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Chapter 2
Quasicontinuous functions with small set of discontinuity points

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2.1 Introduction

The definition of quasicontinuity for real functions of real variable was given in [34] by S. Kempisty. Nevertheless, R. Baire in his work [1] has shown that a function of two variables continuous at each variable is quasicontinuous. An independent definition was given by W. W. Bledsoe [2] in 1952 under the name neighborly function. S. Marcus in [49] proved that the notions of neighborly and quasicontinuous functions are equivalent and he developed further properties of quasicontinuous functions. He showed that quasicontinuous functions need not be (Lebesgue) measurable and for each countable ordinal α there is a quasicontinuous function in the Baire class $\alpha+1$ which does not belong to Baire class α.

N. Levine in [44] introduced the notion of semi-continuous function as a function for which the inverse image of every open set is a semi-open set (a set A is semi-open if A is a subset of the closure of the interior of A). A. Neubrunnová in her paper [53] has shown that the notions of quasicontinuity and semi-continuity in the sense of Levine are equivalent. Z. Grande in [33] has shown that a function f is quasicontinuous if and only if the graph of the function f restricted to the set of all continuity points of f is dense in the graph of f.

A fundamental result concerning continuity points is due to N. Levine [44] for functions with values in a second countable space (and for functions with values in a metric space [53]) is that the set of discontinuity points of a quasicontinuous function is small.

Theorem 2.1. *Let X be a topologocal space and let Y be a second countable space ([44]) or let Y be a metric space ([53]). If $f\colon X \to Y$ is a quasicontinuous function then the set of discontinuity points is of first category.*

So, quasicontinuous functions have the Baire property. On the other hand, if $X = \mathbb{R}^2$ [19] or if X is a Baire pseudometrizable space space without isolated points (or X is a Baire resolvable perfectly normal locally connected space) [5] or X is a hereditarily separable perfectly normal Fréchet-Urysohn space [50], then for each F_σ-set A of first category there is a quasicontinuous function $f\colon X \to \mathbb{R}$ such that A is the set of all discontinuity points of this function. Points of quasicontinuity were characterized in [45]. Quasicontinuous functions were investigated very intensively. We recommend a survey [52] published in 1988 with more than 120 references.

2.2 Basic definitions

Let $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ be the set of all real, rational and positive integer numbers, respectively. For a set $A \subset \mathbb{R}$ denote by $\mathrm{Int}A$ and $\mathrm{Cl}A$ the interior and the closure of A, respectively.

Recall that a function $f\colon X \to Y$ (X and Y are topological spaces) is said to be quasicontinuous at a point x if for each neighbourhood U of x and each neighbourhood V of $f(x)$ there is an open nonempty set $G \subset U$ such that $f(G) \subset V$ [34].

H. P. Thielman introduced cliquish functions:
A function $f\colon X \to Y$ (X is a topological space and (Y,d) is a metric space) is said to be cliquish at a point $x \in X$ if for each neighbourhood U of x and each $\varepsilon > 0$ there is an open nonempty set $G \subset U$ such that $d(f(y), f(z)) < \varepsilon$ for each $y, z \in G$ [63].

Denote by $C(f)$, $D(f)$, $Q(f)$ and $K(f)$ the set of all continuity, discontinuity, quasicontinuity and cliquishness points of f, respectively. A function f is quasicontinuous (cliquish) if $Q(f) = X$ ($K(f) = X$). Further, denote by $\mathscr{C}(X,Y)$, $\mathscr{Q}(X,Y)$ and $\mathscr{K}(X,Y)$ (or briefly $\mathscr{C}$, $\mathscr{Q}$ and $\mathscr{K}$) the family of all continuous, quasicontinuous and cliquish functions. Evidently $C(f) \subset Q(f) \subset K(f)$ and $\mathscr{C} \subset \mathscr{Q} \subset \mathscr{K}$ (if Y is a metric space). The set $D(f)$ for cliquish functions is of first category and if X is a Baire space then f is cliquish if and only if the set $C(f)$ is dense in X. The triplet $(C(f), Q(f), K(f))$ is characterized in [5], [17], [16], [18].

The notion of strongly quasicontinuous function was used by Z. Grande for s.q.c. functions [26]. In this paper, we will use the notion of strong quasicontinuity for any classes functions between continuous and quasicontinuous functions with the set of discontinuity points of measure zero and all such classes of functions will be called stronly quasicontinuous.

2.3 Quasicontinuous functions with sets of discontinuity points of measure zero

In this section we will assume that functions are defined in $\mathbb{R}$ with values in $\mathbb{R}$. Quasicontinuous functions need not be measurable [49]. The set $Q(f)$ need not be measurable, however, if $f\colon \mathbb{R} \to \mathbb{R}$ is measurable, then the set $Q(f)$ is measurable [35]. The set $Q(f) \setminus C(f)$ is of the first category [53], however it need not be measurable nor of measure zero. Even there is a Darboux function such that the measure of $Q(f) \setminus C(f)$ is positive [43]. If f is a quasicontinuous function then the set $D(f)$ is measurable as an F_σ-set, however it need not be of measure zero. Of course, if $D(f)$ is of measure zero then the function f is measurable. In this section we will deal with quasicontinuous functions with sets of discontinuity points of measure zero.

Let ℓ_e (ℓ) denote the outer Lebesgue measure (Lebesgue measure) in $\mathbb{R}$. Denote by

$$d_u(A,x) = \limsup_{h\to 0^+} \frac{\ell_e(A\cap(x-h,x+h))}{2h}$$

the upper outer density of $A \subset \mathbb{R}$ at a point $x \in \mathbb{R}$. Similarly,

$$d_l(A,x) = \liminf_{h\to 0^+} \frac{\ell_e(A\cap(x-h,x+h))}{2h}$$

is the lower outer density of $A \subset \mathbb{R}$ at a point $x \in \mathbb{R}$.

A point $x \in \mathbb{R}$ is called a density point of $A \subset \mathbb{R}$ if there exists a measurable (in the sense of Lebesgue) set $B \subset A$ such that $d_l(B,x) = 1$. The family $\mathscr{T}_d$ of all measurable sets A such that every point $x \in A$ is a density point of A is a topology called the density topology. Denote by $\mathscr{T}_e$ the Euclidean topology on

$\mathbb{R}$. A function f is approximately continuous (at x) if it is continuous (at x) as a function $f\colon (\mathbb{R},\mathscr{T}_d) \to (\mathbb{R},\mathscr{T}_e)$. Denote by $\mathscr{A}$ the family of all approximately continuous functions. Approximately continuous functions need not be quasicontinuous, and quasicontinuous functions need not be approximately continuous.

In [55] O'Malley introduced the topology $\mathscr{T}_{ae}$ as the set of all $A \in \mathscr{T}_d$ for which $\ell(A \setminus \operatorname{Int} A) = 0$ and proved that $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathscr{T}_{ae}$-continuous (i.e. continuous as a mapping from $(\mathbb{R},\mathscr{T}_{ae})$ to $(\mathbb{R},\mathscr{T}_e)$) if and only if it is everywhere approximately continuous and almost everywhere continuous. It is easy to see that every $\mathscr{T}_{ae}$-continuous function is quasicontinuous. Denote by $\mathscr{C}_{ae}$ the family of all $\mathscr{T}_{ae}$-continuous functions.

Z. Grande gave the following definitions

Definition 2.1. [26] A function $f\colon \mathbb{R} \to \mathbb{R}$ is s.q.c. at x if for every $\varepsilon > 0$ and for every $U \in \mathscr{T}_d$ there is a nonempty open set V such that $V \cap U \neq \emptyset$ and $|f(y) - f(x)| < \varepsilon$ for all $y \in V \cap U$.

Definition 2.2. [26] A function $f\colon \mathbb{R} \to \mathbb{R}$ has property $A(x)$ at $x \in \mathbb{R}$ if there exists an open set U such that $d_u(U,x) > 0$ and the restricted function $f\restriction(U \cup \{x\})$ is continuous at x. We will write $f \in A(x)$ if f has the property $A(x)$ at a point x.

Definition 2.3. [26] A function $f\colon \mathbb{R} \to \mathbb{R}$ has property $B(x)$ at $x \in \mathbb{R}$ (abbreviated $f \in B(x)$) if for $\varepsilon > 0$ we have $d_u(\operatorname{Int}\{y\colon |f(y) - f(x)| < \varepsilon\},x) > 0$.

Denote by $Q_s(f)$ the set of all x at which f is s.q.c., by $A(f)$ the set $\{x \in \mathbb{R}\colon f \in A(x)\}$ and by $B(f)$ the set $\{x \in \mathbb{R}\colon f \in B(x)\}$. Obviously,

$$C(f) \subset A(f) \subset B(f) \subset Q_s(f) \subset Q(f).$$

All inclusions can be proper. However, if $Q_s(f) = \mathbb{R}$ then $B(f) = \mathbb{R}$.

The following theorem shows that s.q.c. functions are quasicontinuous functions with the set of discontinuity points of measure zero.

Theorem 2.2. *[26] The set $Q_s(f) \setminus C(f)$ need not have measure zero. However, if $Q_s(f) = \mathbb{R}$ then $\mathbb{R} \setminus C(f) = D(f)$ is of measure zero.*

Theorem 2.3. *[27] The set $B(f) \setminus C(f)$ is of measure zero.*

Moreover, the sets $A(f)$ and $B(f)$ have Baire property, however, they need not be borelian. Further, he gave a characterization of the set $A(f)$.

Theorem 2.4. *[27] Let $A \subset \mathbb{R}$. Then $A = A(f)$ for some $f\colon \mathbb{R} \to \mathbb{R}$ if and only if $A = \bigcup_m \bigcap_n A_{m,n}$, where $A_{m,n}$ be such that there are open sets G_n such that for each $m, n \in \mathbb{N}$ we have $d_u(\mathrm{Int} A_{m,n}, x) \geq 1/m$ for each $x \in A$, $A_{m,n+1} \subset A_{m,n}$, $A_{m,n} \subset A_{m+1,n}$, $G_{n+1} \subset G_n$, $G_n \subset A_{m,n}$ and $d_u(G_n, x) \geq 1/m$ for all $x \in A_{m,n}$.*

Also, there exist the characterizations of the pairs $(C(f), A(f))$ and $(C(f), B(f))$.

Theorem 2.5. *[3] Let A and C be subsets of $\mathbb{R}$. Then $C = C(f)$ and $A = A(f)$ for some function $f\colon \mathbb{R} \to \mathbb{R}$ if and only if there exist open sets G_n such that $C = \bigcap_n G_n \subset A$, $G_{n+1} \subset G_n$ and $\inf\{d_u(G_n, x)\colon n \in \mathbb{N}\} > 0$ for each $x \in A$.*

Theorem 2.6. *[3] Let B and C be subsets of $\mathbb{R}$. Then $C = C(f)$ and $B = B(f)$ for some function $f\colon \mathbb{R} \to \mathbb{R}$ if and only if there exist open sets G_n such that $C = \bigcap_n G_n \subset B$, $G_{n+1} \subset G_n$ and $d_u(G_n, x) > 0$ for each $x \in B$.*

Definition 2.4. [9] Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and let $r \in [0,1)$. We put
$A_r(f) = \{x \in \mathbb{R}\colon$ there is an open set U such that $d_u(U,x) > r$ and $f \restriction (U \cup \{x\})$ is continuous at $x\}$,
$A^l_r(f) = \{x \in \mathbb{R}\colon$ there is an open set U such that $d_l(U,x) > r$ and $f \restriction (U \cup \{x\})$ is continuous at $x\}$,
$B_r(f) = \{x \in \mathbb{R}\colon$ for each $\varepsilon > 0$ there is an open set U such that $d_u(U,x) > r$ and $f(U) \subset (f(x) - \varepsilon, f(x) + \varepsilon)\}$,
$B^l_r(f) = \{x \in \mathbb{R}\colon$ for each $\varepsilon > 0$ there is an open set U such that $d_l(U,x) > r$ and $f(U) \subset (f(x) - \varepsilon, f(x) + \varepsilon)\}$.

The set $A_0(f)$ is the set $A(f)$ from Definition 2.2 and $B_0(f)$ is $B(f)$ from Definition 2.3. We have

Theorem 2.7. *[9] Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and let $0 \leq s < r < 1$. Then*

$$\begin{array}{ccccccc}
 & & A_r(f) & \longrightarrow & B_r(f) & \longrightarrow & A_s(f) & \longrightarrow & Q(f) \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
C(f) & \longrightarrow & A^l_r(f) & \longrightarrow & B^l_r(f) & \longrightarrow & A^l_s(f) & &
\end{array}$$

and each of inclusions can be proper (here, arrows mean inclusions).

For $r \in [0,1)$ let $\mathscr{A}_r = \{f\colon \mathbb{R} \to \mathbb{R}\colon A_r(f) = \mathbb{R}\}$, $\mathscr{A}^l_r = \{f\colon \mathbb{R} \to \mathbb{R}\colon A^l_r(f) = \mathbb{R}\}$, $\mathscr{B}_r = \{f\colon \mathbb{R} \to \mathbb{R}\colon B_r(f) = \mathbb{R}\}$ and $\mathscr{B}^l_r = \{f\colon \mathbb{R} \to \mathbb{R}\colon B^l_r(f) = \mathbb{R}\}$.

Theorem 2.8. *[9] Let $0 \leq s < r < 1$. Then the following inclusions hold*

$$\begin{array}{ccccccccc} & & \mathscr{A}_r & \longrightarrow & \mathscr{B}_r & \longrightarrow & \mathscr{A}_s & \longrightarrow & \mathscr{Q} \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \mathscr{C} & \longrightarrow & \mathscr{A}_r^l & \longrightarrow & \mathscr{B}_r^l & \longrightarrow & \mathscr{A}_s^l & & \end{array}$$

and all inclusions are proper.

According to Theorem 2.3 we have

Theorem 2.9. *All sets $\mathscr{A}_r(f) \setminus C(f)$, $\mathscr{A}_r^l(f) \setminus C(f)$, $\mathscr{B}_r(f) \setminus C(f)$ and $\mathscr{B}_r^l(f) \setminus C(f)$ have measure zero and all families $\mathscr{A}_r$, $\mathscr{A}_r^l$, $\mathscr{B}_r$ and $\mathscr{B}_r^l$ have the set of discontinuity of measure zero.*

Moreover, for $s \in [0,1)$, the set $\bigcup_{1>r>s} \mathscr{B}_r$ is nowhere dense set in $\mathscr{A}_s$ and $\bigcup_{1>r>s} \mathscr{B}_r^l$ is nowhere dense set in $\mathscr{A}_s^l$. So, $(\mathscr{B}_r)_{r\in[0,1)}$ is the family of functions between continuous functions and quasicontinuous almost everywhere continuous functions such that $\mathscr{B}_r$ is nowhere dense subset of $\mathscr{B}_s$ whenever $0 \leq s < r < 1$ (in the topology of uniform convergence).

Sometimes, the density of a set at a point is defined in other way.

$$D_u(A,x) = \limsup_{h\to 0^+,\, k\to 0^+} \frac{\ell_e(A \cap (x-h, x+h))}{k+h}$$

$$D_l(A,x) = \liminf_{h\to 0^+,\, k\to 0^+} \frac{\ell_e(A \cap (x-h, x+h))}{k+h}$$

Evidently, $d_u(A,x) \leq D_u(A,x)$ and $d_l(A,x) \geq D_l(A,x)$. Moreover, $D_l(A,x) = 1$ if and only if $d_l(A,x) = 1$ and $d_u(A,x) > 0$ if and only if $D_u(A,x) > 0$. More we can find in [42].

If we use in Definition 2.4 $D_u(U,x)$ and $D_l(U,x)$ instead of $d_u(U,x)$ and $d_l(U,x)$, respectively, (i.e. let ${}_DA_r(f) = \{x \in \mathbb{R} \colon$ there is an open set U such that $D_u(U,x) > r$ and $f \restriction (U \cup \{x\})$ is continuous at $x\}$), and similarly ${}_DA_r^l(f)$, ${}_DB_r(f)$, ${}_DB_r^l(f)$, ${}_D\mathscr{A}_r$, ${}_D\mathscr{A}_r^l$, ${}_D\mathscr{B}_r$ and ${}_D\mathscr{A}_r^l$, the corresponding Theorems 2.7, 2.8 as well as all remarks remain true, although the classes of functions are different (we have

$$A_r(f) \subset {}_DA_r(f) \text{ and } \mathscr{A}_r \subset {}_D\mathscr{A}_r,$$

with equality only for $r = 0$).

We can use in Definition 2.4 measurable sets instead of open sets.

Definition 2.5. [37] Let $\rho \in (0,1)$. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called ρ-upper continuous at x provided there is a measurable set E such that $x \in E$, $D_u(E,x) > \rho$ and $f \restriction E$ is continuous at x. If f is ρ-upper continuous at every point we say that f is ρ-upper continuous.

Denote the class of all ρ-upper continuous functions by $\mathscr{UC}_\rho$. ρ-upper continuous functions are investigated in [37], [54], [41], [40], [42], [38], [36], [39]. Although the definition seems to be similar to Definition 2.4 and ${}_D\mathscr{A}_\rho \subset \mathscr{UC}_\rho$, the differences are important.

Functions from classes $\mathscr{A}_r$, $\mathscr{A}_r^l$, $\mathscr{B}_r$, $\mathscr{B}_r^l$, ${}_D\mathscr{A}_r$, ${}_D\mathscr{A}_r^l$, ${}_D\mathscr{B}_r$ and ${}_D\mathscr{B}_r^l$ are quasicontinuous the set of discontinuity points is of measure zero and they do not contain approximately continuous functions. Functions from classes $\mathscr{UC}_\rho$ need not be quasicontinuous the measure of the set of discontinuity points can be positive and they contains approximately continuous functions. All classes of functions are measurable.

Z. Grande in [29] has given the following definitions.

Definition 2.6. [29] A function $f\colon \mathbb{R} \to \mathbb{R}$ has property s_0 at a point x if for each positive ε and for each $U \in \mathscr{T}_d$ containing x there is a point $t \in C(f) \cap U$ such that $|f(t) - f(x)| < \varepsilon$.
A function $f\colon \mathbb{R} \to \mathbb{R}$ has property s_1 at a point x if for each positive ε and for each $U \in \mathscr{T}_d$ containing x there is an open interval I such that $\emptyset \neq I \cap U \subset C(f)$ and $|f(t) - f(x)| < \varepsilon$ for all points $t \in I \cap U$.
A function f has property s_0 (s_1) if it has it at each point.

Each function f having property s_1 has also property s_0. Functions with properties s_0 or s_1 are quasicontinuous. Each function with property s_0 at x is s.q.c. at this point. Moreover, a function f has property s_0 if and only if it is s.q.c. Functions with property s_0 have the set $D(f)$ of measure zero and functions with property s_1 have the set $D(f)$ even of measure zero and nowhere dense. The characterization of sets of discontinuity points of these functions is following.

Theorem 2.10. *[20] A set A is the set of points of discontinuity of some function $f\colon \mathbb{R} \to \mathbb{R}$ with property s_0 if and only if A is an F_σ-set of measure zero.*

Theorem 2.11. *[20] A set A is the set of points of discontinuity of some function $f\colon \mathbb{R} \to \mathbb{R}$ with property s_1 if and only if A is an F_σ-set of measure zero and for each nonempty set $U \in \mathscr{T}_d$ contained in the closure of the set A, the set $U \cap A$ is nowhere dense in U.*

E. Strońska investigated maximal families for classes of s.q.c. functions and functions with property s_1. Let X be a topological space and let $\mathscr{F}$ be a nonempty family of real functions defined on X. For $\mathscr{F}$, we define the maximal additive class $\mathscr{M}_{add}(\mathscr{F})$ as
$\mathscr{M}_{add}(\mathscr{F}) = \{f\colon X \to \mathbb{R}\colon f+g \in \mathscr{F}$ for every $g \in \mathscr{F}\}$,
the maximal multiplicative class $\mathscr{M}_{mult}(\mathscr{F})$ as
$\mathscr{M}_{mult}(\mathscr{F}) = \{f\colon X \to \mathbb{R}\colon f\cdot g \in \mathscr{F}$ for every $g \in \mathscr{F}\}$,
the maximal class with respect to maximum $\mathscr{M}_{max}(\mathscr{F})$ as
$\mathscr{M}_{max}(\mathscr{F}) = \{f\colon X \to \mathbb{R}\colon \max(f,g) \in \mathscr{F}$ for every $g \in \mathscr{F}\}$,
the maximal class with respect to minimum $\mathscr{M}_{min}(\mathscr{F})$ as
$\mathscr{M}_{min}(\mathscr{F}) = \{f\colon X \to \mathbb{R}\colon \min(f,g) \in \mathscr{F}$ for every $g \in \mathscr{F}\}$,
and the maximal latticelike class $\mathscr{M}_{latt}(\mathscr{F})$ as
$\mathscr{M}_{latt}(\mathscr{F}) = \{f\colon X \to \mathbb{R}\colon \max(f,g) \in \mathscr{F}$ and $\min(f,g) \in \mathscr{F}$ for every $g \in \mathscr{F}\}$. She proved ($\mathscr{Q}_s$ is the family of all s.q.c. functions and $\mathscr{Q}_{s_1}$ is the family of all functions with property s_1)

Theorem 2.12. *[60]* $\mathscr{M}_{add}(\mathscr{Q}_s) = \mathscr{M}_{max}(\mathscr{Q}_s) = \mathscr{M}_{min}(\mathscr{Q}_s) = \mathscr{M}_{latt}(\mathscr{Q}_s) = \mathscr{Q}_s \cap \mathscr{C}_{ae}$ *and* $\mathscr{M}_{add}(\mathscr{Q}_{s_1}) = \mathscr{M}_{max}(\mathscr{Q}_{s_1}) = \mathscr{M}_{min}(\mathscr{Q}_{s_1}) = \mathscr{M}_{latt}(\mathscr{Q}_{s_1}) = \mathscr{Q}_{s_1} \cap \mathscr{C}_{ae}$.

Let $\mathscr{M}_Q$ denote the family of all functions with this property: if f is not $\mathscr{T}_{ae}$-continuous at $x \in \mathbb{R}$ then $f(x) = 0$ and $d_u(\{t \in \mathbb{R}; f(t) = 0\}, x) > 0$.

Theorem 2.13. *[60]* $\mathscr{M}_{mult}(\mathscr{Q}_s) = \mathscr{Q}_s \cap \mathscr{M}_Q$, $\mathscr{M}_{mult}(\mathscr{Q}_{s_1}) = \mathscr{Q}_{s_1} \cap \mathscr{M}_Q$ *and* $\mathscr{M}_{mult}(\mathscr{Q}_{s_2}) = \mathscr{Q}_{s_2} \cap \mathscr{M}_Q$.

2.4 Quasicontinuous functions with sets of discontinuity points almost of measure zero

Z. Grande in [23] gave the following definition (ℓ is the Lebesgue measure in $\mathbb{R}^n$).

Definition 2.7. [23] A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is R-integrally quasicontinuous at a point x if for each positive ε and for each open set U containing x there is a bounded Jordan measurable set I with nonempty interior such that $I \subset U$, the restricted function $f \restriction I$ is integrable in the sense of Riemann and

$$\left| \frac{\int_I f(t)dt}{\ell(I)} - f(x) \right| < \varepsilon.$$

A function f is R-integrally quasicontinuous if it is such at each point.

Theorem 2.14. *[23] If a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is R-integrally quasicontinuous then there is a dense open set $U \subset \mathbb{R}^n$ such that $\ell(U \setminus C(f)) = 0$.*

Therefore the measure of $D(f)$ is zero on some dense open set. However, there is a R-integrally quasicontinuous nonmeasurable function $f\colon \mathbb{R} \to \mathbb{R}$. Evidently, for such a function the measure of $D(f)$ is positive. Obviously, R-integrally quasicontinuous functions are between continuous and quasicontinuous functions. There are quasicontinuous functions which are not R-integrally-quasicontinuous.

Theorem 2.15. *[23] If $f\colon \mathbb{R}^n \to \mathbb{R}$ is quasicontinuous and if there is a dense open set $G \subset \mathbb{R}^n$ such that $\ell(G \setminus C(f)) = 0$ then f is R-integrally quasicontinuous.*

Therefore, in the family of almost everywhere continuous functions, quasicontinuous and R-integrally quasicontinuous functions coincide.

2.5 Quasicontinuous functions with σ-porous set of discontinuity points

The notion of a σ-porous set was introduced in [15]. For a set $A \subset \mathbb{R}$ and an open interval $I \subset \mathbb{R}$ let $\Lambda(A, I)$ denote the length of the largest subinterval of I having an empty intersection with A. Let $x \in \mathbb{R}$. Then the right-porosity of the set A at x is defined as

$$p^+(A, x_0) = \limsup_{h \to 0^+} \frac{\Lambda\left(A, (x, x+h)\right)}{h},$$

the left-porosity of the set A at x is defined as

$$p^-(A, x_0) = \limsup_{h \to 0^+} \frac{\Lambda\left(A, (x-h, x)\right)}{h},$$

and the porosity of the set A at x is defined as

$$p(A, x_0) = \max\left\{p^-(A, x_0), p^+(A, x_0)\right\}.$$

The set $A \subset \mathbb{R}$ is called right-porous at a point $x \in \mathbb{R}$ if $p^+(A, x) > 0$, left-porous at a point $x \in \mathbb{R}$ if $p^-(A, x) > 0$ and porous at a point $x \in \mathbb{R}$ if $p(A, x) > 0$. The set $A \subset \mathbb{R}$ is called porous if A is porous at each point $x \in A$ and $A \subset \mathbb{R}$ is called σ-porous if A is the countable union of porous sets.

Every σ-porous set is of first category and of measure zero, but there are sets of first category and of measure zero, which are not σ-porous [64].

Definition 2.8. A point $x \in \mathbb{R}$ is called a point of π_r-density of a set $A \subset \mathbb{R}$ for $0 \le r < 1$ (μ_r-density of a set $A \subset \mathbb{R}$ for $0 < r \le 1$) if $p(\mathbb{R} \setminus A, x) > r$, ($p(\mathbb{R} \setminus A, x) \ge r$).

Definition 2.9. [12] Let $r \in [0,1)$. The function $f\colon \mathbb{R} \to \mathbb{R}$ is called
$\mathscr{P}_r$-continuous at a point x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r-density of A and $f \restriction A$ is continuous at a point x,
$\mathscr{S}_r$-continuous at a point x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r-density of A and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

Let $r \in (0,1]$. The function $f\colon \mathbb{R} \to \mathbb{R}$ is called
$\mathscr{M}_r$-continuous at a point x, if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r-density of A and $f \restriction A$ is continuous at a point x,
$\mathscr{N}_r$-continuous at a point x, if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is point of μ_r-density of A and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

All of these functions are called porouscontinuous functions.

Symbols $\mathscr{P}_r(f)$, $\mathscr{S}_r(f)$, $\mathscr{M}_r(f)$ and $\mathscr{N}_r(f)$ will denote the sets of all points at which the function f is $\mathscr{P}_r$-continuous, $\mathscr{S}_r$-continuous, $\mathscr{M}_r$-continuous and $\mathscr{N}_r$-continuous, respectively. Collectively, these sets will be called the sets of porouscontinuity points of the function f.

Porouscontinuity was defined by the set A containing the point x. There is, however, a second option using an open set B where the continuity would be required at a point x for $f \restriction B \cup \{x\}$. In [12] it is shown that it results in the same notion. This is a difference with the measure case.

Theorem 2.16. *[12] Let $f\colon \mathbb{R} \to \mathbb{R}$. Then the set $\mathscr{S}_0(f) \setminus \mathscr{C}(f)$ is σ-porous.*

The following theorem summarizes relations between sets of continuity, porouscontinuity and quasicontinuity of a function $f\colon \mathbb{R} \to \mathbb{R}$.

Theorem 2.17. *[12] Let $0 < r < s < 1$ and $f\colon \mathbb{R} \to \mathbb{R}$. Then*

$$\mathscr{C}(f) \subset \mathscr{M}_1(f) = \mathscr{N}_1(f) \subset \mathscr{P}_s(f) \subset \mathscr{S}_s(f) \subset \mathscr{M}_s(f) = \mathscr{N}_s(f) \subset \mathscr{P}_r(f) \subset \mathscr{P}_0(f) \subset \mathscr{S}_0(f) \subset \mathscr{Q}(f).$$

All inclusions are proper.

Let there be introduced the following denotations:
$\mathscr{M}_r = \{f\colon \mathscr{M}_r(f) = \mathbb{R}\}$, $\mathscr{N}_r = \{f\colon \mathscr{N}_r(f) = \mathbb{R}\}$,
$\mathscr{P}_r = \{f\colon \mathscr{P}_r(f) = \mathbb{R}\}$, $\mathscr{S}_r = \{f\colon \mathscr{S}_r(f) = \mathbb{R}\}$.

Theorem 2.18. *Let* $0 < r < s < 1$. *Then*

$$\mathscr{C} \subset \mathscr{M}_1 = \mathscr{N}_1 \subset \mathscr{P}_s \subset \mathscr{S}_s \subset \mathscr{M}_s = \mathscr{N}_s \subset \mathscr{P}_r \subset \mathscr{P}_0 \subset \mathscr{S}_0 \subset \mathscr{Q}.$$

All inclusions are proper.

Therefore functions in the family $\mathscr{S}_0$, and so all porouscontinuous functions, have σ-porous sets of discontinuity points.

2.6 Limits

It is easy to see that the family of quasicontinuous functions is closed under uniform convergence.

Theorem 2.19. *[9] Let* $s \in [0,1)$. *Then the sets* $\mathscr{B}_r$, $\mathscr{B}_r^l$, $\bigcup_{1>r>s} \mathscr{B}_r$ *and* $\bigcup_{1>r>s} \mathscr{B}_r^l$ *are closed in the topology of the uniform convergence.*

However, the sets $\mathscr{A}_r$ and $\mathscr{A}_r^l$ are not closed.

Theorem 2.20. *[9] For each* $r \in [0,1)$ *there is a sequence* $(f_n)_n$ *of functions belonging to* $\mathscr{A}_r^l$ *such that its uniform limit does not belong to* $\mathscr{A}_r$.

Problem 2.1. Characterize uniform limits of $\mathscr{A}_r$ and $\mathscr{A}_r^l$. Is it true that each function from $\mathscr{B}_r$ ($\mathscr{B}_r^l$) can be written as the uniform limit of functions from $\mathscr{A}_r$ ($\mathscr{A}_r^l$)? (Z. Grande in [26] has shown that this is true for $\mathscr{B}_0$.)

Similarly, by [13], the families $\mathscr{S}_r$ and $\mathscr{M}_r$ are closed under uniform convergence, whereas families $\mathscr{P}_r$ not.

The family of R-integrally quasicontinuous functions is not closed under uniform convergence [23].

Let X be a topological space and (Y,d) a metric one.

We say that a sequence of functions $f_n\colon X \to Y$ discretely converges to the function $f\colon X \to Y$ ([14]) if $\forall x \in X \exists n(x) \forall n \geq n(x)\colon f_n(x) = f(x)$.

Z. Grande in [22] has characterized discrete limits of quasicontinuous almost everywhere continuous functions.

Theorem 2.21. *[22] A function* $f\colon \mathbb{R} \to \mathbb{R}$ *is the discrete limit of a sequence of quasicontinuous almost everywhere continuous functions if and only if the set* $\mathbb{R} \setminus Q(f)$ *is nowhere dense and there is an* F_σ*-set* A *of measure zero such that the restriction* $f \restriction (\mathbb{R} \setminus A)$ *is the discrete limit of a sequence of continuous functions (on* $\mathbb{R} \setminus A$*).*

Recall that a sequence of functions $f_n\colon X \to Y$ quasiuniformly converges to $f\colon X \to Y$ if the sequence $(f_n)_n$ pointwise converges to f and
$\forall \varepsilon > 0 \forall m \in \mathbb{N} \exists p \in \mathbb{N} \forall x \in X\colon \min\{d(f_{m+1}(x), f(x)), \ldots, d(f_{m+p}(x), f(x))\} < \varepsilon.$

The quasiuniform limit of continuous functions is continuous but the quasiuniform limit of quasicontinuous functions need not be quasicontinuous. However, the quasiuniform limit of quasicontinuous functions is cliquish. In [8] it is shown that every cliquish function $f\colon \mathbb{R} \to \mathbb{R}$ can be expressed as the quasiuniform limit of a sequence of quasicontinuous functions. The result was strengthened, by showing it holds for functions defined on more general spaces. Ch. Richter has shown [56] that this is true for functions defined on pseudometrizable spaces and by Z. Grande [21], we can assume moreover that functions are quasicontinuous and Darboux.

The uniform limit of s.q.c. functions $f_n\colon \mathbb{R} \to \mathbb{R}$ is s.q.c. Since s.q.c. functions have the sets of discontinuity points of measure zero (Theorem 2.2), the quasiuniform limit of sequence of s.q.c. functions has the set of discontinuity points of measure zero.

Theorem 2.22. *[29] A function $f\colon \mathbb{R} \to \mathbb{R}$ is almost everywhere continuous if and only if there is a sequence of Darboux s.q.c. functions quasiuniformly convergent to f.*

Similar result we can find for functions with property s_1.

Theorem 2.23. *[58] A function $f\colon \mathbb{R} \to \mathbb{R}$ is almost everywhere continuous if and only if there are functions $f_n\colon \mathbb{R} \to \mathbb{R}$ with property s_1 quasiuniformly converging to f*

Since the set of discontinuity of porouscontinuous functions is σ-porous, the quasiuniform limit of a sequence of some porouscontinuous functions has the set of discontinuity points σ-porous and previous theorem is not true for porouscontinuous functions.

Problem 2.2. Is every function $f\colon \mathbb{R} \to \mathbb{R}$ with σ-porous set of points of discontinuity the quasiunform limit of a sequence (some) porouscontinuous functions?

2.7 Quasicontinuous almost everywhere continuous functions

Evidently, the biggest class of quasicontinuous functions with the set of discontinuity points of measure zero is the family of almost everywhere continuous quasicontinuous functions.

It is easy to see that the uniform limit of quasicontinuous almost everywhere continuous functions is quasicontinuous almost everywhere continuous. From Theorem 2.23 we obtain that each almost everywhere continuous function is the quasiuniform limit of a sequence of quasicontinuous almost everywhere continuous functions.

Almost everywhere continuous function $f\colon \mathbb{R} \to \mathbb{R}$ has dense set of continuity, so it is cliquish. According to [10] (also [24], [47]), each cliquish function $f\colon \mathbb{R} \to \mathbb{R}$ is the sum of two quasicontinuous functions f_1 and f_2 such that $D(f_1) \cap D(f_2) \subset D(f)$. So, immediately we have the characterization of the sums of quasicontinuous almost everywhere continuous functions.

Theorem 2.24. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is almost everywhere continuous if and only if it is the sum of two quasicontinuous functions both with the set of discontinuity points of measure zero.*

However, it need not be the sum of two functions from the family $\mathscr{A}_r$. Similarly, each function with σ-porous set of discontinuity is the sum of two quasicontinuous functions with σ-porous set of discontinuity points.

2.8 Other classes of functions between continuous and quasicontinuous functions

Of course, each family of quasicontinuous functions with some extra property lies between continuous and quasicontinuous functions. For example, Darboux and quasicontinuous functions (see survey paper [51]), strong Świątkowski functions (e.g. [47], [61]), extra strong Świątkowski functions [62], which are both Darboux and quasicontinuous however their set of dicontinuity points can be of positive measure.

Quasicontinuous functions with closed graph [11], [57] or internally quasicontinuous functions (a function f is internally quasicontinuous [48] if is quasicontinuous and its set of points of discontinuity is nowhere dense) are such that the set of discontinuity is nowhere dense, but it can be of positive measure. However, it is a subject for another paper.

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