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THE METHOD OF SOCRATIC PROOFS MEETS CORRESPONDENCE ANALYSIS

Abstract

The goal of this paper is to propose correspondence analysis as a technique for generating the so-called erotetic (*i.e.* pertaining to the logic of questions) calculi which constitute the method of Socratic proofs by Andrzej Wiśniewski. As we explain in the paper, in order to successfully design an erotetic calculus one needs invertible sequent-calculus-style rules. For this reason, the proposed correspondence analysis resulting in invertible rules can constitute a new foundation for the method of Socratic proofs.

Correspondence analysis is Kooi and Tamminga's technique for designing proof systems. In this paper it is used to consider sequent calculi with non-branching (the only exception being the rule of cut), invertible rules for the negation fragment of classical propositional logic and its extensions by binary Boolean functions.

Keywords: Socratic proofs, correspondence analysis, invertible rule, inferential erotetic logic, classical propositional logic, sequent calculus.

*The authors kindly devote this paper
to Andrzej Wiśniewski.*

1. Introduction

1.1. The method of Socratic proofs

The method of Socratic proofs is a proof method grounded in the logic of questions called *inferential erotetic logic* (**IEL**, for short).¹ Developed mainly in the nineties by Andrzej Wiśniewski², the logic focuses its attention on the analysis of inferential relations between questions, distinguishes some classes of inferences with questions involved (henceforth called *erotetic inferences*), and, finally, develops criteria of validity of such erotetic inferences.

Undertaking the task to formally model erotetic inferences has led **IEL** to distinguishing some proof methods, especially the method of Socratic proofs.³ The core of the method is the idea of answering questions by questioning, that is, by transforming the structure of the initially posed question. When the questions concern, for example, validity in a logic **L**, then the method of Socratic proofs constitutes a proof method for **L**. However, the general goal is more ambitious: it is to capture and provide a formal model for a kind of cognitive phenomenon, when an agent tends to solve a problem by consecutive questions.

The fact that we do perform such reasoning is incontestable. The erotetic calculi designed so far may be claimed to successfully model erotetic reasoning such as:

- *Is A a tautology? Well, is $\neg A$ satisfiable?*

But they also shed the light of explanation on such more general examples as:

- From [11, p. 47]: *Let me rephrase my question; what I am really asking is ...*

whereas the very notion of erotetic implication may be adjusted to provide an account of the following:

¹The word *erotetic* comes from the Greek *ερωτημα* which means *question*.

²Tadeusz Kubiński, one of the pioneers in the logic of questions, has already focused on some relations between questions [17], [16]. For **IEL**, see [38] or [35] for a concise introduction. See also [36] for the most recent account of **IEL**.

³Introduced in [37] for the case of classical propositional logic it has been later adjusted to the first-order case (see [40]) as well as various non-classical cases (*int.al.* modal [19, 21], and paraconsistent [41, 6]). The most recent developments of the method are discussed in the monographs [5, 20].

Q_1 : Is $\sqrt{2}^{\sqrt{2}}$ a rational number?

A_1 : (Gelfond-Schneider Theorem) If x and y are algebraic numbers, $x \neq 0$, $x \neq 1$, and y is irrational, then x^y is a transcendental number.

A_2 : Every (real) transcendental number is irrational.

Q_2 : Is $\sqrt{2}$ an algebraic number?

There are two conditions defining the notion of erotetic implication. First, if question Q_1 erotetically implies question Q_2 (on the basis of X), then **soundness of the first question, Q_1 , warrants soundness of the second question, Q_2** . This means that if in a given situation it is reasonable to ask Q_1 , then it is also reasonable to ask Q_2 . (More precisely, soundness of a question under a valuation amounts to the existence of an answer to the question which is true under the valuation. We postpone the technical details to Section 3.) For example, *if* a question:

- *Is Sabrina in the bedroom or in the living room?*

is sound in a given situation, *and if* one can hear Sabrina's voice, then the following question is also sound:

- *Does her voice come from the bedroom or from the living room?*

The second condition defining the notion of erotetic implication amounts to the fact that Q_2 is *asked for a purpose*: **every answer to Q_2 must bring one closer to answering Q_1** . It is the case in our example with Sabrina: every answer to the second question (providing the information that Sabrina's voice comes from the bedroom or from the living room) entails the answer to the first question.

The requirement "**every answer**" is a very strong one, and for this reason it is often weakened: when weak erotetic implication is considered, it is enough that at least **one answer** to the second question is useful in resolving the first one. For example, the affirmative answer to the above question Q_2 : "Yes, $\sqrt{2}$ is an algebraic number",⁴ makes theorem A_1 applicable, and so, together with A_2 , yields the negative answer to Q_1 : the number is not rational. However, in this case the requirement **every answer** is not satisfied, as the negative answer to Q_2 does not entail any solution to the problem expressed by Q_1 . Yet, with the weaker variants of erotetic implication **IEL** can easily deal with that.

⁴It is the true answer, as the polynomial ' $x^2 - 2$ ' witnesses.

What properties should a formal model possess in order to capture at least some of the cognitive phenomena described above? Well, the fundamental properties of the model designed in the framework of **IEL**, that is, of the method of Socratic proofs, are the following:

1. Syntactic, quasi-reductionist approach to questions (see [39] or [36, Chapter 2] for this and other approaches). It means, *int.al.*, that questions are distinguished as separate expressions in the language, expressions differing with respect to statements; and that a question is *identified* (understood) by knowing what counts as an answer. (*Cf.* the so-called Hamblin's postulates, [10].)
2. The rules transforming questions, *i.e.* erotetic rules, are built on a proof-theoretical skeleton of sequent calculus. As we shall see, questions transform certain units composed of sequents.
3. The crucial property: the construction of erotetic rules warrants that they retain the relation of erotetic implication between the question-premise and the question-conclusion.

For the last property to hold, the rules must be *semantically invertible*, that is, semantic correctness of the conclusion of a rule must warrant semantic correctness of its premise. This property is used in proving soundness of the method. However, regardless of their invertibility, the order of the application of erotetic rules is settled.

Each rule of an erotetic calculus transforms a question, but it focuses on a single *constituent* of a question, which is a sequent. When viewed as a rule acting on a sequent, an erotetic rule is a sequent-calculus rule *inverted*, so the derivation process as defined by the rules reflects the backward proof-search in sequent calculi—from the final conclusion in the root to the leaves.

From a purely proof-theoretical point of view, erotetic rules need not be sound in the sense of preserving semantic correctness top-down. Let us observe that the situation is similar in the case of sequent calculi, where, in general, the rules need not be semantically invertible in order to obtain the adequateness result. However, from the erotetic point of view, both soundness (top-down) and invertibility (bottom-up) of erotetic rules are necessary to obtain *erotetic correctness* of the rules. Hence comes the idea to examine the potential of correspondence analysis in the version introduced in the paper [22].

Both directions of applications of the rules open up more opportunities to search for proofs; however, it is probably more important that the two directions give more possibilities in modelling erotetic reasoning. Moreover, invertibility of the rules is essentially used in the completeness proof of the calculi presented in [22].

1.2. The notion of correspondence analysis

Correspondence analysis is Kooi and Tamminga's [15] proof-theoretic approach which, originally, was developed in order to axiomatize via natural deduction systems all the truth-functional unary and binary extensions of three-valued logic **LP** (**L**ogic of **P**aradox) [1, 31]. Later, Tamminga [33], using correspondence analysis, presented natural deduction systems for all the unary and binary extensions of Kleene's strong three-valued logic **K₃** [14, 13].

Further, Petrukhin [23] formulated via correspondence analysis natural deduction systems for all the unary and binary extensions of Belnap-Dunn's four-valued logic **FDE** (**F**irst **D**egree **E**ntailment) [2, 3, 7] supplied with Boolean negation. Petrukhin and Shangin have recently applied correspondence analysis and a proof-searching procedure for **FDE** itself [29]. Petrukhin and Shangin [26] developed a proof-searching algorithm for natural deduction systems for all the binary extensions of **LP**. In [27], the authors extended their proof searching technique to the case of all the binary extensions of **K₃**. Petrukhin [24] presented via correspondence analysis natural deduction systems for all the unary and binary extensions of Kubyshkina and Zaitsev's [18] four-valued logic **LRA** (**L**ogic of **R**ational **A**gent). Besides, he generalized Kooi and Tamminga's ([15], [33]) results for a wider class of three-valued logics [25]. Petrukhin and Shangin [30] used correspondence analysis to syntactically characterize Tomova's natural logics [34, 12]. Petrukhin and Shangin [28] presented correspondence analysis for **PWK** (**P**araconsistent **W**eak **K**leene logic) [9, 4] which is Kleene's weak logic **K₃^w** [14, 13] with two designated values.

Finally, in [22], the authors showed how to use the framework to obtain sequent calculi with the following properties: all the rules are semantically invertible (understood as before, see also explanations below) *and actually inverted*, that is, used in both directions; the rules for connectives (the *logical* rules) are linear, the only branching rule is the rule of the cut, and the rule is not eliminable.

2. Sequent calculi obtained *via* correspondence analysis

We start with a summary of [22]. Some details, that may be found there, are skipped.

Notation. We use \mathcal{P} for a countably infinite set $\{p, q, \dots\}$ of propositional variables and $\mathcal{B} = \{\circ_{\perp}, \wedge, \not\rightarrow, \circ_1, \not\leftarrow, \circ_2, \Downarrow, \vee, \downarrow, \equiv, \circ_{-2}, \leftarrow, \circ_{-1}, \rightarrow, \uparrow, \circ_{\top}\}$ for a set of binary operators, where:

A	B	\circ_{\perp}	\wedge	$\not\rightarrow$	\circ_1	$\not\leftarrow$	\circ_2	\Downarrow	\vee
1	1	0	1	0	1	0	1	0	1
1	0	0	0	1	1	0	0	1	1
0	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0

A	B	\downarrow	\equiv	\circ_{-2}	\leftarrow	\circ_{-1}	\rightarrow	\uparrow	\circ_{\top}
1	1	0	1	0	1	0	1	0	1
1	0	0	0	1	1	0	0	1	1
0	1	0	0	0	0	1	1	1	1
0	0	1	1	1	1	1	1	1	1

Let $\mathcal{L}_{\neg}^{\circ}$ be propositional language with the alphabet $\langle \mathcal{P}, \mathcal{B}, \neg, (,) \rangle$; the set $\mathcal{F}_{\neg}^{\circ}$ of all $\mathcal{L}_{\neg}^{\circ}$'s formulas is defined as usually.

Sequents are introduced as expressions of language $\mathcal{L}_{\neg\Rightarrow}^{\circ}$ which is built upon $\mathcal{L}_{\neg}^{\circ}$ by adding ' \Rightarrow ' (the sequent arrow) and the comma ',' to the alphabet. The only category of a well-formed expression of $\mathcal{L}_{\neg\Rightarrow}^{\circ}$ is that of a *sequent* of $\mathcal{L}_{\neg\Rightarrow}^{\circ}$, which is an expression of the form:

$$\Gamma \Rightarrow \Delta \quad (2.1)$$

where Γ and Δ are finite, possibly empty multisets of formulas of $\mathcal{L}_{\neg}^{\circ}$. We use comma in the antecedent and in the succedent both as a separator of the elements of a multiset and as the sum of multisets (the context makes it clear).

The sequent calculi introduced in [22] are built upon the rules (Ax), ($\Rightarrow \neg$), ($\neg \Rightarrow$), for the negation fragment of **CPL**, together with the structural rule (cut) which is the only branching rule of the system.

$$\begin{array}{c}
(\text{Ax}) \quad A, \Gamma \Rightarrow \Delta, A \quad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ (cut)} \\
\\
\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} (\Rightarrow \neg) \quad \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} (\neg \Rightarrow)
\end{array}$$

The following rules, presented in [22], have been found by correspondence analysis:

$$\begin{array}{ll}
R_{\circ}^{(01)} \frac{A, \Gamma \Rightarrow \Delta, B}{A \circ B, \Gamma \Rightarrow \Delta, B} & R_{\circ}^{(02)} \frac{B, \Gamma \Rightarrow \Delta, \neg(A \circ B)}{A \circ B, \Gamma \Rightarrow \Delta, A} \\
R_{\circ}^{(03)} \frac{B, \Gamma \Rightarrow \Delta, A}{A \circ B, \Gamma \Rightarrow \Delta, A} & R_{\circ}^{(04)} \frac{A, \Gamma \Rightarrow \Delta, B}{\neg B, \Gamma \Rightarrow \Delta, A \circ B} \\
R_{\circ}^{(05)} \frac{B, \Gamma \Rightarrow \Delta, A \circ B}{\neg A, \Gamma \Rightarrow \Delta, A \circ B} & R_{\circ}^{(06)} \frac{A \circ B, \Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta, B} \\
R_{\circ}^{(07)} \frac{B, \Gamma \Rightarrow \Delta, A \circ B}{B, \Gamma \Rightarrow \Delta, A} & R_{\circ}^{(08)} \frac{A \circ B, \Gamma \Rightarrow \Delta, B}{A \circ B, \Gamma \Rightarrow \Delta, A} \\
R_{\circ}^{(09)} \frac{B, \Gamma \Rightarrow \Delta, \neg(A \circ B)}{B, \Gamma \Rightarrow \Delta, A} & R_{\circ}^{(10)} \frac{\neg B, \Gamma \Rightarrow \Delta, A \circ B}{\neg A, \Gamma \Rightarrow \Delta, A \circ B} \\
R_{\circ}^{(11)} \frac{A, \Gamma \Rightarrow \Delta, A \circ B}{A, \Gamma \Rightarrow \Delta, B} & R_{\circ}^{(12)} \frac{A, \Gamma \Rightarrow \Delta, \neg(A \circ B)}{A, \Gamma \Rightarrow \Delta, B} \\
A_{\circ\uparrow}^{(I)} A, B, \Gamma \Rightarrow \Delta, A \circ B & A_{\circ\downarrow}^{(I)} A \circ B, \Delta \Rightarrow \Gamma, A, B \\
A_{\circ\uparrow}^{(II)} \neg A, \neg B, \Gamma \Rightarrow \Delta, A \circ B & A_{\circ\downarrow}^{(II)} A \circ B, \Delta \Rightarrow \Gamma, \neg A, \neg B \\
A_{\circ\uparrow}^{(III)} A, \neg B, \Gamma \Rightarrow \Delta, A \circ B & A_{\circ\downarrow}^{(III)} A \circ B, \Delta \Rightarrow \Gamma, A, \neg B \\
A_{\circ\uparrow}^{(IV)} \neg A, B, \Gamma \Rightarrow \Delta, A \circ B & A_{\circ\downarrow}^{(IV)} A \circ B, \Delta \Rightarrow \Gamma, \neg A, B \\
R_{\circ}^{(I)} \frac{A, B, \Gamma \Rightarrow \Delta, A \circ B}{A \circ B, \Gamma \Rightarrow \Delta, A, B} & R_{\circ}^{(II)} \frac{\neg A, \neg B, \Gamma \Rightarrow \Delta, A \circ B}{A \circ B, \Gamma \Rightarrow \Delta, \neg A, \neg B} \\
R_{\circ}^{(III)} \frac{A, \neg B, \Gamma \Rightarrow \Delta, A \circ B}{A \circ B, \Gamma \Rightarrow \Delta, A, \neg B} & R_{\circ}^{(IV)} \frac{\neg A, B, \Gamma \Rightarrow \Delta, A \circ B}{A \circ B, \Gamma \Rightarrow \Delta, \neg A, B}
\end{array}$$

The tables 1 and 2 (below) summarize the construction of our sequent calculi for various fragments of **CPL** expressed in language $\langle \mathcal{P}, \{\circ\}, \neg, (\cdot) \rangle$.

To increase readability, under a connective \circ we indicate in the table the numbers of the \circ -specific rules instead of their names. In the case of the primary connectives⁵, we define two types of sequent calculi: one can add an axiom or a rule with the respective Roman numeral.

Table 1. Rules for non-primary connectives

$A \circ_{\perp} B$	$A \circ_1 B$	$A \circ_2 B$	$A \vee B$
(02), (08)	(01), (07)	(03), (11)	(01), (09) (02), (10) (03), (12)
$A \equiv B$	$A \circ_{\neg_2} B$	$A \circ_{\neg_1} B$	$A \circ_{\top} B$
(04), (07) (05), (08) (06), (11)	(06), (12)	(04), (09)	(05), (10)

Table 2. Rules for primary connectives

(I) and (08)	$A \circ B = A \wedge B$
(I) and (10)	$A \circ B = A \vee B$
(II) and (08)	$A \circ B = A \downarrow B$
(II) and (10)	$A \circ B = A \uparrow B$
(III) and (02)	$A \circ B = A \not\rightarrow B$
(III) and (05)	$A \circ B = A \leftarrow B$
(IV) and (02)	$A \circ B = A \not\leftarrow B$
(IV) and (05)	$A \circ B = A \rightarrow B$

Semantics for $\mathcal{L}_{\rightarrow}^{\circ}$. As in [22], we will use the symbol ‘ \models ’ for entailment in both languages: $\mathcal{L}_{\rightarrow}^{\circ}$ and $\mathcal{L}_{\neg}^{\circ}$. If v is a valuation, then we say that sequent (2.1) is *true under v* iff if every element of Γ is true under v , then some element of Δ is true under v as well. For example, every sequent of the form (Ax): $A, \Gamma \Rightarrow \Delta, A$ is true under every valuation.

⁵A binary connective \circ is called *primary*, if $\{\neg, \circ\}$ is functionally complete (see [8, p. 13]). In [22] we show the difference between the primary and the non-primary connectives via correspondence analysis.

Finally, by:

$$\Gamma \Rightarrow \Delta \models \Theta \Rightarrow \Lambda$$

we mean that for every valuation v , if sequent $\Gamma \Rightarrow \Delta$ is true under v , then sequent $\Theta \Rightarrow \Lambda$ is true under v .

The notions of a derivation and a proof are defined in a standard manner. By $\vdash_{\mathcal{C}} \Gamma \Rightarrow \Delta$ we mean that sequent $\Gamma \Rightarrow \Delta$ has a proof in sequent calculus \mathcal{C} . In [22] we proved that:

THEOREM 1 (Soundness and Completeness of \mathcal{C}). For each formula $A \in \mathcal{L}_{\rightarrow}^{\circ}$, $\models A$ iff $\vdash_{\mathcal{C}} \Rightarrow A$.

3. Erotetic calculi

3.1. Language

Erotetic calculi are worded in languages containing questions; the declarative expressions are, first of all, sequents.

We enrich language $\mathcal{L}_{\rightarrow}^{\circ}$ with the question forming operator: ‘?’, the semicolon ‘;’, the signs for negation: \underline{ng} and conjunction: $\&$ (in order to build complex declarative formulas from sequents). The resulting language will be called \mathcal{QL} . Atomic declarative formulas of \mathcal{QL} are, simply, sequents of $\mathcal{L}_{\rightarrow}^{\circ}$. The remaining declarative formulas are built from the atomic ones by the use of \underline{ng} and/or $\&$ in a usual way. *Questions of \mathcal{QL}* are expressions of the form:

$$?(\Gamma_1 \Rightarrow \Delta_1; \dots; \Gamma_n \Rightarrow \Delta_n) \quad (3.1)$$

where $\Gamma_i \Rightarrow \Delta_i$ is a sequent, also called a *constituent of question* (3.1).

Erotetic calculus is a set of erotetic rules, that is, rules transforming a question into a question. In the original account, each erotetic step is supposed to simplify the logical structure of the analysed problem by elimination of a logical constant (or better – due to the use of the unified notation⁶ – by *decomposition* of complex α -, β - formulas into their components). The use of correspondence analysis changes this picture since the simplification is sometimes lost.

⁶As far, erotetic calculi have been usually formed with the use of the unified notation: see [36], [5], [20]. However, in [20] the author considers also erotetic calculi where this convention is dropped.

3.2. Rules

One of the characteristic proof-theoretic features of the method of Socratic proofs is that its rules are designed with the aim to capture erotetic implication. For this reason one needs invertibility on the level of declaratives of \mathcal{QL} (that is, sequents). Hence follows the choice of calculi \mathcal{C} introduced in [22] and recalled above as the basis of erotetic calculi $\mathcal{E}_{\mathcal{C}}$.

Greek letters Φ and Ψ are used for finite, possibly empty sequences of sequents. For simplicity, the semicolon is used both as a separator between sequents and as a concatenation symbol between sequences of sequents.

$\mathcal{E}_{\mathcal{C}}$ is any set of rules containing erotetic version of cut, the rules for negation:

$$ER_{cut} \frac{?(\Phi ; \Gamma \Rightarrow \Delta ; \Psi)}{?(\Phi ; \Gamma \Rightarrow \Delta, A ; A, \Gamma \Rightarrow \Delta ; \Psi)}$$

$$ER_{\neg\Rightarrow} \frac{?(\Phi ; \neg A, \Gamma \Rightarrow \Delta ; \Psi)}{?(\Phi ; \Gamma \Rightarrow \Delta, A ; \Psi)} \quad ER_{\Rightarrow\neg} \frac{?(\Phi ; \Gamma \Rightarrow \Delta, \neg A ; \Psi)}{?(\Phi ; A, \Gamma \Rightarrow \Delta ; \Psi)}$$

and a combination of the \circ -specific rules. To save space, these may be given by the following general scheme: if $R = \phi/\psi$ is a \circ -specific rule of \mathcal{C} , that is, R is one of $R_{\circ}^{(01)}$ - $R_{\circ}^{(12)}$ or $R_{\circ}^{(1)}$ - $R_{\circ}^{(IV)}$, then the following:

$$ER \frac{?(\Phi ; \psi ; \Psi)}{?(\Phi ; \phi ; \Psi)}$$

is a rule of $\mathcal{E}_{\mathcal{C}}$. For example, if $R_{\circ}^{(01)}$ belongs to \mathcal{C} , then $ER_{\circ}^{(01)}$ belongs to $\mathcal{E}_{\mathcal{C}}$, where:

$$ER_{\circ}^{(01)} \frac{?(\Phi ; A \circ B, \Gamma \Rightarrow \Delta, B ; \Psi)}{?(\Phi ; A, \Gamma \Rightarrow \Delta, B ; \Psi)}$$

As above, the double line indicates that the rules are not only semantically invertible, but may be applied *in both directions*.

As we can see, calculi \mathcal{C} form the proof-theoretical skeleton of $\mathcal{E}_{\mathcal{C}}$.

DEFINITION 1 (Socratic proof in $\mathcal{E}_{\mathcal{C}}$ v.1). Let $\Gamma \Rightarrow \Delta$ be a sequent of language \mathcal{QL} , and assume that \mathcal{C} does not contain any of axioms: $A_{\circ}^{(I)}$, $A_{\circ}^{(II)}$, $A_{\circ}^{(III)}$, $A_{\circ}^{(IV)}$. A *Socratic proof of $\Gamma \Rightarrow \Delta$ in $\mathcal{E}_{\mathcal{C}}$* is a finite sequence of questions $\langle Q_1, \dots, Q_n \rangle$ such that:

1. $Q_1 = ?(\Gamma \Rightarrow \Delta)$,
2. for each $i, n \geq i > 1$: Q_i results from Q_{i-1} by a rule of $\mathcal{E}_{\mathcal{C}}$, and
3. each constituent of Q_n is of the form (Ax): $A, \Gamma \Rightarrow \Delta, A$.

If there exists a Socratic proof of a sequent in $\mathcal{E}_{\mathcal{C}}$, then we say that the sequent is *provable in $\mathcal{E}_{\mathcal{C}}$* .

Further, in the case of $\mathcal{E}_{\mathcal{C}}$, where \mathcal{C} contains axioms:

DEFINITION 2 (Socratic proof in $\mathcal{E}_{\mathcal{C}}$ v.2). Let $\Gamma \Rightarrow \Delta$ be a sequent of language \mathcal{QL} , where \mathcal{C} contains at least one of axioms: $A_{\circ}^{(I)}$, $A_{\circ}^{(II)}$, $A_{\circ}^{(III)}$, $A_{\circ}^{(IV)}$. A *Socratic proof of $\Gamma \Rightarrow \Delta$ in $\mathcal{E}_{\mathcal{C}}$* is a finite sequence of questions $\langle Q_1, \dots, Q_n \rangle$ such that:

1. $Q_1 = ?(\Gamma \Rightarrow \Delta)$,
2. for each $i, n \geq i > 1$: Q_i results from Q_{i-1} by a rule of $\mathcal{E}_{\mathcal{C}}$, and
3. each constituent of Q_n is of the form (Ax): $A, \Gamma \Rightarrow \Delta, A$, or of the form A_{\circ}^x , where A_{\circ}^x belongs to \mathcal{C} .

If there exists a Socratic proof of a sequent in $\mathcal{E}_{\mathcal{C}}$, then we say that the sequent is *provable in $\mathcal{E}_{\mathcal{C}}$* .

Here is an example of a Socratic proof in $\mathcal{E}_{\mathcal{C}}$. To save some space, $A \equiv B$ stands for $(p \equiv q) \equiv (q \equiv p)$; as soon as a sequent of the form (Ax*) is arrived at, it is represented as Ax_i .

3.3. MiES

This section shows the importance of invertibility of rules in the erotetic context.

Let us start with:

DEFINITION 3. An erotetic rule $?(Φ)/?(Ψ)$ is:

- *sound* iff, for each valuation v , the truth of each constituent of $Φ$ under v warrants the truth of each constituent of $Ψ$ under v ,
- *invertible* iff, for each valuation v , the truth of each constituent of $Ψ$ under v warrants the truth of each constituent of $Φ$ under v .

COROLLARY 1. If a rule $Γ_1 ⇒ Δ_1/Γ_2 ⇒ Δ_2$ of \mathcal{C} is sound and invertible, then an erotetic rule of the form:

$$\frac{?(Φ ; Γ_2 ⇒ Δ_2 ; Ψ)}{?(Φ ; Γ_1 ⇒ Δ_1 ; Ψ)}$$

is sound and invertible.

It is easy to see that ER_{cut} is sound and invertible, hence:

COROLLARY 2. Each rule of $\mathcal{E}_{\mathcal{C}}$ is sound and invertible.

From the fact that axioms (Ax) are true under every valuation, and from the fact that the rules of $\mathcal{E}_{\mathcal{C}}$ are invertible, it follows that:

THEOREM 2 (soundness of $\mathcal{E}_{\mathcal{C}}$). Let $Γ ⇒ Δ$ be a sequent of \mathcal{QL} . If $Γ ⇒ Δ$ has a Socratic proof in $\mathcal{E}_{\mathcal{C}}$, then $Γ ⇒ Δ$ is true under every valuation.

Similarly, completeness of $\mathcal{E}_{\mathcal{C}}$ follows from completeness of \mathcal{C} .

THEOREM 3 (completeness of $\mathcal{E}_{\mathcal{C}}$). If a sequent of \mathcal{QL} is true under every valuation, then it has a Socratic proof in $\mathcal{E}_{\mathcal{C}}$.

Erotetic implication

As we explained in the first section, the construction of erotetic calculi should warrant that the relation of erotetic implication, a central notion for inferential erotetic logic, holds between a question-premise and a question-conclusion. Now we define the notion.

Suppose that we deal with a language rich enough to distinguish between declaratives and questions. Let Q, Q^* stand for questions and dQ, dQ^* for the respective sets of direct answers to these questions. We adjust the definition from [36, p. 67]:

DEFINITION 4 (erotetic implication). A question Q implies a question Q^* on the basis of a set of declaratives X (in symbols: $\text{Im}(Q, X, Q^*)$) iff:

1. for each $A \in dQ$, for each valuation v , if each formula in $X \cup \{A\}$ is true under v , then *some* formula in dQ^* is true under v ,
and
2. for each $B \in dQ^*$, there exists a non-empty proper subset Y of dQ such that, for each valuation v , if each formula in $X \cup \{B\}$ is true under v , then *some* formula in Y is true under v .

Definition 4 is based upon the semantic notion of Boolean valuation; in the case of \mathcal{QL} we need something more general. The notions introduced below are central tools of the so-called *Minimal Erotetic Semantics* (MiES), a very general framework for a semantic analysis of both declaratives and questions developed by Andrzej Wiśniewski.⁷ The primary notion is that of a partition of a language, which comes from [32].

DEFINITION 5 (partition of language \mathcal{QL}). Let $D_{\mathcal{QL}}$ be the set of sequents of language \mathcal{QL} . By a *partition of language \mathcal{QL}* (or a *partition of language \mathcal{QL}*) we mean an ordered pair $P = \langle T_P, U_P \rangle$ such that $T_P \cup U_P = D_{\mathcal{QL}}$ and $T_P \cap U_P = \emptyset$.

In the case of complex languages with questions, the counterpart of the semantic notion of Boolean valuation is that of an *admissible partition*.

DEFINITION 6 (admissible partition of language \mathcal{QL}). Let $P = \langle T_P, U_P \rangle$ be a partition of language \mathcal{QL} . We say that P is *admissible for \mathcal{QL}* iff the following conditions hold:

1. $\Gamma \Rightarrow \Delta' \in T_P$ iff both $\Gamma \Rightarrow \Delta, A' \in T_P$ and $\langle A, \Gamma \Rightarrow \Delta' \rangle \in T_P$, for each formula A ;
2. $\langle \Gamma, \neg A, \Delta \Rightarrow \Theta, A, \Lambda' \rangle \in T_P$ iff $\langle \Gamma, \Delta \Rightarrow \Theta, A, \Lambda' \rangle \in T_P$;
3. $\langle \Gamma, \Delta \Rightarrow \Theta, \neg A, \Lambda' \rangle \in T_P$ iff $\langle \Gamma, A, \Delta \Rightarrow \Theta, \Lambda' \rangle \in T_P$;
4. $\langle \Gamma, A, \Delta \Rightarrow \Theta, B, \Lambda' \rangle \in T_P$ iff $\langle \Gamma, \neg B, \Delta \Rightarrow \Theta, A \equiv B, \Lambda' \rangle \in T_P$;
5. $\langle \Gamma, B, \Delta \Rightarrow \Theta, A \equiv B, \Lambda' \rangle \in T_P$ iff $\langle \Gamma, B, \Delta \Rightarrow \Theta, A, \Lambda' \rangle \in T_P$.

To save some space, in the above definition we have specified only the machinery for ' \circ ' = ' \equiv ' corresponding to rules $R_{\circ}^{(04)}$ and $R_{\circ}^{(07)}$. It is analogous in the remaining cases. Let us also observe that the above definition does not take into account the axioms that may be present in \mathcal{L} .

⁷For the details see [36] or [39].

The reason for their absence is that on the level of language \mathcal{QL} we are interested in entailment only, not in validity.

DEFINITION 7 (entailment in \mathcal{QL}). Suppose that \mathfrak{X} is a set of sequents of language \mathcal{QL} and \mathfrak{t} is a single sequent. We say that *set \mathfrak{X} entails formula \mathfrak{t} in language \mathcal{QL}* , symbolically:

$$\mathfrak{X} \vDash_{\mathcal{QL}} \mathfrak{t}$$

iff there is no admissible partition P for language \mathcal{QL} such that $\mathfrak{X} \subseteq \mathsf{T}_{\mathsf{P}}$ and $\mathfrak{t} \notin \mathsf{T}_{\mathsf{P}}$.

As before, dQ stands for the set of direct answers to Q . In the case of questions of \mathcal{QL} of the form (3.1) (see page 107), the set is composed of two declarative formulas of the language: the affirmative answer (3.2) and the negative answer (3.3).

$$(\Gamma_1 \Rightarrow \Delta_1) \ \& \ (\dots \& ((\Gamma_{n-1} \Rightarrow \Delta_{n-1}) \ \& \ (\Gamma_n \Rightarrow \Delta_n) \dots)) \quad (3.2)$$

$$\underline{ng}((\Gamma_1 \Rightarrow \Delta_1) \ \& \ (\dots \& ((\Gamma_{n-1} \Rightarrow \Delta_{n-1}) \ \& \ (\Gamma_n \Rightarrow \Delta_n) \dots)) \quad (3.3)$$

DEFINITION 8 (erotetic implication in \mathcal{QL}). Suppose that Q and Q^* are questions of \mathcal{QL} and that \mathfrak{X} is a set of sequents. We say that *question Q implies question Q^* on the basis of set \mathfrak{X} of sequents* iff, for each admissible partition P of language \mathcal{QL} , the following holds:

1. for each $\mathfrak{t} \in dQ$: if $\mathfrak{X} \cup \{\mathfrak{t}\} \subseteq \mathsf{T}_{\mathsf{P}}$, then $dQ^* \cap \mathsf{T}_{\mathsf{P}} \neq \emptyset$; and
2. for each $\mathfrak{u} \in dQ^*$: there is a non-empty proper subset \mathfrak{X}^* of dQ such that if $\mathfrak{X} \cup \{\mathfrak{u}\} \subseteq \mathsf{T}_{\mathsf{P}}$, then $\mathfrak{X}^* \cap \mathsf{T}_{\mathsf{P}} \neq \emptyset$.

The above construction leads to the following:

COROLLARY 3. Suppose that a sequence of questions $\langle Q_1, \dots, Q_n \rangle$ is a Socratic proof of a certain sequent in $\mathcal{E}_{\mathcal{Q}}$. Then question Q_1 implies question Q_n on the basis of the empty set of sequents.

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