The jump of Milnor number for linear deformations of plane curve singularities

Aleksandra Zakrzewska

Summary of the PhD thesis

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an **isolated singularity**, i.e. there exists a representative $\widehat{f}_0 : U \to \mathbb{C}$ of f_0, holomorphic in an open neighbourhood U of the point $0 \in \mathbb{C}^n$ such that:

1. $\widehat{f}_0(0) = 0$,
2. $\nabla \widehat{f}_0(0) = 0$,
3. $\nabla \widehat{f}_0(z) \neq 0$ for $z \in U \setminus \{0\}$,

where for a holomorphic function f we put $\nabla f := \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$. In the sequel a singularity means an isolated singularity.

A **deformation of the singularity** f_0 is the germ of a holomorphic function $f = f(s, z) : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that

1. $f(0, z) = f_0(z)$,
2. $f(s, 0) = 0$.

The deformation $f(s, z)$ of the singularity f_0 will also be treated as a family (f_s) of functions germs, taking $f_s(z) := f(s, z)$. Since f_0 is an isolated singularity, f_s for sufficiently small s also has isolated singularities near 0 ([5] Theorem 2.6 I). By the above for sufficiently small s one can define μ_s

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}_n / (\nabla f_s),$$

called the **Milnor number of** f_s, where $\mathcal{O}_n$ is the ring of holomorphic function germs at 0, and (∇f_s) is the ideal in $\mathcal{O}_n$ generated by $\frac{\partial f_0}{\partial z_1}, \ldots, \frac{\partial f_0}{\partial z_n}$.

Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities ([5], Theorem 2.6 I), there exists an open neighbourhood S ([5], Theorem 2.6 I and Proposition 2.57 II) of the point 0 such that

1. $\mu_s =$ const. for $s \in S \setminus \{0\}$,
2. $\mu_0 \geq \mu_s$ for $s \in S$.

The constant difference $\mu_0 - \mu_s$ (for $s \neq 0$) will be called **the jump of the deformation** (f_s) and denoted by $\lambda((f_s))$. The smallest non-zero value among all the jumps of deformations of the singularity f_0 will be called **the jump of the Milnor number of the singularity** f_0 and denoted by $\lambda(f_0)$.

From now on we will consider only plane curve singularities $f_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$.

The first general result concerning the jump of the Milnor number was obtained by Sabir Gusein-Zade([6]), who proved that there exist singularities f_0 for which $\lambda(f_0) > 1$. Later the same problem was consider by:

- A. Bodin ([2]) who gave a formula for $\lambda(f_0)$ for f_0 convenient with its Newton polygon reduced to one segment for non-degenerate deformations,
- J. Walewska ([7]) who generalized Bodin's results to the non-convenient case for non-degenerate deformations,
- S. Brzostowski, T. Krasiński and J. Walewska ([3]) who calculated all possible Milnor numbers of all non-degenerate deformations of homogeneous singularities,
- in the same paper they proved that for the singularity $f_0^n(x,y) = x^n + y^n$ $(n \geq 2)$ we have $\lambda(f_0) = \left[\frac{n}{2}\right]$.

In this paper we consider the jumps of the Milnor number of singularities for all **linear deformations** of f_0 i.e. deformations of the form $f_s = f_0 + sg$, where g is a holomorphic function in a neighbourhood of 0 such that $g(0) = 0$. The smallest non-zero value among all the jumps of linear deformations of the singularity f_0 will be denoted by $\lambda^{lin}(f_0)$.

The main result is a formula for $\lambda^{lin}(f_0)$ in the class of homogeneous and semi-homogeneous singularities.

Theorem 1 *For every homogeneous singularity f_0 of order $n \geq 2$ the jump of Milnor number for linear deformations is given by formula*

$$\lambda^{lin}(f_0) = \begin{cases} n-2, & n \geq 3 \\ 1, & n = 2 \end{cases}.$$

From this we immediately obtain.

Theorem 2 *For every semi-homogeneous singularity f_0 of order $n \geq 2$ the jump of Milnor number for linear deformations is given by formula*

$$\lambda^{lin}(f_0) = \begin{cases} n-2, & n \geq 3 \\ 1, & n = 2 \end{cases}.$$

To get main result the **Enriques diagrams** will be used. For any singularity we can associate its weighed Enriques diagram (D, ν) which represents the whole resolution process of this singularity ([4] Chapter 3.9). It is a finite graph with two types of edges and a weight function $\nu : D \to \mathbb{Z}$ on vertices of the diagram. M. Alberich-Carramiñana and J.Roé ([1] Theorem 1.3) gave a necessary and sufficient condition for two Enriques diagrams of singularities to be linear adjacent, which means for singularities that one singularity is a linear deformation of another. Using their result we get the formula for $\lambda^{lin}(f_0)$.

Chapter 1 is an Introduction. In Chapter 2 the Enriques diagrams of singularities are briefly presented. In Chapter 3 abstract Enriques diagrams are introduced and the result of M. Alberich-Carramiñana and J.Roé [1] is recalled. Some properties of abstract Enriques diagrams, needed in the main theorem, are also proved. From them we obtain an interesting result that $\lambda^{lin}(f_0)$ is a topological invariant for any singularity f_0. In Chapter 4 the main theorem i.e. the formula for $\lambda^{lin}(f_0)$ for homogeneous and semi-homogeneous singularities is proved.

Bibliography

[1] M. Alberich-Carramiñana, J.Roé: *Enriques Diagrams and Adjacency of Planar Curve Singularities*, Canad. J. Math., Vol 57(1):3-16, 2005.

[2] A. Bodin: *Jump of Milnor numbers*, Bull. Braz. Math. Soc. (N.S.), 38(3):389-396, 2007.

[3] S. Brzostowski, T. Krasiński, J. Walewska: *Milnor numbers in deformations of homogeneous singularities* ArXiv (1104.7704), 2014.

[4] E. Casas-Alvero: *Singularities of Plane Curve*, Cambridge University Press, 2000.

[5] G.-M. Gruel, C. Lossen, E. Shustin: *Introduction to Singularities and Deformation*, Springer Verlag, 2006.

[6] S. Gusein-Zade: *On singularities from which an A_1 can be split off.*, Funct. Anal. Appl., 27(1):57-60, 1993.

[7] J.Walewska: *The jump of Milnor numbers in families of non-degenerate and non-convenient singularities*, Analytic and Algebraic Geometry, University of Łódź Press, 141-153, 2013.