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INF-HESITANT FUZZY IDEALS IN BCK/BCI-ALGEBRAS[1]

Abstract

Based on the hesitant fuzzy set theory which is introduced by Torra in the paper [12], the notions of Inf-hesitant fuzzy subalgebras, Inf-hesitant fuzzy ideals and Inf-hesitant fuzzy p-ideals in BCK/BCI-algebras are introduced, and their relations and properties are investigated. Characterizations of an Inf-hesitant fuzzy subalgebras, an Inf-hesitant fuzzy ideals and an Inf-hesitant fuzzy p-ideal are considered. Using the notion of BCK-parts, an Inf-hesitant fuzzy ideal is constructed. Conditions for an Inf-hesitant fuzzy ideal to be an Inf-hesitant fuzzy p-ideal are discussed. Using the notion of Inf-hesitant fuzzy (p-) ideals, a characterization of a p-semisimple BCI-algebra is provided. Extension properties for an Inf-hesitant fuzzy p-ideal is established.

Keywords: p-semisimple BCI-algebra, Inf-hesitant fuzzy subalgebra, Inf-hesitant fuzzy ideal, Inf-hesitant fuzzy p-ideal.

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1. Introduction

Several generalizations and extensions of Zadeh's fuzzy sets have been introduced in the literature, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. As another generalization of fuzzy sets, Torra [12] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Xu and Xia [17] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can be obtained. They investigated the connections of the aforementioned distance measures and further develop a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [10, 14, 15, 16, 18]). In the algebraic structures, Jun et al. [6, 8] applied the hesitant fuzzy sets to BCK/BCI-algebras and MTL-algebras. They introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras, and the notions of a (Boolean, prime, ultra, good) hesitant fuzzy filter and a hesitant fuzzy MV-filter of MTL-algebras. They investigated related relations and properties, and considered characterizations of hesitant fuzzy subalgebras, hesitant fuzzy ideals, (Boolean, ultra) hesitant fuzzy filters in BCK/BCI-algebras and MTL-algebras. Recently BCK/BCI-algebras have been widely applied to soft set theory, cubic structure, bipolar and m-polar fuzzy set theory etc. (see [1], [2], [3], [4], [7], [11]).

In this paper, based on the hesitant fuzzy set theory which is introduced by Torra [12], we introduce the notions of Inf-hesitant fuzzy subalgebras, Inf-hesitant fuzzy ideals and Inf-hesitant fuzzy p-ideals in BCK/BCI-algebras. We investigate their relations and properties, and find conditions for an Inf-hesitant fuzzy ideal to be an Inf-hesitant fuzzy p-ideal. We discuss caracterizations of an Inf-hesitant fuzzy subalgebras, an Inf-hesitant fuzzy ideals and an Inf-hesitant fuzzy p-ideal. We construct an Inf-hesitant fuzzy ideal by using the notion of BCK-parts. Using the notion of Inf-hesitant fuzzy (p-) ideals, we provide a characterization of a p-semisimple BCI-algebra. Finally, we establish the extension properties for an Inf-hesitant fuzzy p-ideal.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK-algebra. A BCK-algebra X is said to be positive implicative if it satisfies:

$$(\forall x, y, z \in X)\, ((x * y) * z = (x * z) * (y * z))\,. \tag{2.1}$$

A BCK-algebra X is said to be implicative if it satisfies:

$$(\forall x, y \in X)\, (x = x * (y * x))\,. \tag{2.2}$$

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\, (x * 0 = x)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\, (x \leq y \;\Rightarrow\; x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.4}$$
$$(\forall x, y, z \in X)\, ((x * y) * z = (x * z) * y)\,, \tag{2.5}$$
$$(\forall x, y, z \in X)\, ((x * z) * (y * z) \leq x * y) \tag{2.6}$$

where $x \leq y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions:

$$(\forall x, y, z \in X)\, (0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))\,, \tag{2.7}$$
$$(\forall x, y \in X)\, (0 * (0 * (x * y)) = (0 * y) * (0 * x))\,, \tag{2.8}$$
$$(\forall x \in X)\, (0 * (0 * (0 * x)) = 0 * x)\,. \tag{2.9}$$

A BCI-algebra X is said to be p-semisimple (see [5]) if $0 * (0 * x) = x$ for all $x \in X$.

Every p-semisimple BCI-algebra X satisfies:

$$(\forall x, y, z \in X)\,((x * z) * (y * z) = x * y)\,. \tag{2.10}$$

A nonempty subset S of a BCK/BCI-algebra X is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an ideal of X if it satisfies:

$$0 \in A, \tag{2.11}$$

$$(\forall x \in X)\,(x * y \in A,\ y \in A\ \Rightarrow\ x \in A)\,. \tag{2.12}$$

A subset A of a BCI-algebra X is called a p-ideal of X (see [19]) if it satisfies (2.11) and

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \in A,\ y \in A\ \Rightarrow\ x \in A)\,. \tag{2.13}$$

Note that every p-ideal is an ideal, but the converse is not true in general (see [19]). Note that an ideal A of a BCI-algebra X is a p-ideal of X if and only if the following assertion is valid:

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \in A\ \Rightarrow\ x * y \in A)\,. \tag{2.14}$$

We refer the reader to the books [5, 9] for further information regarding BCK/BCI-algebras.

3. Inf-hesitant fuzzy subalgebras and ideals

Torra [12] introduced a new extension for fuzzy sets to manage those situations in which several values are possible for the definition of a membership function of a fuzzy set.

Definition 3.1 ([12, 13]). Let X be a reference set. A hesitant fuzzy set on X is defined in terms of a function that when applied to X returns a subset of $[0, 1]$, which can be viewed as the following mathematical representation:

$$H := \{(x, h(x)) \mid x \in X\}$$

where $h : X \to \mathscr{P}([0, 1])$.

In what follows, the power set of $[0, 1]$ is denoted by $\mathscr{P}([0, 1])$ and

$$\mathscr{P}^*([0, 1]) = \mathscr{P}([0, 1]) \setminus \{\emptyset\}.$$

For any element $D \in \mathscr{P}^*([0,1])$, the infimum of D is denoted by $\inf D$. For any hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ and $D \in \mathscr{P}^*([0,1])$, consider the set

$$\mathrm{Inf}[H; D] := \{x \in X \mid \inf h(x) \geq \inf D\}.$$

Definition 3.2. Let X be a BCK/BCI-algebra. Given an element $D \in \mathscr{P}^*([0,1])$, a hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ is called an Inf-hesitant fuzzy subalgebra of X related to D (briefly, D-Inf-hesitant fuzzy subalgebra of X) if the set $\mathrm{Inf}[H; D]$ is a subalgebra of X whenever it is non-empty. If $H := \{(x, h(x)) \mid x \in X\}$ is a D-Inf-hesitant fuzzy subalgebra of X for all $D \in \mathscr{P}^*([0,1])$ with $\mathrm{Inf}[H; D] \neq \emptyset$, then we say that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X.

Example 3.3.

(1) Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, (0.8, 1]), (a, (0.3, 0.5) \cup \{0.9\}), (b, [0.5, 0.7]), (c, (0.3, 0.5) \cup \{0.7\})\}.$$

Since $\inf h(0) = 0.8$, $\inf h(a) = 0.3 = \inf h(c)$ and $\inf h(b) = 0.5$, it is routine to verify that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X.

(2) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	c	c	0	0
d	d	c	c	a	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, \{0.8, 0.9\}), (a, [0.2, 0.9)), (b, (0.7, 0.8]), (c, \{0.5\} \cup (0.7, 0.9)), (d, [0.1, 0.5])\}.$$

Note that $\inf h(0) = 0.8$, $\inf h(a) = 0.2$, $\inf h(b) = 0.7$, $\inf h(c) = 0.5$ and $\inf h(d) = 0.1$. It is easy to check that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X.

(3) Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table.

$*$	0	1	a	b	c
0	0	0	c	c	a
1	1	0	c	c	a
a	a	a	0	0	c
b	b	a	1	0	c
c	c	c	a	a	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, [0.8, 0.9]), (1, (0.6, 0.7]), (a, [0.5, 0.6]), (b, [0.5, 0.6]), (c, [0.3, 0.7])\}.$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is a D_1-Inf-hesitant fuzzy subalgebra of X with $D_1 := [0.55, 0.65]$. But it is not a D_2-Inf-hesitant fuzzy subalgebra of X with $D_2 := [0.4, 0.6]$ since $\mathrm{Inf}[H; D_2] = \{0, 1, a, b\}$ is not a subalgebra of X.

(4) Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	a	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, [0.7, 0.8]), (a, (0.6, 0.7]), (b, [0.3, 0.6]), (c, [0.5, 0.7]), (d, [0.2, 0.4])\}.$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is a D_1-Inf-hesitant fuzzy subalgebra of X with $D_1 := [0.2, 0.4]$. If we take $D_2 := (0.4, 0.6]$, then $\mathrm{Inf}[H; D_2] = \{0, a, c\}$ which is not a subalgebra of X. Hence $H := \{(x, h(x)) \mid x \in X\}$ is not a D_2-Inf-hesitant fuzzy subalgebra of X.

THEOREM 3.4. *A hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on a BCK/BCI-algebra X is an Inf-hesitant fuzzy subalgebra of X if and only if the following assertion is valid:*

$$(\forall x, y \in X) \left(\inf h(x * y) \geq \min\{\inf h(x), \inf h(y)\}\right). \tag{3.1}$$

PROOF: Assume that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X. Assume that there exists $Q \in \mathscr{P}^*([0, 1])$ such that

$$\inf h(x * y) < \inf Q \leq \min\{\inf h(x), \inf h(y)\}.$$

Then $x, y \in \mathrm{Inf}[H; D]$ and $x * y \notin \mathrm{Inf}[H; D]$. This is a contradiction, and so

$$\inf h(x * y) \geq \min\{\inf h(x), \inf h(y)\}$$

for all $x, y \in X$.

Conversely, suppose that (3.1) is valid. Let $D \in \mathscr{P}^*([0, 1])$ and $x, y \in \mathrm{Inf}[H; D]$. Then $\inf h(x) \geq \inf D$ and $\inf h(y) \geq \inf D$. It follows from (3.1) that

$$\inf h(x * y) \geq \min\{\inf h(x), \inf h(y)\} \geq \inf D$$

and that $x * y \in \mathrm{Inf}[H; D]$. Hence the set $\mathrm{Inf}[H; D]$ is a subalgebra of X, and so $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X. □

LEMMA 3.5. *If $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of a BCK/BCI-algebra X, then*

$$(\forall x \in X) \left(\inf h(0) \geq \inf h(x)\right). \tag{3.2}$$

PROOF: Using (III) and (3.1), we have

$$\inf h(0) = \inf h(x * x) \geq \min\left\{\inf h(x), \inf h(x)\right\} = \inf h(x)$$

for all $x \in X$. □

Proposition 3.6. *Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy subalgebra of a BCK-algebra X. For any elements $a_1, a_2, \cdots, a_n \in X$, if there exists $a_k \in \{a_1, a_2, \cdots, a_n\}$ such that $a_1 = a_k$, then*

$$(\forall x \in X)\,(\inf h((\cdots((a_1 * a_2) * a_3) * \cdots) * a_n) \geq \inf h(x))\,.$$

Proof: Using (2.5), (III) and (IV), we have $(\cdots((a_1*a_2)*a_3)*\cdots)*a_n = 0$. Thus the desired result follows from Lemma 3.5. □

Definition 3.7. Let X be a BCK/BCI-algebra. Given an element $D \in \mathscr{P}^*([0,1])$, a hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ is called an Inf-hesitant fuzzy ideal of X related to D (briefly, D-Inf-hesitant fuzzy ideal of X) if the set $\mathrm{Inf}[H; D]$ is an ideal of X whenever it is non-empty. If $H := \{(x, h(x)) \mid x \in X\}$ is a D-Inf-hesitant fuzzy ideal of X for all $D \in \mathscr{P}^*([0,1])$ with $\mathrm{Inf}[H; D] \neq \emptyset$, then we say that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X.

Example 3.8.

(1) The hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ in Example 3.3(1) is an Inf-hesitant fuzzy ideal of X.

(2) Let $(Y, *, 0)$ be a BCI-algebra and $(\mathbb{Z}, +, 0)$ an additive group of integers. Let $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of $(\mathbb{Z}, +, 0)$ and let $X := Y \times \mathbb{Z}$. Then $(X, \otimes, (0,0))$ is a BCI-algebra where the operation $\otimes$ is given by

$$(\forall (x,m), (y,n) \in X)\,((x,m) \otimes (y,n) = (x * y, m - n))\,.$$

For a subset $A := Y \times \mathbb{N}_0$ of X where $\mathbb{N}_0$ is the set of nonnegative integers, let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(x, (0.5, 1]), (y, [0.4, 0.9]) \mid x \in A,\ y \in X \setminus A\}\,.$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X.

(3) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	b	a	0	0
d	d	d	d	d	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, [0.8, 1)), (a, [0.4, 0.7]), (b, \{0.3\} \cup (0.4, 0.6]), \\ (c, [0.6, 0.9]), (d, [0.1, 0.5])\}.$$

If $D_1 := [0.5, 0.8)$, then $\mathrm{Inf}[H; D_1] = \{0, c\}$ which is not an ideal of X since $b * c = 0 \in \mathrm{Inf}[H; D_1]$ but $b \notin \mathrm{Inf}[H; D_1]$. Thus $H := \{(x, h(x)) \mid x \in X\}$ is not a D_1-Inf-hesitant fuzzy ideal of X. We can easily verify that $H := \{(x, h(x)) \mid x \in X\}$ is a D_2-Inf-hesitant fuzzy ideal of X with $D_2 = [0.25, 0.5]$.

THEOREM 3.9. *A hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on a BCK/BCI-algebra X is an Inf-hesitant fuzzy ideal of X if and only if it satisfies* (3.2) *and*

$$(\forall x, y \in X)\,(\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\})\,. \tag{3.3}$$

PROOF: Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy ideal of X. If (3.2) is not valid, then there exists $D \in \mathscr{P}^*([0, 1])$ and $a \in X$ such that $\inf h(0) < \inf D \leq \inf h(a)$. It follows that $a \in \mathrm{Inf}[H; D]$ and $0 \notin \mathrm{Inf}[H; D]$. This is a contradiction, and so (3.2) is valid. Now assume that there exist $a, b \in X$ such that $\inf h(a) < \min\{\inf h(a * b), \inf h(b)\}$. Then there exists $K \in \mathscr{P}^*([0, 1])$ such that

$$\inf h(a) < \inf K \leq \min\{\inf h(a * b), \inf h(b)\},$$

which implies that $a * b \in \mathrm{Inf}[H; K]$, $b \in \mathrm{Inf}[H; K]$ but $a \notin \mathrm{Inf}[H; K]$. This is a contradiction, and thus (3.3) holds.

Conversely, suppose that $H := \{(x, h(x)) \mid x \in X\}$ satisfies two conditions (3.2) and (3.3). Let $K \in \mathscr{P}^*([0, 1])$ be such that $\mathrm{Inf}[H; K] \neq \emptyset$. Obviously, $0 \in \mathrm{Inf}[H; K]$. Let $x, y \in X$ be such that $x * y \in \mathrm{Inf}[H; K]$ and $y \in \mathrm{Inf}[H; K]$. Then $\inf h(x * y) \geq \inf K$ and $\inf h(y) \geq \inf K$. It follows from (3.3) that

$$\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\} \geq \inf K$$

and that $x \in \mathrm{Inf}[H; K]$. Hence $\mathrm{Inf}[H; K]$ is an ideal of X for all $K \in \mathscr{P}^*([0, 1])$, and therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X. □

Theorem 3.10. *Let* $H := \{(x, h(x)) \mid x \in X\}$ *be a hesitant fuzzy set on a* BCI*-algebra* X *defined by*

$$H = \{(x, D), (y, E) \mid x \in B,\ y \in X \setminus B,\ \inf D \geq \inf E\}$$

where $D, E \in \mathscr{P}^*([0,1])$ *and* B *is the* BCK*-part of* X. *Then* $H := \{(x, h(x)) \mid x \in X\}$ *is an Inf-hesitant fuzzy ideal of* X.

Proof: Since $0 \in B$, we have $\inf h(0) = \inf D \geq \inf h(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in B$, then it is clear that

$$\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\}.$$

Assume that $x \in X \setminus B$. Since B is an ideal of X, it follows that $x * y \in X \setminus B$ or $y \in X \setminus B$ and that

$$\inf h(x) = \min\{\inf h(x * y), \inf h(y)\}.$$

Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Int-hesitant fuzzy ideal of X by Theorem 3.9. □

Proposition 3.11. *Every Inf-hesitant fuzzy ideal* $H := \{(x, h(x)) \mid x \in X\}$ *of a* BCK/BCI*-algebra* X *satisfies:*

$$(\forall x, y \in X)\,(x \leq y \ \Rightarrow\ \inf h(x) \geq \inf h(y))\,. \tag{3.4}$$

Proof: Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\} = \min\{\inf h(0), \inf h(y)\} = \inf h(y) \tag{3.5}$$

by (3.3) and (3.2). □

Theorem 3.12. *Let* $H := \{(x, h(x)) \mid x \in X\}$ *be a hesitant fuzzy set on a* BCK/BCI*-algebra* X *which satisfies the condition* (3.2). *Then* $H := \{(x, h(x)) \mid x \in X\}$ *is an Inf-hesitant fuzzy ideal of* X *if and only if the following assertion is valid.*

$$(\forall x, y, z \in X)\,(x * y \leq z \ \Rightarrow\ \inf h(x) \geq \min\{\inf h(y), \inf h(z)\})\,. \tag{3.6}$$

Proof: Assume that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X and let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and thus

$$\begin{aligned}\inf h(x*y) &\geq \min\{\inf h((x*y)*z), \inf h(z)\} \\ &= \min\{\inf h(0), \inf h(z)\} \\ &= \inf h(z).\end{aligned} \tag{3.7}$$

It follows that $\inf h(x) \geq \min\{\inf h(x*y), \inf h(y)\} \geq \min\{\inf h(y), \inf h(z)\}$.

Conversely, suppose that the condition (3.6) is valid. Since $x*(x*y) \leq y$ for all $x, y \in X$, it follows from (3.6) that $\inf h(x) \geq \min\{\inf h(x*y), \inf h(y)\}$ for all $x, y \in X$. Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X. □

PROPOSITION 3.13. *For any Inf-hesitant fuzzy ideal $H := \{(x, h(x)) \mid x \in X\}$ of a BCK/BCI-algebra X, the following assertions are equivalent.*

(1) $\inf h((x*y)*y) \leq \inf h(x*y)$,
(2) $\inf h((x*y)*z) \leq \inf h((x*z)*(y*z))$

for all $x, y, z \in X$.

PROOF: Assume that (1) holds. Note that

$$((x*(y*z))*z)*z = ((x*z)*(y*z))*z \leq (x*y)*z$$

for all $x, y, z \in X$. It follows from Proposition 3.11, (1) and (2.5) that

$$\begin{aligned}\inf h((x*y)*z) &\leq \inf h(((x*(y*z))*z)*z) \\ &\leq \inf h((x*(y*z))*z) \\ &= \inf h((x*z)*(y*z))\end{aligned} \tag{3.8}$$

for all $x, y, z \in X$.

Conversely, suppose that (2) is valid and if we put $z := y$ in (2), then

$$\begin{aligned}\inf h((x*y)*y) &\leq \inf h((x*y)*(y*y)) \\ &= \inf h((x*y)*0) \\ &= \inf h(x*y)\end{aligned} \tag{3.9}$$

for all $x, y \in X$. □

THEOREM 3.14. *In a BCK-algebra X, every Inf-hesitant fuzzy ideal is an Inf-hesitant fuzzy subalgebra.*

PROOF: Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy ideal of a BCK-algebra X. Using (3.3), (2.5), (III), (V) and (3.2), we have

$$\begin{aligned}\inf h(x*y) &\geq \min\{\inf h((x*y)*x), \inf h(x)\}\\ &\geq \min\{\inf h((x*x)*y), \inf h(x)\}\\ &= \min\{\inf h(0*y), \inf h(x)\}\\ &= \min\{\inf h(0), \inf h(x)\}\\ &\geq \min\{\inf h(x), \inf h(y)\}\end{aligned}$$

for all $x, y \in X$. Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X. □

The converse of Theorem 3.14 is not true in general as seen in the following example.

EXAMPLE 3.15. The Inf-hesitant fuzzy subalgebra $H := \{(x, h(x)) \mid x \in X\}$ in Example 3.3(2) is not an Inf-hesitant fuzzy ideal of X since

$$\inf h(d) = 0.1 < 0.5 = \min\{\inf h(d*b), \inf h(b)\}.$$

In a BCI-algebra X, Theorem 3.14 is not true. In fact, the Inf-hesitant fuzzy ideal $H := \{(x, h(x)) \mid x \in X\}$ in Example 3.8 is not an Inf-hesitant fuzzy subalgebra of X since

$$\begin{aligned}\inf h((0,0)\otimes(0,1)) &= \inf h(0,-1) = 0.4\\ &< 0.5 = \min\{\inf h(0,0), \inf h(0,1)\}.\end{aligned}$$

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X. For any $a, b \in X$ and $n \in \mathbb{N}$, let

$$\mathrm{Inf}[b; a^n] := \{x \in X \mid \inf h((x*b)*a^n) = \inf h(0)\}$$

where $(x*b)*a^n = ((\cdots((x*b)*a)*a)*\cdots)*a$ in which a appears n-times. Obviously, $a, b, 0 \in \mathrm{Inf}[b; a^n]$.

PROPOSITION 3.16. *Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X in which the condition* (3.2) *is valid and*

$$(\forall x, y \in X)\,(\inf h(x*y) \geq \max\{\inf h(x), \inf h(y)\})\,. \tag{3.10}$$

For any $a, b \in X$ and $n \in \mathbb{N}$, if $x \in \mathrm{Inf}[b; a^n]$ then $x*y \in \mathrm{Inf}[b; a^n]$ for all $y \in X$.

PROOF: Let $x \in \inf h[b; a^n]$. Then $\inf h((x*b)*a^n) = \inf h(0)$, and thus

$$\begin{aligned}\inf h(((x*y)*b)*a^n) &= \inf h(((x*b)*y)*a^n)\\ &= \inf h(((x*b)*a^n)*y)\\ &\geq \max\{\inf h((x*b)*a^n), \inf h(y)\}\\ &= \max\{\inf h(0), \inf h(y)\} = \inf h(0)\end{aligned}$$

for all $y \in X$. Hence $\inf h(((x * y) * b) * a^n) = \inf h(0)$, that is, $x * y \in \inf h[b; a^n]$ for all $y \in X$. $\square$

PROPOSITION 3.17. *Let* $H := \{(x, h(x)) \mid x \in X\}$ *be a hesitant fuzzy set on a* BCK*-algebra* X*. If an element* $a \in X$ *satisfies:*

$$(\forall x \in X)\,(x \leq a)\,, \tag{3.11}$$

then $\mathrm{Inf}[b; a^n] = X = \mathrm{Inf}[a; b^n]$ *for all* $b \in X$ *and* $n \in \mathbb{N}$*.*

PROOF: Let $b, x \in X$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} \inf h((x * b) * a^n) &= \inf h(((x * b) * a) * a^{n-1}) \\ &= \inf h(((x * a) * b) * a^{n-1}) \\ &= \inf h((0 * b) * a^{n-1}) \\ &= \inf h(0) \end{aligned}$$

by (2.5), (3.11) and (V), and so $x \in \mathrm{Inf}[b; a^n]$, which shows that $\mathrm{Inf}[b; a^n] = X$. Similarly $\mathrm{Inf}[a; b^n] = X$. $\square$

COROLLARY 3.18. *If* $H := \{(x, h(x)) \mid x \in X\}$ *is a hesitant fuzzy set on a bounded* BCK*-algebra* X*, then* $\mathrm{Inf}[b; u^n] = X = \mathrm{Inf}[u; b^n]$ *for all* $b \in X$ *and* $n \in \mathbb{N}$ *where* u *is the unit of* X*.*

PROPOSITION 3.19. *Let* $H := \{(x, h(x)) \mid x \in X\}$ *be a hesitant fuzzy subalgebra of a* BCK*-algebra* X *satisfying the condition* (3.4)*. Then the following assertion is valid.*

$$(\forall a, b, c \in X)\,(\forall n \in \mathbb{N})\,(b \leq c \;\Rightarrow\; \mathrm{Inf}[b; a^n] \subseteq \mathrm{Inf}[c; a^n])\,. \tag{3.12}$$

PROOF: Let $b, c \in X$ be such that $b \leq c$. For any $a \in X$ and $n \in \mathbb{N}$, if $x \in \mathrm{Inf}[b; a^n]$ then

$$\begin{aligned} \inf h(0) = \inf h((x * b) * a^n) &= \inf h((x * a^n) * b) \\ &\leq \inf h((x * a^n) * c) = \inf h((x * c) * a^n) \end{aligned}$$

by (2.4) and (3.4), and so $\inf h((x * c) * a^n) = \inf h(0)$. Thus $x \in \mathrm{Inf}[c; a^n]$, and therefore $\mathrm{Inf}[b; a^n] \subseteq \mathrm{Inf}[c; a^n]$ for all $a \in X$ and $n \in \mathbb{N}$. $\square$

Corollary 3.20. *Every Inf-hesitant fuzzy ideal* $H := \{(x, h(x)) \mid x \in X\}$ *of a* BCK*-algebra* X *satisfies the condition* (3.12).

The following example shows that there exists a hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on a BCK-algebra X such that

(1) $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X,

(2) There exist $a, b \in X$ and $n \in \mathbb{N}$ such that the set $\mathrm{Inf}[b; a^n]$ is not an ideal of X.

Example 3.21. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, (0.8, 0.9]), (a, [0.6, 0.8]), (b, [0.6, 0.8]), (c, \{0.3\} \cup [0.4, 0.6))\}.$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of of X and

$$\mathrm{Inf}[a; c^n] = \{x \in X \mid \inf h((x * a) * c^n) = \inf h(0)\} = \{0, a, c\}$$

which is not an ideal of X for any $n \in \mathbb{N}$ since $b * a = a \in \mathrm{Inf}[a; c^n]$ but $b \notin \mathrm{Inf}[a; c^n]$.

We now consider conditions for a set $\mathrm{Inf}[b; a^n]$ to be an ideal of X.

Theorem 3.22. *Let* $H := \{(x, h(x)) \mid x \in X\}$ *be a hesitant fuzzy set on a* BCK*-algebra* X *such that*

$$(\forall x, y \in X)\,(\inf h(x) = \inf h(y) \;\Rightarrow\; x = y)\,. \tag{3.13}$$

If X *is positive implicative, then* $\mathrm{Inf}[b; a^n]$ *is an ideal of* X *for all* $a, b \in X$ *and* $n \in \mathbb{N}$.

Proof: Let $a, b, x, y \in X$ and $n \in \mathbb{N}$ be such that $x * y \in \mathrm{Inf}[b; a^n]$ and $y \in \mathrm{Inf}[b; a^n]$. Then $\inf h((y * b) * a^n) = \inf h(0)$, which implies from (3.13) that $(y * b) * a^n = 0$. Hence

$$\begin{aligned}\inf h(0) &= \inf h(((x*y)*b)*a^n)\\ &= \inf h((((x*y)*b)*a)*a^{n-1})\\ &= \inf h((((x*b)*(y*b))*a)*a^{n-1})\\ &= \inf h(((((x*b)*a)*((y*b)*a))*a)*a^{n-2})\\ &= \cdots\\ &= \inf h(((x*b)*a^n)*((y*b)*a^n))\\ &= \inf h(((x*b)*a^n)*0)\\ &= \inf h((x*b)*a^n),\end{aligned}$$

which shows that $x \in \mathrm{Inf}[b; a^n]$. Therefore $\mathrm{Inf}[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$. □

Since every implicative BCK-algebra is a positive implicative BCK-algebra, we have the following corollary.

Corollary 3.23. *Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X satisfying* (3.13). *If X is implicative, then* $\mathrm{Inf}[b; a^n]$ *is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.*

Theorem 3.22 is illustrated by the following example.

Example 3.24. Let $X = \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Then X is a positive implicative BCK-algebra. Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H = \{(0, (0.6, 0.9]), (a, [0.7, 0.8)), (b, \{0.4, 0.5, 0.6\}), (c, (0.2, 0.4])\}.$$

Then $\inf h(0) = 0.6$, $\inf h(a) = 0.7$, $\inf h(b) = 0.4$ and $\inf h(c) = 0.2$. Thus $H := \{(x, h(x)) \mid x \in X\}$ satisfies the condition (3.13), but it does not satisfy the condition (3.2). Hence $H := \{(x, h(x)) \mid x \in X\}$ is not an Inf-hesitant fuzzy ideal of X. Note that
$\mathrm{Inf}[0 : 0^n] = \{0\}$, $\mathrm{Inf}[0; a^n] = \{0, a\}$, $\mathrm{Inf}[0; b^n] = \{0, a, b\}$, $\mathrm{Inf}[0; c^n] = \{0, c\}$,

$\mathrm{Inf}[a;0^n] = \{0,a\}$, $\mathrm{Inf}[a;a^n] = \{0,a\}$, $\mathrm{Inf}[a;b^n] = \{0,a,b\}$, $\mathrm{Inf}[a;c^n] = \{0,a,c\}$,
$\mathrm{Inf}[b;0^n] = \{0,a,b\}$, $\mathrm{Inf}[b;a^n] = \{0,a,b\}$, $\mathrm{Inf}[b;b^n] = \{0,a,b\}$, $\mathrm{Inf}[b;c^n] = X$,
$\mathrm{Inf}[c;0^n] = \{0,c\}$, $\mathrm{Inf}[c;a^n] = \{0,a,c\}$, $\mathrm{Inf}[c;b^n] = X$, $\mathrm{Inf}[c;c^n] = \{0,c\}$,
and they are ideals of X.

Proposition 3.25. *Let $H := \{(x,h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X in which the condition* (3.13) *is valid. If J is an ideal of X, then the following assertion holds.*

$$(\forall a,b \in J)\,(\forall n \in \mathbb{N})\,(\mathrm{Inf}[b;a^n] \subseteq J)\,. \tag{3.14}$$

Proof: For any $a,b \in J$ and $n \in \mathbb{N}$, let $x \in \mathrm{Inf}[b;a^n]$. Then

$$\inf h(((x*b)*a^{n-1})*a) = \inf h((x*b)*a^n) = \inf h(0)$$

and so $((x*b)*a^{n-1})*a = 0 \in J$ by (3.13). Since J is an ideal of X, it follows from (2.12) that $(x*b)*a^{n-1} \in J$. Continuing this process, we have $x*b \in J$ and thus $x \in J$. Therefore $\mathrm{Inf}[b;a^n] \subseteq J$ for all $a,b \in J$ and $n \in \mathbb{N}$. □

Theorem 3.26. *Let $H := \{(x,h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X. For any subset J of X, if the condition* (3.14) *holds, then J is an ideal of X.*

Proof: Suppose that the condition (3.14) is valid. Not that $0 \in \mathrm{Inf}[b;a^n] \subseteq J$. Let $x,y \in X$ be such that $x*y \in J$ and $y \in J$. Taking $b := x*y$ implies that

$$\begin{aligned}\inf h((x*b)*y^n) &= \inf h((x*(x*y))*y^n)\\ &= \inf h(((x*(x*y))*y)*y^{n-1})\\ &= \inf h(((x*y)*(x*y))*y^{n-1})\\ &= \inf h(0*y^{n-1}) = \inf h(0),\end{aligned}$$

and so $x \in \mathrm{Inf}[b;y^n] \subseteq J$ with $b = x*y$. Therefore J is an ideal of X. □

Theorem 3.27. *If $H := \{(x,h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of a BCK/BCI-algebra X, then the set*

$$H_a := \{x \in X \mid \inf h(a) \leq \inf h(x)\}$$

is an ideal of X for all $a \in X$.

PROOF: Let $x, y \in X$ be such that $x * y \in H_a$ and $y \in H_a$. Then $\inf h(a) \leq \inf h(x * y)$ and $\inf h(a) \leq \inf h(y)$. It follows from (3.3) and (3.2) that

$$\inf h(a) \leq \min\{\inf h(x * y), \inf h(y)\} \leq \inf h(x) \leq \inf h(0)$$

and that $0 \in H_a$ and $x \in H_a$. Therefore H_a is an ideal of X for all $a \in X$. □

COROLLARY 3.28. *If $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of a BCK/BCI-algebra X, then the set*

$$H_0 := \{x \in X \mid \inf h(0) = \inf h(x)\}$$

is an ideal of X for all $a \in X$.

THEOREM 3.29. *Let $a \in X$ and let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK/BCI-algebra X. Then*

(1) *If H_a is an ideal of X, then $H := \{(x, h(x)) \mid x \in X\}$ satisfies:*

$$\begin{aligned}(\forall x, y \in X)(\inf h(a) \leq \min\{\inf h(x * y), \inf h(y)\} \\ \Rightarrow \inf h(a) \leq \inf h(x)).\end{aligned} \tag{3.15}$$

(2) *If $H := \{(x, h(x)) \mid x \in X\}$ satisfies two condition* (3.2) *and* (3.15)*, then H_a is an ideal of X.*

PROOF:

(1) Assume that H_a is an ideal of X and let $x, y \in X$ be such that $\inf h(a) \leq \min\{\inf h(x * y), \inf h(y)\}$. Then $x * y \in H_a$ and $y \in H_a$, which imply that $x \in H_a$, that is, $\inf h(a) \leq \inf h(x)$.

(2) Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X in which two conditions (3.2) and (3.15) are valid. Then $0 \in H_a$. Let $x, y \in X$ be such that $x * y \in H_a$ and $y \in H_a$. Then $\inf h(a) \leq \inf h(x * y)$ and $\inf h(a) \leq \inf h(y)$, and so $\inf h(a) \leq \min\{\inf h(x * y), \inf h(y)\}$. It follows from (3.15) that $\inf h(a) \leq \inf h(x)$, that is, $x \in H_a$. Therefore H_a is an ideal of X. □

4. Inf-hesitant fuzzy p-ideals

In what follows, we take a BCI-algebra X as a reference set unless otherwise specified.

DEFINITION 4.1. Given an element $D \in \mathscr{P}^*([0,1])$, a hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on X is called an Inf-hesitant fuzzy p-ideal of X related to D (briefly, D-Inf-hesitant fuzzy p-ideal of X) if the set $\mathrm{Inf}[H; D]$ is a p-ideal of X whenever it is non-empty. If $H := \{(x, h(x)) \mid x \in X\}$ is a D-Inf-hesitant fuzzy p-ideal of X for all $D \in \mathscr{P}^*([0,1])$ with $\mathrm{Inf}[H; D] \neq \emptyset$, then we say that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X.

EXAMPLE 4.2.

(1) Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(0, (0.7, 0.9]), (a, (\{0.5\} \cup (0.6, 0.7)), (b, [0.3, 0.6]), (c, [0.3, 0.6])\}.$$

It is easy to verify that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X.

(2) Let $X = \{0, 1, a, b\}$ be a BCI-algebra with the following Cayley table.

$*$	0	1	a	b
0	0	0	a	a
1	1	0	b	a
a	a	a	0	0
b	b	a	1	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(0, [0.7, 0.9]), (1, ([0.3, 0.6)), (a, [0.5, 0.8]), (b, [0.3, 0.6))\}.$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is a D_1-Inf-hesitant fuzzy p-ideal of X with $D_1 = [0.3, 0.6)$. But if $D_2 = (0.4, 0.7]$, then $\mathrm{Int}[H; D_2] = \{0, a\}$ is not a

p-ideal of X since $(1 * b) * (a * b) = a \in \mathrm{Int}[H; D_2]$ and $b \notin \mathrm{Int}[H; D_2]$. Hence $H := \{(x, h(x)) \mid x \in X\}$ is not a D_2-Inf-hesitant fuzzy p-ideal of X.

THEOREM 4.3. *A hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on X is an Inf-hesitant fuzzy p-ideal of X if and only if if it satisfies* (3.2)

$$(\forall x, y, z \in X)\,(\min\{\inf h((x * z) * (y * z)), \inf h(y)\} \le \inf h(x))\,. \tag{4.1}$$

PROOF: Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy p-ideal of X. If (3.2) is not valid, then there exists $D \in \mathscr{P}^*([0,1])$ and $a \in X$ such that $\inf h(0) < \inf D \le \inf h(a)$. It follows that $a \in \mathrm{Inf}[H; D]$ and $0 \notin \mathrm{Inf}[H; D]$. This is a contradiction, and so (3.2) is valid. Now assume that (4.1) is not valid. Then

$$\min\{\inf h((a * c) * (b * c)), \inf h(b)\} > \inf h(a)$$

for some $a, b, c \in X$. Thus there exists $B \in \mathscr{P}^*([0,1])$ such that

$$\min\{\inf h((a * c) * (b * c)), \inf h(b)\} \ge \inf B > \inf h(a).$$

which implies that $(a * c) * (b * c) \in \mathrm{Inf}[H; B]$, $b \in \mathrm{Inf}[H; B]$ but $a \notin \mathrm{Inf}[H; B]$. This is a contradiction, and thus (4.1) holds.

Conversely, suppose that $H := \{(x, h(x)) \mid x \in X\}$ satisfies two conditions (3.2) and (4.1). Let $D \in \mathscr{P}^*([0,1])$ be such that $\mathrm{Inf}[H; D] \neq \emptyset$. Obviously, $0 \in \mathrm{Inf}[H; D]$. Let $x, y, z \in X$ be such that $(x*z)*(y*z) \in \mathrm{Inf}[H; D]$ and $y \in \mathrm{Inf}[H; D]$. Then $\inf h((x*z)*(y*z)) \ge \inf D$ and $\inf h(y) \ge \inf D$. It follows from (4.1) that

$$\inf h(x) \ge \min\{\inf h((x * z) * (y * z)), \inf h(y)\} \ge \inf D$$

and that $x \in \mathrm{Inf}[H; D]$. Hence $\mathrm{Inf}[H; D]$ is a p-ideal of X for all $D \in \mathscr{P}^*([0,1])$ with $\mathrm{Inf}[H; D] \neq \emptyset$, and therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X. $\square$

THEOREM 4.4. *The hesitant fuzzy set $H := \{(x, h(x)) \mid x \in X\}$ on X which is described in Theorem 3.10 is an Inf-hesitant fuzzy p-ideal of X.*

PROOF: In the proof of Theorem 3.10, we know that the condition (3.2) is valid. Let $x, y, z \in X$. If $(x * z) * (y * z) \in X \setminus B$ or $y \in X \setminus B$, then we have

$$\min\{\inf h((x * z) * (y * z)), \inf h(y)\} \le \inf h(x).$$

Assume that $(x * z) * (y * z) \in B$ and $y \in B$. Since $(x * z) * (y * z) \le x * y$ and B is the BCK-part of X, it follows from (2.4) and (III) that

$(x * y) * ((x * z) * (y * z)) \in B$ and from (2.12) that $x \in B$ since B is an ideal of X. Hence

$$\min\{\inf h((x * z) * (y * z)), \inf h(y)\} = \inf h(x).$$

Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X by Theorem 4.3. □

PROPOSITION 4.5. *Every Inf-hesitant fuzzy p-ideal $H := \{(x, h(x)) \mid x \in X\}$ of X satisfies:*

$$(\forall x \in X)\,(\inf h(0 * (0 * x)) \leq \inf h(x))\,. \tag{4.2}$$

PROOF: If we put $z := x$ and $y := 0$ in (4.1), then

$$\begin{aligned} \inf h(x) &\geq \min\{\inf h((x * x) * (0 * x)), \inf h(0)\} \\ &= \min\{\inf h(0 * (0 * x)), \inf h(0)\} \\ &= \inf h(0 * (0 * x)) \end{aligned}$$

for all $x \in A$ by (III) and (3.2). □

THEOREM 4.6. *Every Inf-hesitant fuzzy p-ideal of X is an Inf-hesitant fuzzy ideal of X.*

PROOF: Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy p-ideal of X. Since $x * 0 = x$ for all $x \in X$, it follows from (4.1) that

$$\begin{aligned} \inf h(x) &\geq \min\{\inf h((x * 0) * (y * 0)), \inf h(y)\} \\ &= \min\{\inf h(x * y), \inf h(y)\} \end{aligned}$$

for all $x, y \in X$. Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X. □

The converse of Theorem 4.6 is not true in general as seen in the following example.

EXAMPLE 4.7. Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table.

$*$	0	1	a	b	c
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

Let $H := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$\begin{aligned}H = \{&(0, [0.6, 0.7] \cup \{0.8, 0.9\}), (1, (\{0.5, 0.6, 0.7, 0.8\}),\\ &(a, [0.4, 0.9]), (b, [0.4, 0.9]), (c, [0.4, 0.9])\}.\end{aligned}$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X, but it is not an Inf-hesitant fuzzy p-ideal of X since

$$\inf h(1) = 0.5 < 0.6 = \min\{\inf h((1 * a) * (0 * a)), \inf h(0)\}.$$

PROPOSITION 4.8. *Every Inf-hesitant fuzzy p-ideal $H := \{(x, h(x)) \mid x \in X\}$ of X satisfies:*

$$(\forall x, y, z \in X) \left(\inf h(x * y) \leq \inf h((x * z) * (y * z))\right). \tag{4.3}$$

PROOF: Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy p-ideal of X. Then it is an Inf-hesitant fuzzy ideal of X by Theorem 4.6. Hence

$$\begin{aligned}\inf h((x * z) * (y * z)) &\geq \min\{\inf h(((x * z) * (y * z)) * (x * y)), \inf h(x * y)\}\\ &= \min\{\inf h(0), \inf h(x * y)\} = \inf h(x * y)\end{aligned}$$

for all $x, y, z \in X$. □

We provide conditions for an Inf-hesitant fuzzy ideal to be an Inf-hesitant fuzzy p-ideal.

THEOREM 4.9. *Let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy ideal of X such that*

$$(\forall x, y, z \in X) \left(\inf h(x * y) \geq \inf h((x * z) * (y * z))\right). \tag{4.4}$$

Then $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X.

Proof: If the condition (4.4) is valid, then

$$\begin{aligned}\inf h(x) &\geq \min\{\inf h(x * y), \inf h(y)\} \\ &\geq \min\{\inf h((x * z) * (y * z)), \inf h(y)\}\end{aligned}$$

for all $x, y, z \in X$. Therefore $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X. $\square$

Lemma 4.10. *Every Inf-hesitant fuzzy ideal $H := \{(x, h(x)) \mid x \in X\}$ of X satisfies the following condition:*

$$(\forall x \in X)\,(\inf h(x) \leq \inf h(0 * (0 * x)))\,.$$

Proof: For every $x \in X$, we have

$$\begin{aligned}\inf h(x) &= \min\{\inf h(0), \inf h(x)\} \\ &= \min\{\inf h((0 * (0 * x)) * x), \inf h(x)\} \\ &\leq \inf h(0 * (0 * x))\end{aligned}$$

which is the desired result. $\square$

Theorem 4.11. *If an Inf-hesitant fuzzy ideal $H := \{(x, h(x)) \mid x \in X\}$ of X satisfies the condition* (4.2)*, then it is an Inf-hesitant fuzzy p-ideal of X.*

Proof: Let $x, y, z \in A$. Using Lemma 4.10, (2.7), (2.8) and (4.2), we have

$$\begin{aligned}\inf h((x * z) * (y * z)) &\leq \inf h(0 * (0 * ((x * z) * (y * z)))) \\ &= \inf h((0 * y) * (0 * x)) \\ &= \inf h(0 * (0 * (x * y))) \\ &\leq \inf h(x * y).\end{aligned}$$

It follows from Theorem 4.9 that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X. $\square$

Theorem 4.12. *If $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X, then the set*

$$I := \{x \in X \mid \inf h(x) = \inf h(0)\}$$

is a p-ideal of X.

Proof: Obviously $0 \in I$. Let $x, y, z \in X$ be such that $(x * z) * (y * z) \in I$ and $y \in I$. Then

$$\inf h(x) \geq \min\{\inf h((x * z) * (y * z)), \inf h(y)\} = \inf h(0),$$

and so $\inf h(x) = \inf h(0)$, that is, $x \in I$. Therefore I is a p-ideal of X. $\square$

For any subset I of X, let $H_X^I = \big\{\big(x, \inf h^I(x)\big) \mid x \in X\big\}$ be a hesitant fuzzy set on X defined by

$$\inf h^I(x) = \begin{cases} \{1\} & \text{if } x \in I, \\ [0,1] & \text{otherwise.} \end{cases}$$

LEMMA 4.13. *For any subset I of X, the following are equivalent:*

(1) *I is an ideal (resp. p-ideal) of X.*
(2) *The hesitant fuzzy set $H_X^I = \big\{(x, \inf h^I(x)) \mid x \in X\big\}$ on X is an Inf-hesitant fuzzy ideal (resp. Inf-hesitant fuzzy p-ideal) of X.*

PROOF: The proof is straightforward. □

THEOREM 4.14. *A BCI-algebra X is p-semisimple if and only if every Inf-hesitant fuzzy ideal of X is an Inf-hesitant fuzzy p-ideal of X.*

PROOF: Assume that X is a p-semisimple BCI-algebra and let $H := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy ideal of X. Then

$$\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\} = \min\{\inf h((x * z) * (y * z)), \inf h(y)\}$$

by using (3.3) and (2.10). Hence $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X.

Conversely, suppose that every Inf-hesitant fuzzy ideal of X is an Inf-hesitant fuzzy p-ideal of X. Since the hesitant fuzzy set $H_X^{\{0\}} = \big\{\big(x, \inf h^{\{0\}}(x)\big) \mid x \in X\big\}$ on X is an Inf-hesitant fuzzy ideal of X, it is also an Inf-hesitant fuzzy p-ideal of X. It follows from Lemma 4.13 that $\{0\}$ is a p-ideal of X. For any $x \in X$, we have

$$\begin{aligned} ((x * (0 * (0 * x))) * x) * (0 * x) &= ((x * x) * (0 * (0 * x))) * (0 * x) \\ &= (0 * (0 * (0 * x))) * (0 * x) \\ &= (0 * (0 * x)) * (0 * (0 * x)) = 0 \in \{0\} \end{aligned}$$

by using (2.5) and (III), which implies from (2.13) that $x*(0*(0*x)) \in \{0\}$. Hence $x * (0 * (0 * x)) = 0$, that is, $x \leq 0 * (0 * x)$. Since $0 * (0 * x) \leq x$, we get $0 * (0 * x) = x$. Therefore X is a p-semisimple BCI-algebra. □

THEOREM 4.15. (Extension property for Inf-hesitant fuzzy p-ideals) *Let*

$$H := \{(x, h(x)) \mid x \in X\} \text{ and } G := \{(x, g(x)) \mid x \in X\}$$

be Inf-hesitant fuzzy ideals of X such that $\inf h(0) = \inf g(0)$ *and* $\inf h(x) \subseteq \inf g(x)$ *for all $x \in X$. If $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X, then so is $G := \{(x, g(x)) \mid x \in X\}$.*

PROOF: Assume that $H := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X. Using (2.8), (2.9) and (III), we have $0*(0*(x*(0*(0*x)))) = 0$ for all $x \in X$. It follows from hypothesis and (4.2) that

$$\begin{aligned}\inf g(x*(0*(0*x))) &\geq \inf h(x*(0*(0*x))) \\ &\geq \inf h(0*(0*(x*(0*(0*x))))) \\ &= \inf h(0) = \inf g(0).\end{aligned}$$

Hence

$$\begin{aligned}\inf g(x) &\geq \min\{\inf g(x*(0*(0*x))), \inf g(0*(0*x))\} \\ &\geq \min\{\inf g(0), \inf g(0*(0*x))\} \\ &= \inf g(0*(0*x)),\end{aligned}$$

and thus $G := \{(x, g(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy p-ideal of X by Theorem 4.11. □

5. Conclusions

Since hesitant fuzzy set theory was introduced by Torra in 2010, this concept has been applied to many areas including algebraic structures. The aim of this paper is introduce the notion of Inf-hesitant fuzzy set, and applied it to BCK/BCI-algebras. We have introduced the notions of Inf-hesitant fuzzy subalgebras, Inf-hesitant fuzzy ideals and Inf-hesitant fuzzy p-ideals in BCK/BCI-algebras, and have investigated their relations and properties. We have discussed caracterizations of an Inf-hesitant fuzzy subalgebras, an Inf-hesitant fuzzy ideals and an Inf-hesitant fuzzy p-ideal, and have constructed an Inf-hesitant fuzzy ideal by using the notion of BCK-parts. We have provided conditions for an Inf-hesitant fuzzy ideal to be an Inf-hesitant fuzzy p-ideal, and have provided a characterization of a p-semisimple BCI-algebra. We have considered characterizations of Inf-hesitant fuzzy p-ideals. We finally have established extension property for an Inf-hesitant fuzzy p-ideal. Future research will focus on applying the notions/contents to other types of ideals in BCK/BCI-algebras and related algebraic structures.

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