{-1}(Z)$, and $\pi^{-1}(\Gamma \cup Z) = \bar{\Gamma} \cup \pi^{-1}(Z)$ (by Propositions 6 and 4). If Γ is smooth, crosswise to Z and transversal to Z (in case Z is not one-point set) then $\bar{\Gamma}$ is smooth, crosswise and transversal to $\pi^{-1}(Z)$ (by Propositions 5 and 4). If Γ_i are crosswise to Z then: Γ_i are separated by Z implies $\bar{\Gamma}_i$ are separated by $\pi^{-1}(Z)$. If Γ_i are disjoint then $\bar{\Gamma}_i$ are disjoint.

A multiple blowing-up over $E \subset M$ is a composition of blowings-up $\pi = \pi_1 \circ ... \circ \pi_r : \bar{M} \to M$, where

$$\begin{array}{ccccccccc} & & E_{r-1} & & & & E_1 & & E_0 = E \\ & & \cap & & & & \cap & & \cap \\ \bar{M} = M_r & \xrightarrow{\pi_r} & M_{r-1} & \to & \dots & \to & M_1 & \xrightarrow{\pi_1} & M_0 = M \end{array}$$

$\pi_i : M_i \to M_{i-1}$ is the blowing-up at a point of $E_{i-1}, i = 1, ..., r$, and $E_i = \pi_i^{-1}(E_{i-1}), i = 1, ..., r-1$. Then π is also a multiple blowing-up over $F \supset E$. If E is analytic and nowhere dense then π is a modification in E. Obviously:

(3) If $\pi : \bar{M} \to M$ is a multiple blowing-up over E and $\bar{\pi} : \mathring{M} \to \bar{M}$ – over $\pi^{-1}(E)$ then $\pi \circ \bar{\pi} : \mathring{M} \to M$ is a multiple blowing-up over E.

(4) If M is open in a manifold N and $\pi : \bar{M} \to M$ is a multiple blowing-up over $E \subset M$ then $\pi = \pi_1^M$, where $\pi_1 : \bar{N} \to N$ is a multiple blowing-up over E, $\bar{M}$ is open in $\bar{N}$ (by Proposition 1 and Remark 1).

(5) The inverse image of a set of type τ' by a multiple blowing-up is a set of type τ.

(6) Let $\pi : \bar{M} \to M$ be a multiple blowing-up over a set Z of type τ'. If Γ is crosswise to Z then consecutively using the operation of taking proper inverse images by $\pi_1, ..., \pi_r$ we obtain, according to (2), a curve $\bar{\Gamma} \subset \bar{M}$ which is crosswise to $\pi^{-1}(Z)$. It is called the proper inverse image of the curve Γ by the multiple blowing-up π, and then $\pi^{-1}(\Gamma \cup Z) = \bar{\Gamma} \cup \pi^{-1}(Z)$ (hence $\bar{\Gamma} = \overline{\pi^{-1}(\Gamma) \setminus \pi^{-1}(Z)}$). By (2):

(a) Γ smooth, crosswise to Z and transversal to Z (in case Z is not a one-point set) implies $\bar{\Gamma}$ is smooth, crosswise and transversal to $\pi^{-1}(Z)$. If Γ_i are crosswise to Z then:

(b) Γ_i separated by Z implies $\bar{\Gamma}_i$ separated by $\pi^{-1}(Z)$,

(c) Γ_i disjoint implies $\bar{\Gamma}_i$ disjoint. Moreover:

(d) If Γ is crosswise to Z and $\mathring{\Gamma}$ is the proper inverse image of $\bar{\Gamma}$ by a multiple blowing-up $\bar{\pi} : \mathring{M} \to \bar{M}$ over $\pi^{-1}(Z)$ then $\mathring{\Gamma}$ is the proper inverse image of Γ by $\pi \circ \bar{\pi}$.

(7) Let Γ be crosswise to a. By the first implication in (2) we recursively define a sequence of blowings-up $... \to M_i \overset{\pi_i}{\to} M_{i-1} \to ... \to M_1 \overset{\pi_1}{\to} M$ and a sequence of triplets $a_i \in \Gamma_i \subset M_i$, where Γ_i is crosswise to a_i, where $a_0 = a$, $\Gamma_0 = \Gamma$, $M_0 = M$, in such a way that: π_i is the blowing-up at a_{i-1}, Γ_i is the proper inverse image of Γ_{i-1} and $\{a_i\} = \Gamma_i \cap \pi_i^{-1}(a_{i-1})$. Then $\pi_{(k)} = \pi_1 \circ ... \circ \pi_k : M_k \to M$ is a multiple blowing-up over a by which Γ_k is the proper inverse image of Γ.

(A) If Γ is crosswise to a then there exists a multiple blowing-up over a such that the proper inverse image $\bar{\Gamma}$ is smooth.

In fact, let us take a sequence of blowings-up as in (7) for Γ. We will show that for some i the proper inverse image Γ_i of Γ by $\pi_{(i)}$ belongs to $\mathcal{C}_1(a_i)$, and so it is smooth. Namely $\Gamma = \Gamma_0$ belongs to some $\mathcal{C}_r(a_0)$ (see Section 6). By Proposition 6, if $\Gamma_v \in \mathcal{C}_{p,q}(a_v)$, $p > 1$, then Γ_{v+1} belongs to $\mathcal{C}_{p,q-p}(a_{v+1})$ if $q > 2p$, and to $\mathcal{C}_{q-p}(a_{v+1})$ if $q < 2p$ (and then $q - p < p$). So, if $\Gamma_i \in \mathcal{C}_p(a_i)$, $p > 1$, then some Γ_j $(j > i)$ belongs to $\mathcal{C}_s(a_j)$, where $s < p$.

(B) If Γ, Γ' are smooth, crosswise to a and separated by a then there exists a multiple blowing-up over a such that proper inverse images $\bar{\Gamma}, \bar{\Gamma}'$ are disjoint.

In fact, let us consider constructions of sequences π_i, Γ_i, a_i for Γ and π'_i, Γ'_i, a'_i for Γ' described in (7). We may take the same first blowing-up $\pi_1 = \pi'_1$ at $a_0 = a'_0 = a$, and (by the assumption) the curves $\Gamma_0, \Gamma'_0 \ni a_0$ are separated by a_0; let p be their order of tangency. Let us consider the following condition:

(σ_k) for $i \leqslant k$ we can take the same blowings-up $\pi_i = \pi'_i$ at $a_{i-1} = a'_{i-1}$ and $\Gamma_{i-1}, \Gamma'_{i-1}$ are separated by a_{i-1} and tangent at a_{i-1} of order $p - i + 1$.

By the above (σ_1) holds. Suppose (σ_k) holds for $k < p$; then (σ_{k+1}) holds; in fact, $\Gamma_{k-1}, \Gamma'_{k-1}$ are tangent at a_{k-1} of order $p - k + 1$, so by Proposition 7 there is $a_k = a'_k$, and taking the same blowing-up $\pi_{k+1} = \pi'_{k+1}$ at a_k we have Γ_k, Γ'_k are tangent of order $p-k$ at a_k, crosswise to $\pi_k^{-1}(a_{k-1})$ and separated by $\pi_k^{-1}(a_{k-1})$ (see (2)), and so separated by a_k. In consequence (σ_p) holds, that is we may have $\pi_i = \pi'_i$ for $i \leqslant p$ and curves $\Gamma_{p-1}, \Gamma'_{p-1}$ are separated by a_{p-1} and tangent of order 1 at a_{p-1}. Hence by Proposition 7 the curves Γ_p, Γ'_p have different points a_p, a'_p in $\pi_p^{-1}(a_{p-1})$, but (see (2)) they are separated by $\pi_p^{-1}(a_{p-1})$ and so they are disjoint. Hence $\pi_{(p)}$ is a required multiple blowing-up over a.

(C) If Γ is smooth and crosswise at a to Z of type τ' then there exists a multiple blowing-up π over a such that the proper inverse image $\bar{\Gamma}$ of Γ intersect transversally $\pi^{-1}(Z)$.

In fact, let us take a sequence of blowings-up as in (7) for Γ (treated as crosswise to a). Then Γ_k are smooth and transversal to $\pi_k^{-1}(a_{k-1})$ (Proposition 4). The sets $Z_k = \pi_{(k)}^{-1}(Z)$ are of type τ. Since (see (6)) Γ_k is crosswise to $Z_k \ni a_k$ then $Z_k \cap \Gamma_k = \{a_k\}$. Let $N_1, ..., N_r$ be irreducible components of Z_k and consider the following condition

$$(\tau_p) \qquad N_i \ni a_k \Longrightarrow \Gamma_k \text{ is tangent of order} \leqslant p \text{ at } a_k \text{ to } N_i,$$

and notice that if $N_i \not\ni a_k$ then $\Gamma_k \cap N_i = \emptyset$. By (1) the irreducible components of Z_{k+1} are proper inverse images by $\pi_{k+1} : \bar{N}_1, ..., \bar{N}_r$ and $\pi_{k+1}^{-1}(a_k)$ (the latter is transversal to Γ_{k+1} at a_{k+1}). Hence, by Proposition 8, if Γ_k is tangent at a_k of order q to $N_i \ni a_k$ then Γ_{k+1} is tangent at a_{k+1} of order $q-1$ to $\bar{N}_i \ni a_{k+1}$ when $q > 1$, and $\bar{N}_i \not\ni a_{k+1}$ when $q = 1$. So, if (τ_p), $p > 1$, holds for k, then (τ_{p-1}) holds for $k+1$. Hence for some k the condition (τ_1) holds, and then Γ_{k+1} is disjoint with $\bar{N}_1, ..., \bar{N}_r$ and transversal to $\pi_{k+1}^{-1}(a_k)$ i.e. intersect transversally Z_{k+1}. Then $\pi_{(k+1)}$ is a required multiple blowing-up over a.

(8) If Γ is crosswise at a to Z of type τ' then there exists a multiple blowing-up π over a such that proper inverse image $\bar{\Gamma}$ of Γ is smooth, crosswise and transversal to $\pi^{-1}(Z)$.

In fact, by (A) there exists a multiple blowing-up $\pi_1 : M_1 \to M$ over a such that the proper inverse image $\bar{\Gamma} \subset M_1$ is smooth; by (6) it is crosswise to $\pi^{-1}(Z)$ of type τ (see (5)) at $c \in \pi_1^{-1}(\Gamma) \cap \pi_1^{-1}(Z) = \pi_1^{-1}(a)$, so by (C) there exists a multiple blowing-up $\pi_2 : M_2 \to M_1$ over c such that the proper inverse image $\mathring{\Gamma} \subset M_2$ of the curve $\bar{\Gamma}$ is smooth, transversal and croosswise (by (6) and $c \in \pi_1^{-1}(Z)$) to $\pi_2^{-1}(\pi_1^{-1}(Z)) = \pi^{-1}(Z)$, where $\pi = \pi_1 \circ \pi_2 : M_2 \to M$ is a multiple blowing-up over a (by (3) and $c \in \pi^{-1}(a)$), which satisfies the assertion (by (6) (d)).

Proposition 9. *If $\Gamma_1, ..., \Gamma_r$ are crosswise to a and separated by a then there exist a multiple blowing-up π over a such that the proper inverse images $\bar{\Gamma}_1, ..., \bar{\Gamma}_r$ are smooth, disjoint, and crosswise and transversal to $\pi^{-1}(a)$.*

Dowód. For the case $r = 1$ it is precisely (8) taking $Z = \{a\}$. Assume the proposition is true for $r-1$, $(r > 1)$; so there exists a multiple blowing-up $\pi_1 : M_1 \to M$ over a such that, if $\bar{\Gamma}_i \subset M_1$ are proper inverse images of Γ_i then $\bar{\Gamma}_1, ..., \bar{\Gamma}_{r-1}$

are smooth, disjoint, and crosswise and transversal to $Z_1 = \pi_1^{-1}(a)$ of type τ (see (5)). Then (see (6)) $\bar{\Gamma}_r$ is crosswise to Z_1 and we have $\bar{\Gamma}_r \cap Z_1 = \{a_1\}$. By (8) there exists a multiple blowing-up $\pi_2 : M_2 \to M_1$ over a_1 such that if $\mathring{\Gamma}_i \subset M_2$ are proper inverse images of $\bar{\Gamma}_i$ then $\mathring{\Gamma}_r$ is smooth, crosswise and transversal to $Z_2 = \pi_2^{-1}(Z_1) = \pi_0^{-1}(a)$, where $\pi_0 = \pi_1 \circ \pi_2 : M_2 \to M$ is the multiple blowing-up over a (see (3)). Then $\mathring{\Gamma}_1, ..., \mathring{\Gamma}_{r-1}$ are smooth, disjoint, and crosswise and transversal to Z_2 (see (6) (a) and (c)); moreover (see (6) (d)) the curves $\mathring{\Gamma}_i$ are proper inverse images of Γ_i by π_0 and so they are separated by Z_2 (see (6) (b)). If they are disjoint, π_0 satisfies the condition of the proposition. In the remaining cases is for example $\mathring{\Gamma}_r \cap \mathring{\Gamma}_1 = \{a_2\}$, $a_2 \in Z_2$, and then $\mathring{\Gamma}_r$ is disjoint with $\mathring{\Gamma}_2, ..., \mathring{\Gamma}_{r-1}$, that is $\mathring{\Gamma}_2, ..., \mathring{\Gamma}_r$ are disjoint. By (B) there exists a multiple blowing-up $\pi_3 : M_3 \to M_2$ over a_2 such that if $\Gamma_i' \subset M_3$ are proper inverse images of $\mathring{\Gamma}_i$ then Γ_r' and Γ_1' are disjoint. But (see (6) (a)) Γ_i' are smooth, crosswise and transversal to $\pi_3^{-1}(Z_2) = \pi^{-1}(a)$, where $\pi = \pi_0 \circ \pi_3 : M_3 \to M$ is a multiple blowing-up over a (see (3)), under which Γ_i' are proper inverse images of Γ_i (see (6) (d)); moreover (see (6) (c)) $\Gamma_1', ..., \Gamma_{r-1}'$ and $\Gamma_2', ..., \Gamma_r'$ are disjoint and so Γ_i' are disjoint. Then π satisfies the condition of the proposition. □

Proposition 10. *If $\Gamma \subset M$ is a closed curve and the set of its singular points Γ^* is finite then there exists a multiple blowing-up π over Γ^* such that $\pi^{-1}(\Gamma) = \Lambda \cup Z$, where $Z = \pi^{-1}(\Gamma^*)$ is of type τ, and Λ a smooth, closed curve which intersects transversally Z. In other words: $\pi^{-1}(\Gamma) = N_1 \cup ... \cup N_r \cup \Lambda$, where N_i are smooth, compact hypersurfaces, Λ a smooth, closed curve, $N_1, ..., N_r, \Lambda$ are mutually tranversal and $\pi^{-1}(\Gamma^*) = N_1 \cup ... \cup N_r$.*

Dowód. Let $\Gamma^* = \{a_1, ..., a_k\}$ and assume the proposition is true for $k-1$ provided $k > 1$. There exists an open neighbourhood U of the point a_k such that $a_1, ..., a_{k-1} \notin U$ and $\Gamma \cap U = \Gamma_1 \cup ... \cup \Gamma_r$, where Γ_i are closed curves in U, crosswise to a_k and separated by a_k. By Proposition 9 there exists a multiple blowing-up $\pi_1 : M_1 \to M$ over a_k such that the proper inverse images $\bar{\Gamma}_i$ of the curves Γ_i by the multiple blowing-up π_1^U are closed in $U_1 = \pi_1^{-1}(U)$, smooth, disjoint and transversal to $Z_1 = \pi_1^{-1}(a_k)$ and (by (6)) $\pi^{-1}(\Gamma_i) = \bar{\Gamma}_i \cup Z_1$. Then $\bar{\Gamma}_0 = \bigcup \bar{\Gamma}_i$ is a closed curve in U_1, smooth and intersect transversally Z_1, and $\pi_1^{-1}(\Gamma \cap U) = \bar{\Gamma}_0 \cup Z_1$. The curve $\pi_1^{-1}(\Gamma) \backslash Z_1$ is closed in $M_1 \backslash Z_1$ and its all singular points are $b_i = \pi_1^{-1}(a_i)$, $i = 1, ..., k-1$. Since $\bar{\Gamma}_0 \cap (U_1 \backslash Z_1) = \pi_1^{-1}(\Gamma \cap U) \backslash Z_1 = (\pi_1^{-1}(\Gamma) \backslash Z_1) \cap (U_1 \backslash Z_1)$ then $\bar{\Gamma} = (\pi_1^{-1}(\Gamma) \backslash Z_1) \cup \bar{\Gamma}_0$ is closed in M_1 which intersects transversally Z_1 and $\bar{\Gamma}^* = \{b_1, ..., b_{k-1}\}$. It is $\pi_1^{-1}(\Gamma) = \bar{\Gamma} \cup Z_1$ (since $\pi_1^{-1}(\Gamma) = \pi_1^{-1}(\Gamma \cap U) \cup \pi_1^{-1}(\Gamma \backslash a_k)$). If $k = 1$ then π_1 satisfies the conditions of the proposition. So, let us assume $k > 1$. Then (by the induction hypothesis) there exists a multiple blowing-up $\pi_2 : M_2 \to M_1$ over $\bar{\Gamma}^*$ such that $\pi_2^{-1}(\bar{\Gamma}) = \Lambda \cup Z_2$, where $\Lambda \subset M_2$ is a closed, smooth, intersect transversally $Z_2 = \pi_2^{-1}(\bar{\Gamma}^*)$ of type τ. Then $\pi = \pi_1 \circ \pi_2 : M_2 \to M$ is a multiple blowing-up over Γ^* (see (3)) and $\pi^{-1}(\Gamma) = \pi_2^{-1}(\bar{\Gamma}) \cup \pi_2^{-1}(Z_1) = \Lambda \cup Z$, where $Z = Z_2 \cup \pi_2^{-1}(Z_1) = \pi^{-1}(\Gamma^*)$. Since $Z_1 \subset U_1$ is disjoint with Γ^* then $\pi_2^{-1}(Z_1) \subset \pi_2^{-1}(U_1)$ is disjoint with Z_2 and $\pi_2^{U_1}$ is a biholomorphism. Then $\pi_2^{-1}(\bar\Gamma \cap U_1) = \Lambda \cap \pi_2^{-1}(U_1)$ intersect transversally $\pi_2^{-1}(Z_1)$ and so Λ intersect transversally $\pi_2^{-1}(Z_1)$. Then Λ intersect transversally Z and π satisfies the conditions of the proposition. □

8. Blowing-up of submanifolds

Let M be a n-dimensional manifold and $n = p + q$. Let $f_1, ..., f_q \in \mathcal{O}(M)$ and assume $f = (f_1, ..., f_q) : M \to \mathbb{C}^q$ is a submersion. Then $L = f^{-1}(0)$ is a submanifold of dimension p. The subset

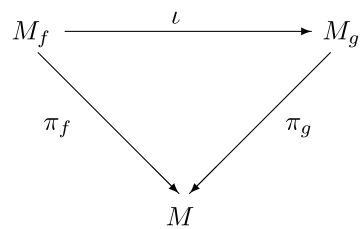
$$M_f = \{(z, \lambda) : f(z) \in \lambda\} \subset M \times \mathbb{P}_{q-1},$$

that is $M_f = \phi^{-1}(\Pi_q)$, where $\phi = f \times e : M \times \mathbb{P}_{q-1} \to \mathbb{C}^q \times \mathbb{P}_{q-1}$, $e = \mathrm{id}\, \mathbb{P}_{q-1}$, is also a submersion, is a closed submanifold of dimension n. The canonical projection

$$\pi_f : M_f \to M$$

is called an elementary blowing-up by functions $f_1, ..., f_q$. It is a modification in the set L called the centre of blowing-up. It is so because π_f is a proper mapping, $(M_f)_{M\setminus L} = M_f \setminus \pi_f^{-1}(L)$ is the graph of the holomorphic mapping $M\setminus L \ni z \to \mathbb{C}f(z) \in \mathbb{P}_{q-1}$, hence $\pi_f^{M\setminus L} : M_f \setminus \pi_f^{-1}(L) \to M\setminus L$ is a biholomorphism, and $\pi_f^{-1}(L) = L \times \mathbb{P}_{q-1}$ is a closed, smooth hypersurface called the exceptional set of the blowing-up. Of course $\pi_f^G = \pi_{f_G}$ is an elementary blowing-up by $(f_i)_G$ with centre $L \cap G$.

Proposition 11. *If additionally $g = (g_1, ..., g_q) : M \to \mathbb{C}^q$ is a submersion and $\sum \mathcal{O}(M) f_i = \sum \mathcal{O}(M) g_i$ (i.e. f_i and g_j generate the same ideal in $\mathcal{O}(M)$; then $g^{-1}(0) = f^{-1}(0) = L$), then the blowings-up π_f and π_g are isomorphic: the diagram*



is commutative, where ι is a biholomorphism.

Corollary 5. *If $\pi_i : M_i \to M$ are elementary blowings-up with the centre L then arbitrary point $a \in L$ has an open neighbourhood U in M such that $\pi_1^U \approx \pi_2^U$.*

In particular we have the elementary blowing-up of $\mathbb{C}^n$ by $v = (z_{p+1}, ..., z_n)$:

$$\mathbb{C}^n_v = \{(z, \lambda) \in \mathbb{C}^n \times P_{q-1} : v \in \lambda\} = \mathbb{C}^p \times \Pi_q$$

and

$$\pi_v = (\mathrm{id}\,\mathbb{C}^p) \times \pi_q : \mathbb{C}^p \times \Pi_q \to \mathbb{C}^p \times \mathbb{C}^q.$$

The blowings-up π_v^Ω, where Ω is an open neighbourhood of 0 in $\mathbb{C}^n$ is called standard.

(#) Let $L \subset M$ be a p-dimensional submanifold. If $\phi : U \to \Omega$ is a chart at a ($\phi(a) = 0$) such that $\phi(L \cap U) = \{v = 0\} \cap \Omega$ then $\psi = (\phi_{p+1}, ..., \phi_n)$ is a submersion and the blowing-up π_ψ is isomorphic to the elementary blowing-up π_v^Ω

$$\begin{array}{ccc} U_\psi & \xrightarrow{(\phi \times e)_{U_\psi}} & (\mathbb{C}^p \times \pi_{q-1})_\Omega \\ \downarrow{\scriptstyle \pi_\psi} & & \downarrow{\scriptstyle \pi_v^\Omega} \\ U & \xrightarrow{\phi} & \Omega \end{array}$$

Notice that if $f : \bar{M} \to M$ is a modification in an analytic set $Z \subset M$ and $G \subset M$ is an open set then $f^G : f^{-1}(G) \to G$ is a modification in $Z \cap G$.

Proposition 12. *If $f_i : M_i \to M$ are modifications ($i = 1, 2$) and $M = \bigcup G_\iota$ is an open cover then $f_1 \approx f_2$ if and only if $f_1^{G_\iota} \approx f_2^{G_\iota}$ for every ι.*

Corollary 6. *Elementary blowings-up of a manifold with the same centre are isomorphic.*

Proposition 13. *(on gluing modifications). If $M = \bigcup M_\iota$ is an open cover and $f_\iota : \bar{M}_\iota \to M$ are modifications such that $f_\iota^{M_\iota \cap M_k} \approx f_k^{M_\iota \cap M_k}$, then there exists a unique (up to isomorphism) modification $f : \bar{M} \to M$ such that $f^{M_\iota} \approx f_\iota$.*

———— * ————

Let $L \subset M$ be a p-dimensional closed submanifold.

There exists a unique (up to an isomorphism) modification $\pi : \bar{M} \to M$ in L such that each point $a \in L$ has an open neighbourhood U_a such that π^{U_a} is isomorphic to an elementary blowing-up of U_a with the centre $L \cap U_a$. We will call it the blowing-up of the manifold M in the submanifold L (the latter is called the centre of blowing-up).

In fact, the uniqueness follows from Proposition 12 (applied to the cover: $M \setminus L$ and U_a for $a \in L$). For the existence: for every $a \in L$ we take an elementary blowing-up $\pi_a : M_a \to U_a$ of an open neighbourhood U_a of the point a with the centre $L \cap U_a$. By Proposition 12 and Corollary 6 we have $\pi_a^{U_a \cap U_b} \approx \pi_b^{U_a \cap U_b}$ (as blowings-up with the common centre $L \cap U_a \cap U_b$); we take also $e = \mathrm{id}\, M \setminus L$; then obviously $\pi_a^{U_a \setminus L} \approx e^{U_a \setminus L}$. By Proposition 13, there exists a modification $\pi : \bar{M} \to M$ such that $\pi^{U_a} \approx \pi_a$.

The subset $\pi^{-1}(L)$ is a closed and smooth hypersurface. (There is: $\pi^{-1}(V)$ is isomorphic to $V\times\mathbb{P}_q$, where V are sufficently small open neighbourhoods of points in L; moreover $\pi^L : \pi^{-1}(L) \to L$ is a locally trivial fibration with the fiber $\mathbb{P}_{q-1}$). It is called the exceptional set of the blowing-up π.

Proposition 14. *If $\pi : \bar{M} \to M$ is a blowing-up in L and $N \subset M$ is a submanifold of dimension s which intersect transversally L, then $\pi^{-1}(N)$ is a submanifold of dimension s which intersect transversally $\pi^{-1}(L)$.*

Dowód. Let $a \in N \cap L$. By Proposition 3 we take a chart $\phi : U \to \Omega$ at a such that $\phi(L\cap U) = \{v = 0\} \cap \Omega$ and $\phi(N \cap U) = \{t = 0\} \cap \Omega$, where $t = (z_1, ..., z_r)$, $r = n-s \leqslant p$; then $L\cap U = \psi^{-1}(0)$, where $\psi = (\phi_{p+1}, ..., \phi_n)$. Shrinking U we may assume that π^U is isomorphic to an elementary blowing-up π_f of U, isomorphic in turn to π_ψ (Proposition 11) which is isomorphic to π_v^Ω (by (#)), that is π^U is isomorphic to π_v^Ω over ϕ

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\text{bihol.}} & \pi_v^{-1}(\Omega) \\ \pi^U \downarrow & & \downarrow \pi_v^\Omega \\ U & \xrightarrow{\phi} & \Omega \end{array}$$

Then $\pi^{-1}(L\cap U)$, $\pi^{-1}(N\cap U)$ correspond to $\pi_v^{-1}(\{v = 0\}\cap\Omega)$, $\pi_v^{-1}(\{t = 0\}\cap\Omega)$ by the biholomorphism $\pi^{-1}(U) \to \pi_v^{-1}(\Omega)$. But $\pi_v^{-1}(\{v = 0\}) = \mathbb{C}^p\times(0\times\mathbb{P}_{q-1})$ and $\pi_v^{-1}(\{t = 0\}) = \{u \in \mathbb{C}^r : t = 0\} \times \Pi_q$ (a submanifold of dimension s), where $u = (z_1, ..., z_p)$, intersect transversally in $\mathbb{C}^p\times\Pi_q$, so the inverse images $\pi^{-1}(L)$, $\pi^{-1}(N)$ in $\pi^{-1}(U)$ (the second is a submanifold of dimension s) intersect transversally which implies that $\pi^{-1}(N)$ is a submanifold of dimension s and intersect transversally $\pi^{-1}(L)$ (because the sets of the form $\pi^{-1}(U)$ cover $\pi^{-1}(L)\cap\pi^{-1}(N)$). □

Theorem 1. *If $\Gamma \subset M$ is a closed curve with Γ^* finite then there exists a modification $\pi : \bar{M} \to M$ in Γ such that $\pi^{-1}(\Gamma)$ is a finite union of smooth, closed and mutually transversal hypersurfaces in M.*

Dowód. Let us take a multiple blowing-up $\pi_1 : M_1 \to M$ as in Proposition 10 and the blowing-up $\pi_2 : M_2 \to M_1$ of the curve Λ. Then $\pi = \pi_1\circ\pi_2 : M_2 \to M$ is a modification in Γ. Submanifolds $N_1, ..., N_r \subset M_1$ are mutually transversal in M_1 and pairs N_i, N_j $(i \neq j)$ intersect outside Λ. Hence $\pi_2^{-1}(N_i) \subset M_2$ are smooth hypersurfaces (Proposition 14), compact, mutually transversal in $M_2\backslash\pi_2^{-1}(\Lambda)$ and pairs $\pi_2^{-1}(N_i)$, $\pi_2^{-1}(N_j)$, $i \neq j$, intersect only outside $\pi_2^{-1}(\Lambda)$; moreover by Proposition 14 each $\pi_2^{-1}(N_i)$ intersect transversally $\pi^{-1}(\Lambda)$. Then smooth, closed hypersurfaces $\pi_2^{-1}(N_1), ..., \pi_2^{-1}(N_r), \pi_2^{-1}(\Lambda)$ with the union equal to $\pi^{-1}(\Gamma)$ are mutually transversal in M_2. □

9. Desingularization of a coherent sheaf of ideals on a 2-dimensional manifold

Let M be a 2-dimensional manifold.

A parameter at a point $a \in M$ is a germ $\phi \in \mathfrak{m}_a$ such that $d_a\phi \neq 0$. We say ϕ correspond to a germ of smooth curve A if $V(\phi) = A$; then it is a generator of $I(A)$ unique up to an invertible factor. We say parameters ϕ, ψ at a are transversal, if $V(\phi), V(\psi)$ are transversal, which means $d_a\phi, d_a\psi$ are linearly independent, or equivalently $(\bar{\phi}, \bar{\psi})$ is a chart (a system of coordinates at a) for some representatives $\bar{\phi}, \bar{\psi}$.

We say a germ $f \in \mathcal{O}_a$ is of type (NC) if $f \sim \phi^\alpha \psi^\beta$, where ϕ, ψ are transversal parameters at a. (It means that in some chart it has the form $az_1^\alpha z_2^\beta$, $a(0) \neq 0$). It holds if and only if $V(f) = A \cup B$ or $= A$ or $= \emptyset$, where A, B are germs of transversal smooth curves. Then, respectively to the above cases, $f \sim \phi^\alpha \psi^\beta$ or $f \sim \phi^\alpha$ or $f \sim 1$, where ϕ, ψ are parameters corresponding to A, B.

We say a function $f \in \mathcal{O}_M$ is of type (NC) if its all germs f_z, $z \in M$ are of type (NC). Then by Proposition 10 we have

Proposition 15. *If $f \in \mathcal{O}_M$ and $V(f)^*$ is finite, then there exists a blowing-up $\pi : \bar{M} \to M$ over $V(f)^*$ such that $f \circ \pi$ is of type (NC).*

——————— * ———————

By a coherent sheaf of ideals on M we mean a family $\mathcal{T}$ of ideals $\mathcal{T}_z \subset \mathcal{O}_z$, $z \in M$, such that each point in M has an open neighbourhood U in which $\mathcal{T}$ has a finite set of generators i.e. there exist $\phi_1, ..., \phi_r \in \mathcal{O}_U$ such that $(\phi_1)_z, ..., (\phi_r)_z$ generate $\mathcal{T}_z$ for every $z \in U$ ($\mathcal{T}$ corresponds to a sheaf according to the standard definition - obtained by the presheaf: $\{f \in \mathcal{O}_G : f_z \in \mathcal{T}_z \text{ for } z \in G\}_{G \text{ open in } M}$). The set of its zeros is defined by $V(\mathcal{T}) = \{z \in M : \mathcal{T}_z \neq \mathcal{O}_z\}$; since $V(\mathcal{T}) \cap U = \{\phi_1 = ... = \phi_r = 0\}$ if $\phi_1, \ldots, \phi_r$ generate $\mathcal{T}$ in U, then it is an analytic subset of the manifold M.

If $f : N \to M$ is a holomorphic mapping between manifolds we define the coherent sheaf $f^*\mathcal{T}$ on N (called the inverse image of the sheaf $\mathcal{T}$) by: $(f^*\mathcal{T})_\xi \subset \mathcal{O}_\xi$ is the ideal generated by $\mathcal{T}_{f(\xi)} \circ f_\xi$ that is by $\phi_1 \circ f_\xi, ..., \phi_r \circ f_\xi$, provided $\phi_1, ..., \phi_r$ generate $\mathcal{T}_{f(\xi)}$ (so, if ψ_i generate $\mathcal{T}$ in U then $\psi_i \circ f$ generate $f^*\mathcal{T}$ in $f^{-1}(U)$). It is obviously $V(f^*\mathcal{T}) = f^{-1}(V(\mathcal{T}))$. If $g : L \to N$ is a holomorphic mapping between manifolds then

$$(f \circ g)^*\mathcal{T} = g^*(f^*\mathcal{T}).$$

We say a sequence of germs $\phi_1, ..., \phi_r \in \mathcal{O}_a$ is of type (NC) if $\phi_i \sim \phi^{\alpha_i}\psi^{\beta_i}$, where ϕ, ψ are transversal parameters at a. We say a sequence of functions $f_1, ..., f_r \in \mathcal{O}_M$ is of type (NC) if each sequence of germs $(f_1)_z, ..., (f_r)_z$, $z \in M$, is of type (NC). Notice that if $f_1, ..., f_r \in \mathcal{O}_M$ then if the sequence $(f_1)_a, ..., (f_r)_a$ is of type (NC) then for an open neighbourhood U of the point a the sequence $(f_1)_U, ..., (f_r)_U$ is of type (NC).

We say an ideal of the ring $\mathcal{O}_a$ is of type (NC*), respectively (NC), if there exists a sequence of generators of the ideal of type (NC), respectively one generator of type (NC). We say a sheaf $\mathcal{T}$ is of type (NC*), respectively (NC), at a point $z \in M$ if $\mathcal{T}_z$ is of type (NC*), respectively (NC). At last we say a sheaf $\mathcal{T}$ is of type (NC*), respectively (NC) if it is of type (NC*), respectively (NC), in each point $z \in M$.

By $\sigma\mathcal{T}$ we will denote the set of points in which $\mathcal{T}$ is not of type (NC). Obviously $\sigma\mathcal{T} \subset V(\mathcal{T})$ (in general the inclusion $\sigma\mathcal{T} \subset V(\mathcal{T})^*$ does not hold, for example the point $0 \in \mathbb{C}^2$ and the sheaf generated in $\mathbb{C}^2$ by z_1^2 and $z_1 z_2$).

Lemma 1. *If $\phi_1, ..., \phi_r$ are holomorphic in an open neighbourhood U of a point a and $(\phi_i)_a \neq 0$, then after shrinking U there is $\phi_i = \psi_1^{\alpha_{i1}} ... \psi_s^{\alpha_{is}}$ for some $\psi_v \in O(U)$ such that $V(\psi_v)$ are (in U) crosswise to a, separated by a and $d_z\psi_v \neq 0$ for $z \in V(\psi_v)\backslash a$.*

Dowód. In fact, it suffices to take as ψ_v representatives, in a sufficiently small neighbourhood U, of all non-associated, irreducible divisors of the germs $(\phi_i)_a$. □

Hence

(1) The set $\sigma\mathcal{T}$ is isolated.

It suffices to take generators ϕ_i in U and ψ_v as above and let $z \in U\backslash a$. If $z \in V(\mathcal{T})$ then z belongs to a unique $V(\psi_s)$ and then $\mathcal{T}_z = O_z(\psi_s)_z^\alpha$, where $\alpha = \min(\alpha_{1s}, ..., \alpha_{rs})$.

Proposition 16. *If $\mathcal{T}$ is a coherent sheaf of ideals in M with $\sigma\mathcal{T}$ finite then there exists a multiple blowing-up $\pi : \bar{M} \to M$ over $\sigma\mathcal{T}$ such that $\pi^*\mathcal{T}$ is of type (NC*).*

Dowód. Let $a_1, ..., a_k$ be all the points in which $\mathcal{T}$ is not of type (NC*) (their number is finite because they belong to $\sigma\mathcal{T}$). Using induction with respect to k, by (3) in Section 7, it suffices to show that there exists a multiple blowing-up $\pi : \bar{M} \to M$ over a_k such that $\pi^{-1}(a_1), ..., \pi^{-1}(a_{k-1})$ are unique points of $\bar{M}$ in which $\pi^*\mathcal{T}$ is not of type (NC*). Really, let us take generators $\phi_1, ..., \phi_r$ of the sheaf $\mathcal{T}$ in an open neighbourhood U of the point a_k, and $\psi_1, ..., \psi_s \in \mathcal{O}(U)$ as in Lemma 1 (after shrinking U). By Proposition 9 applied to $V(\psi_v)$ (and by (4) and (6) in Section 7), there exists a multiple blowing-up $\pi : \bar{M} \to M$ over a_k and curves $L_1, ..., L_q \subset U$ smooth, closed and mutually tranversal in U, such that each $V(\psi_v \circ \pi) = \pi^{-1}(V(\psi_v))$ is the union of some of them. Let $c \in \pi^{-1}(U)$. It suffices to show that the sequence $(\psi_1 \circ \pi)_c, ..., (\psi_s \circ \pi)_c$ is of type (NC). If $c \notin \cup L_i$ then $V(\psi_v \circ \pi) = \emptyset$, so $(\psi_v \circ \pi)_c \sim 1$. If c belongs to a unique L_i then $V(\psi_v \circ \pi) = \emptyset$ or $= (L_i)_c$, so $(\psi_v \circ \pi)_c \sim \phi^{\alpha_v}$, where ϕ is a parameter corresponding to L_c. If at last $c \in L_i \cap L_j$, $i \neq j$, then $V(\psi_v \circ \pi) = \emptyset$ or $= (L_i)_c$ or $= (L_i)_c \cup (L_j)_c$, so $(\psi_v \circ \pi)_c \sim \phi^{\alpha_v}\psi^{\beta_v}$, where ψ, ϕ are parameters corresponding to $(L_i)_c, (L_j)_c$. □

Let $\pi : \bar{M} \to M$ be a blowing-up at a. Let σ_ξ be a parameter corresponding to S_ξ for $\xi \in S$.

Let ϕ be a parameter at a. The inverse image $\bar{\Gamma}$ of a representative of $V(\phi)$ intersects S precisely in one point a_ϕ, and the parameter $\bar{\phi}$ at a_ϕ corresponding to $\bar{\Gamma}_{a_\phi}$ is tranversal to σ_{a_ϕ} (see Proposition 4); notice that if parameters ϕ, ψ at a are transversal then $a_\phi \neq a_\psi$. It is

$$\phi \circ \pi_\xi \sim \begin{cases} \sigma_\xi \text{ for } \xi \in S \setminus a_\phi, \\ \bar{\phi}\sigma_\xi \text{ for } \xi = a_\phi. \end{cases}$$

In fact, it suffices to consider the canonical blowing-up p and $\phi = (z_1)_0$. Then $\phi \circ (p \circ \chi_1)_u = (z_1)_u$ for $u \in \{z_1 = 0\}$ and $\phi \circ (p \circ \chi_2)_v = (z_2 w_1)_v$ for $v \in \{z_2 = 0\}$, and $a_\phi = \chi_2(0)$ and $\bar{\phi} \circ \chi_2 = (w_1)_0$ (because $\chi_2^{-1}(p^{-1}(V(z_1))) = \{z_2 = 0\} \cup \{w_1 = 0\}$). It implies that if $f \sim \phi^\alpha \psi^\beta$, where ϕ, ψ are transversal parameters at a, then, putting $c = a_\phi, d = a_\psi$,

$$(\#) \qquad f \circ \pi_\xi \sim \begin{cases} \sigma_\xi^{\alpha+\beta} & \text{if } \xi \in S \setminus (a_\phi, a_\psi), \\ \bar{\phi}^\alpha \sigma_\xi^{\alpha+\beta} & \text{if } \xi = a_\phi, \\ \bar{\psi}^\beta \sigma_\xi^{\alpha+\beta} & \text{if } \xi = a_\psi. \end{cases}$$

A pair $f, g \in \mathcal{O}_a$ of type (NC): $f \sim \phi^\alpha \psi^\beta$, $g \sim \phi^{\alpha'} \psi^{\beta'}$, ϕ, ψ transversal parameters at a, is called unessential, if $(\alpha' - \alpha)(\beta' - \beta) \geqslant 0$; then f is a divisor of g or g is a divisor of f. If $(\alpha' - \alpha)(\beta' - \beta) < 0$ then we call the pair f, g essential of type (p, q), where $p = \min(|\alpha' - \alpha|, |\beta' - \beta|)$, $q = \max(|\alpha' - \alpha|, |\beta' - \beta|)$.

(2) Let $f, g \in O_a$ be a pair of type (NC). Then

(a) All the pairs $P_\xi = (f \circ \pi_\xi, g \circ \pi_\xi)$, $\xi \in S$, are of type (NC).

(b) If the pair f, g is unessential then all the pairs P_ξ, $\xi \in S$, are unessential.

(c) If the pair f, g is essential of type (p, p), then all the pairs $P_\xi, \xi \in S$, are unessential.

(d) If the pair f, g is essential of type (p, q), $p < q$, then there exists $c \in S$ such that all the pairs $P_\xi, \xi \in S \setminus c$, are unessential, and the pair P_c is essential of type $(p, q - p)$ or $(q - p, p)$ depending on whether $q \geqslant 2p$ or $q \leqslant 2p$.

Dowód. (a) and (b) are obvious by (#). The case (c) follows (by (#)) from the fact that then $\alpha + \beta = \alpha' + \beta'$. Let us pass to the proof of (d). We may assume (changing f and g if necessary) that $\alpha + \beta < \alpha' + \beta'$. If $\alpha < \alpha'$ then $\beta > \beta'$ and P_{a_ϕ} is unessential; then $p = \beta - \beta'$, $q = \alpha' - \alpha$, $q - p = (\alpha' + \beta') - (\alpha + \beta)$ and the pair P_{a_ψ} is essential of type – as in (d). If $\alpha > \alpha'$ then $\beta < \beta'$, so P_{a_ψ} is unessential: then $p = \alpha - \alpha'$, $q = \beta' - \beta$, $q - p = (\alpha' + \beta') - (\alpha + \beta)$ and the pair P_{a_ϕ} is essential of type – as in (d). □

Let $f, g \in \mathcal{O}_M$ be a pair of type (NC). We say it is unessential at a point $z \in M$, respectively, essential of type (p, q), if the pair of germs f_z, g_z is such a pair. Let us notice that each point has a neighbourhood U such that the pair f, g is unessential at each point $z \in U \setminus a$. From (2) it follows:

(3) Let $f, g \in \mathcal{O}_M$ be a pair of type (NC). Then the pair $f \circ \pi,\ g \circ \pi$ is also of type (NC). If the pair f, g is unessential in M then the pair $f \circ \pi,\ g \circ \pi$ is unessential in $\bar{M}$. Assume that the pair f, g is unessential at all the points of $M \backslash a$ and essential of type (p, q) at a. If $p = q$ then $f \circ \pi,\ g \circ \pi$ is unessential at all the points of $\bar{M}$; if $p < q$ then there exists $c \in \bar{M}$ such that $f \circ \pi,\ g \circ \pi$ is unessential at all the points $\bar{M} \backslash c$ and essential at c, of type $(p, q - p)$ or $(q - p, p)$ depending on whether $q \geqslant 2p$ or $q \leqslant 2p$.

(4) If the pair $f, g \in \mathcal{O}_M$ of type (NC) is unessential at all the points of $M \backslash a$ then there exists a multiple blowing-up $\pi : \bar{M} \to M$ over a such that the pair $f \circ \pi$, $g \circ \pi$ (also of type (NC) by (3)) is unessential at all the points of $\bar{M}$.

In fact, if the pair f, g is essential at a, we may define (by (3)) a sequence $\bar{M} = M_r \overset{\pi_r}{\to} \dots \overset{\pi_1}{\to} M_0 = M$, where $\pi_i : M_i \to M_{i-1}$ is the blowing-up at a_{i-1} $(i = 1, \dots, r)$, $a_0 = a$, and $a_i \in \pi_i^{-1}(a_{i-1})$ is the unique point of M_i in which the pair $f \circ \pi_1 \circ \dots \circ \pi_i,\ g \circ \pi_1 \circ \dots \circ \pi_i$ is essential $(i = 1, \dots, r - 1)$, and in particular of type (p, p) if $i = r - 1$ (because if $0 < p \leqslant q$ and the sequence $(p_i, q_i) \in \mathbb{N}^2$ is defined by $(p_0, q_0) = (p, q)$ and

$$(p_i, q_i) = \begin{cases} (p_{i-1}, q_{i-1} - p_{i-1}), & \text{if } \ q_{i-1} \geqslant 2p_{i-1}, \\ (q_{i-1} - p_{i-1}, p_{i-1}), & \text{if } \ q_{i-1} \leqslant 2p_{i-1}, \end{cases}$$

then there must be $p_{r-1} = q_{r-1}$ for some r).

Let $\pi : \bar{M} \to M$ be a multiple blowing-up. From (#) it follows:

(5) If $\xi \in \bar{M}$ and the sequence $f_1, \dots, f_r \in O_{\pi(\xi)}$ is of type (NC) then also the sequence $f_1 \circ \pi_\xi, \dots, f_r \circ \pi_\xi \in O_\xi$.

For it suffices to check it for a blowing-up. Hence (taking $r = 1$):

(6) If $\mathcal{T}$ is a coherent sheaf of ideals then $\sigma(\pi^* \mathcal{T}) \subset \pi^{-1}(\sigma \mathcal{T})$. Hence (by (1)), if $\sigma \mathcal{T}$ is finite then also the set $\sigma(\pi^* \mathcal{T})$ is finite.

(For if ϕ is a generator of type (NC) of the ideal $\mathcal{T}_{\pi(\xi)}$ then $\phi \circ \pi_\xi$ is a generator of type (NC) of the ideal $(\pi^* \mathcal{T})_\xi$).

Theorem 2 (Hironaka Theorem on 2-dimensional manifold)**.** *If $\mathcal{T}$ is a coherent sheaf of ideals on M for which $\sigma \mathcal{T}$ is finite, then there exists a multiple blowing-up $\pi : \bar{M} \to M$ over $\sigma \mathcal{T}$ such that $\pi^* \mathcal{T}$ is of type (NC).*

Dowód. By Proposition 12 (and by (3) in Section 7 and (6)) we may assume that $\mathcal{T}$ is of type (NC*).

Let us introduce the following definitions: An ideal $I \subset \mathcal{O}_z$ is of type (n), where $n \geqslant 1$, if I has a sequence at most n generators of type (NC). A sheaf $\mathcal{T}$ on M is of type (n) if $\sigma \mathcal{T}$ is finite and each $\mathcal{T}_z$, $z \in M$, is of type (n). Then (by (5) and (6)) for every multiple blowing-up $\pi : \bar{M} \to M$ the sheaf $\pi^* \mathcal{T}$ is also of type (n). A sheaf $\mathcal{T}$ of type (n) is of type (n, r), where $r \geqslant 0$, if, with exception of r points, each $\mathcal{T}_z$ is of type $(n - 1)$. Each sheaf $\mathcal{T}$ of type (n) is (because $\sigma \mathcal{T}$ is finite) of type (n, r) for some $r \geqslant 0$. A sheaf of type $(n, 0)$ is of type $(n - 1)$ and a sheaf of type (1) is of type (NC). Since $\mathcal{T}$ is of some type (n) (because $\sigma\mathcal{T}$ is finite) then it suffices (by (3) in Section 7 and (6)) to prove that if $\mathcal{T}$ is of type (n, r), $n \geqslant 2$, $r \geqslant 1$, then there exists a multiple blowing-up $\pi : \bar{M} \to M$ over $\sigma\mathcal{T}$ such that $\pi^*\mathcal{T}$ is of type $(n, r-1)$.

So, let $\mathcal{T}$ be of type (n, r), $n \geqslant 2$, $r \geqslant 1$. Then there exist points $a_1, ..., a_r \in M$ such that $\mathcal{T}_{a_i}$ are of type (n), and for $z \neq a_1, ..., a_r$ the ideals $\mathcal{T}_z$ are of type $(n-1)$. There exists a sequence $f_1, ..., f_n \in \mathcal{O}_U$ of type (NC) of generators of $\mathcal{T}$ in an open neighbourhood U of the point a_r, and (shrinking U) we may additionally assume that the pair f_1, f_2 is unessential at each points of the set $U \backslash a_r$. By (5) (and by (4) in Section 7) there exists a multiple blowing-up $\pi : \bar{M} \to M$ over a_r such that the pair $f_1 \circ \pi$, $f_2 \circ \pi \in \mathcal{O}_{\pi^{-1}(U)}$ is unessential at all the points of the set $\pi^{-1}(U)$. Then, if $\xi \in \pi^{-1}(U)$ then in the sequence $(f_i \circ \pi)_\xi$ of generators of the ideal $(\pi^*\mathcal{T})_\xi$ we may omit one of the generators $(f_1 \circ \pi)_\xi$, $(f_2 \circ \pi)_\xi$, that is $(\pi^*\mathcal{T})_\xi$, $\xi \in \pi^{-1}(U)$, are of type $(n-1)$. Since for $\xi \in \bar{M} \backslash \pi^{-1}(a_r)$ different of $\pi^{-1}(a_1), ..., \pi^{-1}(a_{r-1})$, the ideals $(\pi^*\mathcal{T})_\xi$ are obviously of type $(n-1)$ then $\pi^*\mathcal{T}$ is of type $(n, r-1)$. □

Literatura

[L1] Łojasiewicz Stanisław, *Wstęp do geometrii analitycznej zespolonej*, PWN, Warszawa 1988 (in Polish).

[L2] Łojasiewicz Stanisław, *Introduction to Complex Analytic Geometry*, Birkhäuser, 1991.