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# Chapter 5
# On equivalence of topological and restrictional continuity

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## 5.1 Introduction

Let $\mathbb{R}$ denote the set of reals and $\mathbb{N}$ the set of positive integers. By $\tau_0$ we shall denote the natural topology on $\mathbb{R}$. Let $\mathscr{B}(\tau)$, $\mathbb{K}(\tau)$, $\mathscr{B}a(\tau)$ denote the family of all Borel sets, meager sets and sets having the Baire property in a topological space $(\mathbb{R},\tau)$, respectively. A $\tau$-open set $A \subset \mathbb{R}$ is $\tau$-regular if $A = int_\tau cl_\tau A$ , where $int_\tau$ and $cl_\tau$ mean the interior and closure with respect to the topology $\tau$. If $\tau = \tau_0$ then we shall use the notation $\mathscr{B}$, $\mathbb{K}$ and $\mathscr{B}a$, respectively. The symmetric difference of sets $A$, $B$ is denoted by $A \triangle B$.

Let $\Phi\colon \tau_0 \to 2^{\mathbb{R}}$ be an operator satisfying the following conditions:

$(i)$ $\quad \Phi(\emptyset) = \emptyset, \quad \Phi(\mathbb{R}) = \mathbb{R},$
$(ii)$ $\quad \underset{A\in\tau_0}{\forall}\ \underset{B\in\tau_0}{\forall}\ \Phi(A\cap B) = \Phi(A)\cap\Phi(B),$
$(iii)$ $\quad \underset{A\in\tau_0}{\forall}\ A \subset \Phi(A).$

Let $\Phi$ stand for the family for all operators satisfying conditions $(i)-(iii)$.

**Remark 5.1.** If $\Phi \in \Phi$ then $\Phi(A) \subset cl_{\tau_0}A$ for every $A \in \tau_0$.

It is well known that every set $A \in \mathscr{B}a$ has the unique representation

$$A = G(A) \triangle B$$

where $G(A)$ is a regular open set and $B \in \mathbb{K}$ (cf. [4]). In particular, if $V \in \tau_0$ then $V = W \setminus P$ where $W$ is regular open and $P$ is a nowhere dense closed set (see [5]).

Let $\Phi \in \Phi$ and $\Phi_r \colon \mathscr{B}a \to 2^{\mathbb{R}}$ be defined by formula

$$\underset{A \in \mathscr{B}a}{\forall} \Phi_r(A) = \Phi(G(A)).$$

The following theorems are a special case of similar theorems in [1] concerning arbitrary topological Baire spaces.

**Theorem 5.1.** *For every $\Phi \in \Phi$, the operator $\Phi_r$ is a lower density operator on $(\mathbb{R}, \mathscr{B}a, \mathbb{K})$. This means that the following conditions are satisfied:*

1° $\Phi_r(\emptyset) = \emptyset$, $\Phi_r(\mathbb{R}) = \mathbb{R}$,
2° $\underset{A \in \mathscr{B}a}{\forall} \; \underset{B \in \mathscr{B}a}{\forall} \Phi_r(A \cap B) = \Phi_r(A) \cap \Phi_r(B)$,
3° $\underset{A \in \mathscr{B}a}{\forall} \; \underset{B \in \mathscr{B}a}{\forall} A \triangle B \in \mathbb{K} \Rightarrow \Phi_r(A) = \Phi_r(B)$,
4° $\underset{A \in \mathscr{B}a}{\forall} A \triangle \Phi_r(A) \in \mathbb{K}$.

**Theorem 5.2.** *For every operator $\Phi \in \Phi$, the family $\mathscr{T}_{\Phi_r} = \{A \in \mathscr{B}a \colon A \subset \Phi_r(A)\}$ is a topology on $\mathbb{R}$ strictly stronger than $\tau_0$.*

*Proof.* Since the pair $(\mathscr{B}a, \mathbb{K})$ has the hull property, what means that every family of pairwise disjont sets having the Baire property but not meager is at most countable, and $\Phi_r$ is a lower density operator on $(\mathbb{R}, \mathscr{B}a, \mathbb{K})$, we infer that the family $\mathscr{T}_{\Phi_r} = \{A \in \mathscr{B}a; A \subset \Phi_r(A)\}$ is a topology on $\mathbb{R}$, called an abstract density topology on $(\mathbb{R}, \mathscr{B}a, \mathbb{K})$ (see [4], p. 208 and p. 213). If $V \in \tau_0$ then by Remark 5.1, $V = W \setminus P$ where $W$ is a regular open set and $P \in \mathbb{K}$. Hence $G(A) = W$ and $\Phi_r(V) = \Phi(W) \supset W \supset V$. Therefore $V \in \mathscr{T}_{\Phi_r}$. Evidently, the set of irrational numbers is a member of $\mathscr{T}_{\Phi_r} \setminus \tau_0$, so the proof is complete. □

The next theorem lists properties of the topological space $(\mathbb{R}, \mathscr{T}_{\Phi_r})$. For the proofs and some related comments see Theorem 4 in [1].

**Theorem 5.3.** *Let $\Phi \in \Phi$. Then the topological space $(\mathbb{R}, \mathscr{T}_{\Phi_r})$ has the following properties:*

*a)* $A \in \mathbb{K}$ *iff* $A$ *is* $\mathscr{T}_{\Phi_r}$*-nowhere dense and closed,*
*b)* $\mathbb{K}(\mathscr{T}_{\Phi_r}) = \mathbb{K}$,

*c*) $\mathscr{B}a(\mathscr{T}_{\Phi_r}) = \mathscr{B}(\mathscr{T}_{\Phi_r}) = \mathscr{B}a$,
*d*) $(\mathbb{R}, \mathscr{T}_{\Phi_r})$ *is the Baire space,*
*e*) $A \subset X$ *is compact iff* $A$ *is finite,*
*f*) $(\mathbb{R}, \mathscr{T}_{\Phi_r})$ *is neither separable, nor first countable or second countable,*
*g*) $(\mathbb{R}, \mathscr{T}_{\Phi_r})$ *is not a Lindelöf space,*
*h*) *if* $A \subset \mathbb{R}$ *then* $\mathrm{Int}_{\Phi_r}(A) = A \cap \Phi_r(B)$, *where* $B \in \mathscr{B}a$ *is a kernel of* $A$.

Some examples of operators belonging to $\Phi$ have already been considered in the literature.

*Example 5.1.* Let $\Phi = \Phi_d$, where $\Phi_d$ denotes the density operator on the family of Lebesgue measurable sets in $\mathbb{R}$. Then $\Phi \in \Phi$; the topology $\mathscr{T}_{\Phi_r} = \{A \in \mathscr{B}a \colon A \subset \Phi_r(A)\}$ was intensively investigated in [11] and some generalization of this approach is presented in [10].

*Example 5.2.* Let $\Phi = \Phi_\Psi$, where $\Phi_\Psi$ denote the $\Psi$-density operator on the family of Lebesgue measurable sets in $\mathbb{R}$ (see [11]). Then $\Phi \in \Phi$; the topology $\mathscr{T}_{\Phi_r} = \{A \in \mathscr{B}a \colon A \subset \Phi_\Psi(A)\}$ was investigated in [8].

*Example 5.3.* Let $\Phi(A) = A$ for every $A \in \tau_0$. Then $\Phi \in \Phi$ and $\mathscr{T}_{\Phi_r} = \{B \subset \mathbb{R} \colon B = C \setminus D,\ C \in \tau_0,\ D \in \mathbb{K}\}$, (see in [1] and [3]).

*Example 5.4.* Let $\Phi = \Phi_{\mathscr{I}}$, where $\Phi_{\mathscr{I}}$ denote the $\mathscr{I}$-density operator on the family $\mathscr{B}a$ in $\mathbb{R}$ (see [5]). Then $\Phi \in \Phi$ and for every set $A \in \mathscr{B}a$, $\Phi_r(A) = \Phi(G(A)) = \Phi(A)$. This implies that $\mathscr{T}_{\Phi_r} = \mathscr{T}_{\mathscr{I}}$, where $\mathscr{T}_{\mathscr{I}}$ is the $\mathscr{I}$-density topology (see [6]).

## 5.2 The main results

In the following part we shall focus on two kinds of continuity: topological and restrictional. Let $\Phi \in \Phi$.

**Definition 5.1.** A function $f \colon \mathbb{R} \to \mathbb{R}$ is $\mathscr{T}_{\Phi_r}$-topologically continuous at $x_0 \in \mathbb{R}$ if

$$\underset{\varepsilon>0}{\forall}\ \underset{A\in\mathscr{T}_{\Phi_r}}{\exists}\ (x_0 \in A \wedge A \subset \{x \colon |f(x) - f(x_0)| < \varepsilon\}).$$

Obviously, a function $f \colon X \to \mathbb{R}$ is $\mathscr{T}_{\Phi_r}$-topologically continuous at every point $x \in X$ if and only if it is continuous as a transformation from the topological space $(X, \mathscr{T}_{\Phi_r})$ to $(\mathbb{R}, \tau_0)$.

**Definition 5.2.** We shall say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathscr{T}_{\Phi_r}$-restrictionally continuous at $x_0 \in \mathbb{R}$ if there exists a set $E \in \mathscr{B}a$ such that $x_0 \in \Phi_r(E)$ and $f|_E$ is $\tau_0$-continuous at $x_0$.

*Property 5.1.* (cf. [1]) Let $\Phi \in \Phi$. If $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathscr{T}_{\Phi_r}$-restrictionally continuous at $x_0 \in \mathbb{R}$ then $f$ is $\mathscr{T}_{\Phi_r}$-topologically continuous at $x_0$.

*Proof.* Assume that $f$ is $\mathscr{T}_{\Phi_r}$-restricionally continuous at $x_0 \in \mathbb{R}$. Then there exists a set $E \in \mathscr{B}a$ such that $x_0 \in \Phi_r(E)$ and $f|_E$ is $\tau_0$- continuous at $x_0$. Thus, for every $\varepsilon > 0$ there exist $V \in \tau_0$ such that $x_0 \in V$ and $E \cap V \subset \{x \in \mathbb{R}\colon |f(x) - f(x_0)| < \varepsilon\}$. Then $x_0 \in A = E \cap \Phi_r(E) \cap V \in \mathscr{T}_{\Phi_r}$ and $A \subset \{x \in \mathbb{R}\colon |f(x) - f(x_0)| < \varepsilon\}$. This means that $f$ is $\mathscr{T}_{\Phi_r}$-topologically continuous at $x_0$. □

The converse is not true. Namely, if $\Phi = \Phi_{\mathscr{I}}$ then $\mathscr{T}_{\Phi_r} = \mathscr{T}_{\mathscr{I}}$, and it was proved in [6] that $\mathscr{T}_{\mathscr{I}}$-topological continuity and $\mathscr{T}_{\mathscr{I}}$-restrictional continuity are not equivalent.It is also worth mentioning that the topologies in papers [12] and [9] are such that topological and restrictional continuity are not equivalent. However, if $\Phi = \Phi_d$ or $\Phi = \Phi_\Psi$, the paper [8] contains the proof of equivalence of both kinds of continuity.

By Corollary 3 in [1] we obtain the following theorem giving equivalence of topological and restrictional continuity on residual sets.

**Theorem 5.4.** *Let $\Phi \in \Phi$ and $f\colon \mathbb{R} \to \mathbb{R}$. If $C_1(f)$ and $C_2(f)$ are the sets of $\mathscr{T}_{\Phi_r}$-topological continuity and $\mathscr{T}_{\Phi_r}$- restrictional continuity respectively, then $C_1(f)$ is residual if and only if $C_2(f)$ is residual with respect to topology $\tau_0$.*

Now, we characterize the equivalence of topological and restrictional continuity in terms of the $\mathscr{T}_{\Phi_r}$-topology for every $\Phi \in \Phi$.

**Theorem 5.5.** *Let $f\colon \mathbb{R} \to \mathbb{R}$, $\Phi \in \Phi$ and $x_0 \in \mathbb{R}$. The following conditions are equivalent:*

$(a)$ *$f$ is $\mathscr{T}_{\Phi_r}$-topologically continuous at $x_0$ if and only if $f$ is $\mathscr{T}_{\Phi_r}$-restrictionally continuous at $x_0$;*

$(b)$ *for every decreasing sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathscr{B}a$ such that $x_0 \in \bigcap_{n=1}^{\infty} \Phi_r(E_n)$ there exists a sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $r_n \searrow 0$ such that*
$$x_0 \in \Phi_r\left(\bigcup_{n=1}^{\infty} E_n \cap (\mathbb{R} \setminus (x_0 - r_n, x_0 + r_n))\right);$$

$(c)$ *for every decreasing sequence $\{E_n\}_{n\in\mathbb{N}} \subset \tau_0$ such that $x_0 \in \bigcap_{n=1}^{\infty} \Phi_r(E_n)$ there exists a sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $r_n \searrow 0$ such that*
$$x_0 \in \Phi_r\left(\bigcup_{n=1}^{\infty}(E_n \cap (\mathbb{R} \setminus (x_0 - r_n, x_0 + r_n)))\right);$$

$(d)$ *for every decreasing sequence* $\{E_n\}_{n\in\mathbb{N}}$ *of* $\tau_0$*-regular open sets such that* $x_0 \in \bigcap_{n=1}^{\infty} \Phi_r(E_n)$ *there exists a sequence* $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ *with* $r_n \searrow 0$ *such that* $x_0 \in \Phi_r(\bigcup_{n=1}^{\infty}(E_n \cap (\mathbb{R} \setminus (x_0 - r_n, x_0 + r_n))))$.

*Proof.* By Theorem 4 in [2] (see also Theorem 3.1 in [7]) conditions (a) and (b) are equivalent. Obviously, $(b) \Rightarrow (c) \Rightarrow (d)$. We shall prove $(d) \Rightarrow (b)$.

Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathscr{B}a$ be a decreasing sequence such that $x_0 \in \bigcap_{n=1}^{\infty} \Phi_r(E_n)$. Then $\{G(E_n)\}_{n\in\mathbb{N}}$ is a decreasing sequence of regular open sets such that $\Phi_r(E_n) = \Phi_r(G(E_n))$ for all $n \in \mathbb{N}$, and $x_0 \in \bigcap_{n=1}^{\infty} \Phi_r(G(E_n))$. Then there exists a sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $r_n \searrow 0$ such that $x_0 \in \Phi_r(\bigcup_{n=1}^{\infty}(G(E_n) \cap (\mathbb{R} \setminus (x_0 - r_n, x_0 + r_n)))) = \Phi_r(\bigcup_{n=1}^{\infty}(E_n \cap (\mathbb{R} \setminus (x_0 - r_n, x_0 + r_n))))$. □

*Property 5.2.* If $\Phi(A) = A$ for every $A \in \tau_0$, then $\Phi \in \Phi$ and for every function $f\colon \mathbb{R} \to \mathbb{R}$ , $\mathscr{T}_{\Phi_r}$-topological continuity and $\mathscr{T}_{\Phi_r}$-restrictional continuity are equivalent.

*Proof.* Evidently $\Phi \in \Phi$. It is sufficient to prove condition (a) of Theorem 5. Let $\{E_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of $\tau_0$-regular open sets such that $x_0 \in \bigcap_{n=1}^{\infty} \Phi_r(E_n)$ for every $n \in \mathbb{N}$. Since $\Phi_r(E_n) = \Phi(G(E_n)) = \Phi(E_n) = E_n$ for every $n \in \mathbb{N}$, we have that $x_0 \in \bigcap_{n=1}^{\infty} E_n$. Let $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be a sequence with $c_n \searrow 0$ and $(x_0 - c_n, x_o + c_n) \subset E_n$ for every $n \in \mathbb{N}$. Putting $r_n = c_{n+1}$ for every $n \in \mathbb{N}$ we have that $(x_0 - c_1, x_0 + c_1) \setminus \{x_0\} \subset \bigcup_{n=1}^{\infty}(E_n \cap (\mathbb{R} \setminus (x_0 - r_n, x_0 + r_n)))$. Hence $x_0 \in G(\bigcup_{n=1}^{\infty}(E_n \cap (\mathbb{R} \setminus (x_0 - r_n, x_0 + r_n)))) = \Phi_r(\bigcup_{n=1}^{\infty}(E_n \cap (\mathbb{R} \setminus (x_0 - r_n, x_0 + r_n))))$. □

**Theorem 5.6.** *Let* $f\colon \mathbb{R} \to \mathbb{R}$, $\Phi \in \Phi$ *and* $x_0 \in \mathbb{R}$. *If for every decreasing sequence* $\{E_n\}_{n\in\mathbb{N}}$ *of* $\tau_0$*-regular open sets such that* $x_0 \in \bigcap_{n=1}^{\infty} \Phi(E_n)$ *there exists a sequence* $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$*with* $r_n \searrow 0$ *such that* $x_0 \in \Phi(\bigcup_{n=1}^{\infty}(E_n \cap (\mathbb{R} \setminus [x_0 - r_n, x_0 + r_n])))$ *then* $\mathscr{T}_{\Phi_r}$*-topological continuity and* $\mathscr{T}_{\Phi_r}$*-restrictional continuity of the function* $f$ *at* $x_0$ *are equivalent.*

*Proof.* It is sufficient to prove condition (b) of Theorem 5. Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathscr{B}a$ be a decreasing sequence such that $x_0 \in \bigcap_{n=1}^{\infty} \Phi_r(E_n)$. Then $\{G(E_n)\}_{n\in\mathbb{N}}$ is a decreasing sequence of regular open sets such that $x_0 \in \bigcap_{n=1}^{\infty} \Phi(G(E_n))$. Hence there exists a sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $r_n \searrow 0$ such that

$$x_0 \in \Phi(\bigcup_{n=1}^{\infty}(G(E_n) \cap (\mathbb{R} \setminus [x_0 - r_n, x_0 + r_n]))).$$

For every $n \in \mathbb{N}$ we get

$$\begin{aligned} G(E_n \cap (\mathbb{R} \setminus [x_0 - r_n, x_0 + r_n])) &= G(E_n) \cap (\mathbb{R} \setminus [x_0 - r_n, x_0 + r_n]) \\ &\subset G(\bigcup_{n=1}^{\infty}(E_n \cap (\mathbb{R} \setminus [x_0 - r_n, x_0 + r_n]))). \end{aligned}$$

Hence

$$\Phi\left(\bigcup_{n=1}^{\infty}\left(G\left(E_n\right) \cap (\mathbb{R} \setminus [x_0 - r_n, x_0 + r_n])\right)\right) \subset$$

$$\Phi\,(G\,(\bigcup_{n=1}^{\infty}(E_n\cap(\mathbb{R}\setminus[x_0-r_n,x_0+r_n]))))$$

and

$$\begin{aligned}x_0\in\Phi(G(\bigcup_{n=1}^{\infty}(E_n\cap(\mathbb{R}\setminus[x_0-r_n,x_0+r_n]))))\\=\Phi_r(\bigcup_{n=1}^{\infty}(E_n\cap(\mathbb{R}\setminus[x_0-r_n,x_0+r_n])))\\=\Phi_r(\bigcup_{n=1}^{\infty}(E_n\cap(\mathbb{R}\setminus(x_0-r_n,x_0+r_n)))).\end{aligned}$$

□

The converse of Theorem 5.6 is not true. Let $\Phi(A)=A$ for every $A\in\tau_0$ and let $x_0\in\mathbb{R}$. Putting $E_n=(x_0-\varepsilon_n,x_0+\varepsilon_n)$, where $\{\varepsilon_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_+$ is a sequence tending to 0, we have $x_0\in\bigcap_{n=1}^{\infty}\Phi(E_n)$. At the same time for every sequence $\{r_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_+$ with $r_n\searrow 0$ we get that

$$x_0\notin\Phi(\bigcup_{n=1}^{\infty}((E_n\cap(\mathbb{R}\setminus[x_0-r_n,x_0+r_n])))).$$

On the other hand, by Property 2, $\mathscr{T}_{\Phi_r}$-restrictional continuity and $\mathscr{T}_{\Phi_r}$-topological continuity are equivalent. The following theorem establishes the equivalence in Theorem 5.6 under additional assumption.

**Theorem 5.7.** *Let $\Phi\in\Phi$ be an operator such that $\Phi(A)=\Phi(B)$ for every $A,B\in\tau_0$ whenever $A\bigtriangleup B$ is countable. Then for an arbitrary function $f\colon\mathbb{R}\to\mathbb{R}$ and $x_0\in\mathbb{R}$, $\mathscr{T}_{\Phi_r}$-topological continuity and $\mathscr{T}_{\Phi_r}$-restrictional continuity of $f$ at $x_0$ are equivalent if and only if for every decreasing sequence $\{E_n\}_{n\in\mathbb{N}}$ of $\tau_0$-regular open sets such that $x_0\in\bigcap_{n=1}^{\infty}\Phi(E_n)$ there exists a sequence $\{r_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_+$ with $r_n\searrow 0$ such that $x_0\in\Phi(\bigcup_{n=1}^{\infty}(E_n\cap(\mathbb{R}\setminus[x_0-r_n,x_0+r_n]))$.*

*Proof.* Sufficiency is a consequence of the previous theorem.

Necessity. Let us suppose that there exists a decreasing sequence $\{E_n\}_{n\in\mathbb{N}}$ of regular open sets such that $x_0\in\bigcap_{n=1}^{\infty}\Phi(E_n)$ and for every sequence $\{r_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_+$ with $r_n\searrow 0$, we have

$$x_0\notin\Phi(\bigcup_{n=1}^{\infty}(E_n\cap(\mathbb{R}\setminus[x_0-r_n,x_0+r_n]))).$$

Let

$$f(x)=\begin{cases}2 & \text{for } x\notin E_1 \text{ and } x\neq x_0,\\ 1/n & \text{for } x\in E_n\setminus E_{n+1} \text{ and } x\neq x_0,\\ 0 & \text{for } x\in\bigcap_{n=1}^{\infty}E_n \text{ or } x=x_0.\end{cases}$$

Then

$$\forall_{n\in\mathbb{N}}\ E_n\subset\{x\in\mathbb{R}\colon |f(x)-f(x_0)|\le 1/n\}$$

and $x_0\in\Phi(E_n)=\Phi_r(E_n)$. Thus $f$ is $\mathscr{T}_{\Phi_r}$-topologically continuous at $x_0$. Let us suppose that $f$ is $\mathscr{T}_{\Phi_r}$-restrictionally continuous at $x_0$. Then there exists a set $E\in\mathscr{B}a$ such that $x_0\in\Phi_r(E)$ and $f|_E$ is $\tau_0$-continuous at $x_0$. Hence for every $n\in\mathbb{N}$ there exists $r_n>0$ such that

$$E\cap(x_0-r_n,x_0+r_n)\subset\{x\in\mathbb{R}\colon |f(x)-f(x_0)|\le 1/n\}.$$

We can assume that $r_n\searrow 0$. Then for every $n\in\mathbb{N}$,

$$E \cap (\mathbb{R} \setminus [x_0 - r_{n+1}, x_0 + r_{n+1}]) \cap (x_0 - r_n, x_0 + r_n)$$
$$\subset E_{n+1} \cap (\mathbb{R} \setminus [x_0 - r_{n+1}, x_0 + r_{n+1}]).$$

Hence

$$G(E) \cap (\mathbb{R} \setminus [x_0 - r_{n+1}, x_0 + r_{n+1}]) \cap (x_0 - r_n, x_0 + r_n)$$
$$\subset G(E_{n+1}) \cap ((\mathbb{R} \setminus [x_0 - r_{n+1}, x_0 + r_{n+1}])).$$

This implies that

$$G(E) \cap \bigcup_{n=1}^{\infty} ((\mathbb{R} \setminus [x_0 - r_{n+1}, x_0 + r_{n+1}]) \cap (x_0 - r_n, x_0 + r_n))$$
$$\subset \bigcup_{n=1}^{\infty} (E_{n+1} \cap (\mathbb{R} \setminus [x_0 - r_n, x_0 + r_n]))$$
$$\subset \bigcup_{n=1}^{\infty} (E_n \cap (\mathbb{R} \setminus [x_0 - r_n, x_0 + r_n])).$$

Then

$$\Phi(G(E)) \cap \Phi(\bigcup_{n=1}^{\infty} ((\mathbb{R} \setminus [x_0 - r_{n+1}, x_0 + r_{n+1}]) \cap (x_0 - r_n, x_0 + r_n)))$$
$$\subset \Phi(\bigcup_{n=1}^{\infty} (E_n \cap (\mathbb{R} \setminus [x_0 - r_n, x_0 + r_n]))).$$

Since

$$\Phi\left(\bigcup_{n=1}^{\infty} ((\mathbb{R} \setminus [x_0 - r_{n+1}, x_0 + r_{n+1}]) \cap (x_0 - r_n, x_0 + r_n))\right) =$$
$$\Phi\left((x_0 - r_1, x_0 + r_1) \setminus (\bigcup_{n=1}^{\infty} \{r_n\} \cup \{x_0\})\right) = \Phi(x_0 - r_1, x_0 + r_1)$$
$$\supset (x_0 - r_1, x_0 + r_1)$$

and $x_0 \in \Phi_r(E) = \Phi(G(E))$. The contradiction that

$$x_0 \in \Phi\left(\bigcup_{n=1}^{\infty} (E_n \cap (\mathbb{R} \setminus [x_0 - r_n, x_0 + r_n]))\right)$$

ends the proof. □

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