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Consumer Recommendations and
Market Segmentation**

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OPTIMAL GOODWILL MODEL WITH CONSUMER RECOMMENDATIONS AND MARKET SEGMENTATION

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ABSTRACT. We propose a new dynamic model of product goodwill where a product is sold in many market segments, and where the segments are indicated by the usage experience of consumers. The dynamics of product goodwill is described by a partial differential equation of the Lotka–Sharpe–McKendrick type. The main novelty of this model is that the product goodwill in a segment of new consumers depends not only on advertising effort, but also on consumer recommendations, for which we introduce a mathematical representation. We consider an optimal goodwill model where in each market segment the control variable is the company’s advertising efforts in order to maximize its profits. Using the maximum principle, we numerically find the optimal advertising strategies and corresponding optimal goodwill paths. The sensitivity of these solutions is analysed. We identify two types of optimal advertising campaign: ‘strengthening’ and ‘supportive’. They may assume different shapes and levels depending on the market segment. These experiments highlight the need for both researchers and managers to consider a segmented advertising policy.

1. INTRODUCTION

The concept of goodwill is becoming more and more important in modern business management. It can be defined as the value of the intangible assets of the company, and it is usually calculated as the difference between the price paid by the buyer for the company [Kapferer, 2012, p. 18] and the book value of its assets. This value has been created over time by, *inter alia*, the positive experiences of clients, and may be managed through investment in communication with them. Advertising is a form of communication used to encourage consumers to buy or continue to buy products. A consistent focus on consumer satisfaction with the product helps to build its reputation. Such accumulated goodwill will encourage consumers to buy the product. [Cañibano et al., 2000] stated that investment in goodwill is intended to acquire future earning power. The importance of this process may be reflected in mergers and acquisitions made after 1980, e.g. Nestle bought Rowntree Macintosh for three times its stock value and 26 times its earnings [Kapferer, 2012, p. 18]. Although this phenomenon has been studied by many researchers, there are still some gaps that prevent a full understanding of the dynamics of goodwill.

One of the ways to analyse the properties of a firms goodwill is the model approach. The first attempts to model the concept of goodwill were taken in 1962 by Nerlove and Arrow. They

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introduced the definition of goodwill as that part of the demand for a product that is generated by current and past advertising (see [Nerlove and Arrow, 1962]). The authors described the dynamics of goodwill by an ordinary differential equation and assumed that the stock of goodwill depreciates over time at a constant rate and depends positively on advertising effort.

In this paper we study a modified optimal goodwill model describing the dynamics of a product's goodwill by a partial differential equation (PDE). We analyse the evolution of the product's goodwill in a segmented market in which the segments are defined by the usage experiences of the consumers. Numerous empirical studies, summarized by [Bagwell, 2007], indicate that there are decreasing returns to advertising efforts. This observation is included in our model by introducing a non-linear relation between advertising and goodwill i.e. a non-linear advertising response function. A similar observation was used by [Weber, 2005] and by [Mosca and Viscolani, 2004] in a goodwill model expressed by an ordinary differential equation. In addition, we expand the existing models by allowing for a depreciation rate of goodwill that is not homogenous, as has been assumed in the literature up to now. We assume, more realistically, that it can vary with the usage experience of the product. The main difference between the existing models and that presented in this paper is the way goodwill is formed in the segment of consumers with no experience of the product. In our model, the goodwill in the segment of new consumers depends on two factors: advertising directed exclusively to this segment, and consumer recommendations, which can be amplified by advertising aimed at consumers with some experience.

The aim of this paper is to find the optimal advertising strategies and optimal goodwill paths to maximize the profit of the company in a finite time horizon. We analyse the sensitivity of the optimal advertising policies to the goodwill elasticity of demand and a non-linear advertising response function. The tool for this study is the maximum principle (see [Bogusz and Górajski, 2014]) which yields a system of partial differential equations with boundary, initial, and finite conditions, see (13) in Section 3.3. In our experiments we recognize two types of optimal advertising campaign: 'strengthening' and 'supportive'. The supportive optimal advertising policies in contrast to strengthening strategies are not able to increase the level of goodwill. What can be compelling for managers is that using supportive strategies, the company can achieve the greatest relative increase in profits. These experiments highlight the need for both researchers and managers to consider a segmented advertising policy.

This paper is organized as follows. The next section contains the economic background of the extended goodwill model. Section 3 presents a new optimal control model of product goodwill, and formulas for the optimal solutions are derived. We also introduce the mathematical formulation of the process of consumer recommendation and market segmentation based on usage experience. Section 4 includes numerical simulations of the calibrated model for the Polish information and communication sector. We consider a simulation study with several parametrisations of the goodwill model. We examine the influence of the goodwill elasticity of demand, the advertising response function, and consumer recommendations on the shape of the optimal advertising strategies and on the firm's profit. Finally, Section 5 contains some concluding remarks.

2. ECONOMIC BACKGROUND

In this section, in order to justify the components in the new model, we first review the empirical findings on market segmentation and consumer recommendations. Next, we elaborate briefly on the existing literature on goodwill modelling.

2.1. Market Segmentation. In the marketing literature (cf. [Smith, 1956], [Kamakura and Wedel, 1999]), it is believed that market segmentation is a fundamental strategy in modern marketing. This strategy may result in understanding the target group of consumers. This allows firms to more accurately match the needs of each segment by differentiating their products and services. The overall objective of using a market segmentation strategy is to improve the company's competitive position and better serve the needs of customers (see [Jha et al., 2009]). Moreover, advertising directed at each segment is more effective, which may also increase the profits of the enterprise (see [Allenby and Rossi, 1999] and [McDonald and Morris, 2004]). Very extensive psychological studies have been conducted on the effects of advertising on various sectors of society. The differences are perceived in many ways, not least in shaping the advertising response of younger and older consumers (cf. [Drolet et al., 2007]).

2.2. Consumer Recommendation. Consumer recommendations are considered the most trusted source of information about products (see [Brown and Reingen, 1987, Murray, 1991]). [Bruce et al., 2012] and [Agliari et al., 2010] present empirical evidence that consumer recommendations have a strong influence on the level of goodwill. Positive consumer recommendations influence goodwill by enhancing a company's reputation (cf. [Sundaram et al., 1998]). Moreover, recommendations attract more consumers to purchase the product by reducing the risk of the purchase decision and facilitating consumer choice from a differentiated set of products due to price and other attributes (see [Trusov et al., 2009]). Therefore, they were taken into account in modelling the sales of products in [Monahan, 1984], but so far, to the best of our knowledge, they have not been included in models describing the dynamics of goodwill.

2.3. Goodwill Modelling. We mentioned in the Introduction that the first goodwill model was proposed by [Nerlove and Arrow, 1962]. It is a simple model described by an ordinary differential equation and includes only a linear influence of advertising and a constant rate of depreciation of the goodwill. However, it is highly unlikely that all companies observe a linear effect of advertising on goodwill, or that additional factors do not positively (or negatively) influence the dynamics of a products goodwill. Therefore, many researchers have proposed modifications of the classical Nerlove and Arrow goodwill model. [Viscolani and Zaccour, 2009] examine the dynamics of goodwill on the competing market, with the assumption that a competitor's investment in advertising will negatively affect a firms goodwill. [Marinelli and Savin, 2008] consider the effect of spatial diffusion of the goodwill, and describe the dynamics of the goodwill by a partial differential equation in time and space variables. Several papers consider randomness in the dynamics of goodwill (e.g. [Raman, 2006, Gozzi et al., 2009]). These are just some examples of modifications of the model

proposed by Nerlove and Arrow. A detailed summary of the research in this topic can be found in the review papers [Feichtinger et al., 1994, Huang et al., 2012].

Recently, scientists have started to take into account in the goodwill model of one of the most effective marketing activities, namely market segmentation. For this reason, [Grosset and Viscolani, 2005] and [Buratto et al., 2006] consider goodwill models with market segmentation. In [Grosset and Viscolani, 2005], it is assumed that a firm sells one product in infinitely many segments, indicated by the age of the customers a , and wants to choose an advertising strategy to maximize the profit; in [Grosset and Viscolani, 2008], the authors used segmentation to compare different advertising media. Recently, [Faggian and Grosset, 2013] have described a model whereby a firm wants to optimally promote and sell a single product in an age-segmented market, over an infinite time horizon.

Another approach to market segmentation and a goodwill model was proposed by [Barucci and Gozzi, 1999]. They considered market segmentation in a goodwill model by product differentiation and described the situation in which a monopolistic firm sells infinitely many products. Market segmentation is used likewise to analyse advertising strategies affecting the dynamics of goodwill over several geographical regions in [Marinelli and Savin, 2008].

It is worth mentioning that the goodwill equation (2) resembles the Lotka–Sharpe–McKendrick equation used to describe population dynamics (see [Webb, 1985, Chan and Zhu, 1989]) or the capital accumulation process in a vintage capital framework in [Barucci and Gozzi, 2001, Feichtinger et al., 2006].

3. OPTIMAL GOODWILL MODEL WITH CONSUMER RECOMMENDATIONS

3.1. Goodwill Equation. Consider a monopolist selling a product in a segmented market in which the consumers are differentiated by their usage experience, which is equal to the time of using the product $a \in [0, 1]$. We assume that in segment $a = 0$ are the consumers who have already purchased the product. Furthermore, the time of using the product is normalized to 1, thus consumers that have used the product for time 1 leave the market permanently. The duration of the product life cycle is equal to T . In each segment a and at each moment of time $t \in [0, T]$, we model the product goodwill $G(t, a)$, defined in [Nerlove and Arrow, 1962] as that part of the product demand that results from past and current investments in advertising. In the sequel we assume that $G(t, a)$ is equal to the number of consumers who have been using the product for $a \in [0, 1)$ units of time, and that they continue buying the product at time $t \geq 0$ as an effect of advertising.

We denote by $u(t, a)$ and $u_0(t)$ the intensiveness of the advertising efforts at time t directed to consumer segment a , and to new consumers, respectively. We measure the advertising intensity in Gross Rating Points which are defined as impressions divided by the number of people in the audience for an advertisement. An impression is generated each time an advertisement is viewed (see [Farris et al., 2009]). In our model we assume that $u^\rho(t, a)$ and $u_0^\rho(t)$, $\rho \in (0, 1]$ positively influence the product goodwill $G(t, a)$ in segment a and the level of product goodwill $G(0, t)$ of new consumers, respectively. The parameter ρ represents the non-linear-concave $\rho \in (0, 1)$ or

linear $\rho = 1$ effect of advertising efforts on the rate of change of the product goodwill. For many years, empirical studies have contradicted the linearity of the advertising response function. Recall that the advertising response function is the relation between an input of advertising efforts and its effect on demand (cf. [Simon and Arndt, 1980]). They provide over 100 experimental and real evidence studies, and they conclude that advertising has diminishing returns to scale. In view of this finding, we assume that the advertising response function possesses a concave or linear shape. The linear function corresponds to the most effective advertising channel. More precisely, we assume that

$$(1) \quad \Delta G(t, a) \approx u^\rho(t, a), \rho \in (0, 1]$$

Furthermore, there is a natural depreciation rate of goodwill $\delta(a) \geq 0$ in each age generation of consumers a and its dependence on the time of using the product is natural for experience products (see [Nelson, 1974]). For this kind of goods, as time goes by, consumers discover the features of the product and update their judgement on it, which yields changes in the depreciation rate of the goodwill. Therefore, the dynamics of goodwill is described by a PDE of the form:

$$\frac{\partial G(t, a)}{\partial t} + \frac{\partial G(t, a)}{\partial a} + \delta(a)G(t, a) = u^\rho(t, a) \quad (t, a) \in [0, T] \times [0, 1].$$

Moreover, the value of the goodwill of new consumers $G(t, 0)$ is also affected by the recommendations of the older generations of consumers.

Let $N(t, a)$ be the number of individuals who tend to buy the good for the first time at moment t as a result of consumer recommendations coming from segment a . We distinguish two disjoint groups $N_1(t, a)$, $N_2(t, a)$ of new consumers affected by recommendations and denote by $N(t, a)$ the union of these two groups

$$N(t, a) = N_1(t, a) + N_2(t, a).$$

Let $R(a)$ be the relative rate of consumer recommendations in segment a , defined as the ratio of new consumers who buy the product influenced by users with experience a to the total number of consumers in segment a . Recommendations are closely connected with product quality, which we assume to be constant over time, hence $R(a)$ reflects the share of consumers in segment a who have assessed the products quality positively. Since usually the quality of a product can be recognised only after some amount of time (see [Godes and Mayzlin, 2004]), the rate of consumer recommendation $R(a)$ is heterogeneous with respect to the time of using the product a . Hence, the number of new consumers in the first group is given by

$$N_1(t, a) = R(a)G(t, a).$$

Advertising efforts $u(t, a)$ influence not only the level of goodwill $G(t, a)$ but also the strength of consumer recommendations in segment a by reminding consumers in segment a why they like this product, and thus encouraging them to assess the product positively. Thanks to the advertising $u(t, a)$, it is easier for consumers in segment a to recommend the product with a specific rationale

attached to their recommendations (see [Keller and Fay, 2009]). Therefore, advertising efforts $u(t, a)$ strengthen the effectiveness of consumer recommendations in consumer segment a , and as a result, a new group of people $N_2(t, a)$ buy the product, and we obtain

$$N_2(t, a) = \frac{u^\rho(t, a)}{G(t, a)} G(t, a) = u^\rho(t, a),$$

where $\frac{u^\rho(t, a)}{G(t, a)}$ is the rate of advertising effectiveness in consumer generation a .

Finally, we determine that the number of new consumers who buy the product at time t as a result of consumer recommendations is equal to

$$\int_0^1 N(t, a) da = \int_0^1 (N_1(t, a) + N_2(t, a)) da = \int_0^1 (R(a)G(t, a) + u^\rho(t, a)) da.$$

The value of goodwill $G(t, 0)$ in the segment of new consumers is also affected by the advertising campaign $u_0(t)$. Hence, adding the effect of consumer recommendations and advertising effort, we obtain

$$G(t, 0) = \int_0^1 (R(a)G(t, a) + u^\rho(t, a)) da + u_0^\rho(t).$$

From the above considerations, we can describe the dynamics of goodwill by the PDE

$$(2) \quad \begin{cases} \frac{\partial G(t, a)}{\partial t} + \frac{\partial G(t, a)}{\partial a} + \delta(a)G(t, a) = u^\rho(t, a) & (t, a) \in [0, T] \times [0, 1], \\ G(t, 0) = \int_0^1 (R(a)G(t, a) + u^\rho(t, a)) da + u_0^\rho(t) & t \in [0, T], \\ G(0, a) = G_0(a) & a \in [0, 1], \end{cases}$$

where

- $G(t, a)$: is the product goodwill at time t in segment a ,
- $u(t, a)$: is the advertising effort at time t directed to segment a ,
- $u_0(t)$: is the advertising effort directed to new consumers at time t ,
- $R(a)$: is the rate of consumer recommendations in generation a , and
- $\delta(a)$: is the depreciation rate of product goodwill in segment a .

The following assumptions are used in [Bogusz and Górajski, 2014] to prove the existence and uniqueness of optimal advertising strategies.

A0: $G_0(a) > 0$ for a.e. $a \in [0, 1]$ and $G_0 \in L^\infty(0, 1)$.

A1: $R: [0, 1] \rightarrow [0, \infty)$ belongs to $L^\infty(0, 1)$, $\delta: [0, 1] \rightarrow [0, 1]$ is a measurable function such that

$$(3) \quad \int_0^1 R(a) e^{-\int_0^a \delta(s) ds} da < 1.$$

For the maximal advertising intensity (possibly infinite) $I \in (0, \infty]$, denote the sets of admissible controls by

$$U_{ad} = \{u \in L^\infty((0, T) \times (0, 1)) : 0 \leq u(t, a) \leq I \text{ for a.e. } (t, a) \in [0, T] \times [0, 1]\}.$$

and

$$U_{0,ad} = \{u_0 \in L^\infty(0, T) : 0 \leq u_0(t) \leq I \text{ for a.e. } t \in [0, T]\}.$$

3.2. The Firm's Costs and Income. The aim of a monopolistic firm is to choose the advertising strategy that maximizes the sum of the discounted profits in a horizon T . Moreover, we assume, as in [Nerlove and Arrow, 1962], that the demand function q is of the form

$$(4) \quad q(t, a) = p(t, a)^{-\epsilon_p} G(t, a)^{\epsilon_g},$$

where $p(t, a)$ is the price of the product for consumers in segment a , $-\epsilon_p < -1$ is the price elasticity of demand, and ϵ_g is the goodwill elasticity of demand.

Let $C(t, a) = C(q(t, a))$ be the total operating cost function at time t in segment a . Then $p^*(t, a) = mc(t, a) \frac{\epsilon_p}{\epsilon_p - 1}$ is the optimal price, i.e. the solution to the problem:

$$\max_{p(t, a)} (p(t, a)q(t, a) - C(q)),$$

where $mc(t, a) = \frac{dC}{dq}$ is the marginal cost in segment a and $m = \frac{\epsilon_p}{\epsilon_p - 1}$ is a monopolistic mark-up. We assume that the total operating cost is of the form $C(q) = q^\alpha + c_f$, where $c_f > 0$ is fixed cost, $\alpha \geq 1$ is the elasticity of variable cost with respect to output. Thus, the optimal price is given by

$$p^*(t, a) = \alpha (q^*(t, a))^{\alpha-1} m.$$

Hence, and using (4), we have

$$p^*(t, a) = \alpha (p^*(t, a))^{-\epsilon_p} G(t, a)^{\epsilon_g} \alpha^{-1} m,$$

and solving for p^* , we obtain

$$(5) \quad p^*(t, a) = (m \cdot \alpha)^{\frac{1}{1+\epsilon_p(\alpha-1)}} G(t, a)^{\frac{\epsilon_g(\alpha-1)}{1+\epsilon_p(\alpha-1)}}.$$

Note that for a linear production cost ($\alpha = 1$), the optimal price $p^*(t, a) = p^*$ is constant in time and in each consumer segment. The optimal production $q^*(t, a)$ which meets the demand in consumer segment a is given by

$$(6) \quad q^*(t, a) = p^*(t, a)^{-\epsilon_p} G(t, a)^{\epsilon_g} = K G(t, a)^{\frac{\epsilon_g}{1+\epsilon_p(\alpha-1)}},$$

where $K = (m \cdot \alpha)^{\frac{-\epsilon_p}{1+\epsilon_p(\alpha-1)}}$.

We denote the instantaneous costs of advertising efforts in segment $a \in (0, 1]$ and in the segment of new consumers by

$$(7) \quad C_A(u(t, a)) = \frac{\beta}{2}u^2(t, a),$$

$$(8) \quad C_A(u_0(t)) = \frac{\beta}{2}u_0^2(t),$$

respectively, where $\frac{\beta}{2} > 0$ is the unit price of advertising efforts. Therefore, from (5)–(7), it follows that the firm's profit at time t from market segment a is

$$(9) \quad \begin{aligned} \Pi(t, a) &= p^*(t, a) \cdot q^*(t, a) - C(q^*(t, a)) - C_A(u(t, a)) - C_A(u_0(t)) \\ &= K^{1-\frac{1}{\epsilon_p}} G(t, a)^{\frac{\epsilon_g + \epsilon_g(\alpha-1)}{1+\epsilon_p(\alpha-1)}} - K^\alpha G(t, a)^{\frac{\alpha\epsilon_g}{1+\epsilon_p(\alpha-1)}} - c_f - \frac{\beta}{2}(u^2(t, a) + u_0^2(t)) \\ &= \left(K^{1-\frac{1}{\epsilon_p}} - K^\alpha \right) G(t, a)^{\frac{\epsilon_g \alpha}{1+\epsilon_p(\alpha-1)}} - c_f - \frac{\beta}{2}(u^2(t, a) + u_0^2(t)), \end{aligned}$$

for a.e. $(t, a) \in [0, T] \times (0, 1]$.

The firm wants to maximise the sum of discounted profits in the finite horizon T . Hence the goal functional for the firm takes the form

$$(10) \quad \begin{aligned} J(G, u, v) &= \int_0^1 \int_0^T e^{-rt} \Pi(t, a) da dt \\ &= \int_0^1 \int_0^T e^{-rt} \left(K_\Pi * G(t, a)^\gamma - \frac{\beta}{2}(u^2(t, a) + u_0^2(t)) - c_f \right) dt da, \end{aligned}$$

where $K_\Pi = K^{1-\frac{1}{\epsilon_p}} - K^\alpha$, $\gamma = \frac{\epsilon_g \alpha}{1+\epsilon_p(\alpha-1)}$, $r > 0$ is the force of interest. In the sequel, we assume that

$$\mathbf{A3:} \quad K_\Pi = K^{1-\frac{1}{\epsilon_p}} - K^\alpha > 0, \text{ where } K = (m\alpha)^{\frac{-\epsilon_p}{1+\epsilon_p(\alpha-1)}} \text{ and } \gamma = \frac{\epsilon_g \alpha}{1+\epsilon_p(\alpha-1)} \in (0, 1].$$

Definition 3.1. The triple (G^*, u^*, v^*) is an optimal solution to the problem of maximizing (10) subject to (2) if G^* is a generalised mild solution to (2) with $(u^*, v^*) \in U_{ad} \times U_{0,ad}$ and

$$J(G^*, u^*, v^*) \geq J(G, u, v)$$

holds for any admissible controls (u, v) and G satisfying (2).

In [Bogusz and Górajski, 2014], we proved that for any $(u, u_0) \in U_{ad} \times U_{0,ad}$, there exists a generalised mild solution to (2).

3.3. Optimal Advertising Strategies. Relying on the maximum principle introduced in [Feichtinger et al., the optimal solution for problem (10) with (2) satisfies

$$(11) \quad u_0^*(t) = \begin{cases} 0 & \text{for } \xi(t, 0) > 0 \\ \left(-\frac{\rho}{\beta} e^{rt} \xi(t, 0)\right)^{\frac{1}{2-\rho}} & \text{for } \xi(t, 0) \in [0, -\frac{\beta}{\rho} e^{-rt} I^{2-\rho}] \\ I & \text{for } \xi(t, 0) < -\frac{\beta}{\rho} e^{-rt} I^{2-\rho} \end{cases},$$

and

$$(12) \quad u^*(t, a) = \begin{cases} 0 & \text{for } \xi(t, 0) + \xi(t, a) > 0 \\ \left(-\frac{\rho}{\beta} e^{rt} (\xi(t, a) + \xi(t, 0))\right)^{\frac{1}{2-\rho}} & \text{for } \xi(t, 0) + \xi(t, a) \in [0, -\frac{\beta}{\rho} e^{-rt} I^{2-\rho}] \\ I & \text{for } \xi(t, 0) + \xi(t, a) < -\frac{\beta}{\rho} e^{-rt} I^{2-\rho} \end{cases}.$$

for a.e. $(t, a) \in [0, T] \times [0, 1]$ and $\xi: [0, T] \times [0, 1] \rightarrow \mathbb{R}$ is a unique solution to the adjoint system described in (13).

Summarizing the above considerations, the optimal triple (G^*, u^*, u_0^*) is the solution of the following system

$$(13) \quad \begin{cases} \frac{\partial G^*(t, a)}{\partial t} + \frac{\partial G^*(t, a)}{\partial a} + \delta(a)G^*(t, a) = (u^*(t, a))^\rho & (t, a) \in [0, T] \times [0, 1], \\ G^*(t, 0) = \int_0^1 (R(a)G^*(t, a) + (u^*(t, a))^\rho) da + (u_0^*(t))^\rho & t \in [0, T], \\ G^*(0, a) = G_0(a) & a \in [0, 1] \\ \frac{\partial \xi(t, a)}{\partial t} + \frac{\partial \xi(t, a)}{\partial a} = K_{\Pi} e^{-rt} \gamma (G^*(t, a))^{\gamma-1} + \xi(t, a)\delta(a) - \xi(t, 0)R(a) & (t, a) \in [0, T] \times [0, 1], \\ \xi(T, a) = 0 & a \in [0, 1], \\ \xi(t, 1) = 0 & t \in [0, T]. \end{cases}$$

where (u^*, u_0^*) are given by (12) and (11).

4. SIMULATION STUDY: OPTIMAL GOODWILL AND ADVERTISING PATHS

In order to analyse the quantitative behaviour of the optimal advertising strategies and corresponding optimal goodwill paths, we present different numerical experiments involving the goodwill model. The purpose of these simulations is to expose the role of market segmentation and consumers' recommendations in creating advertising policies. We also analyse how the goodwill elasticity of demand ϵ_g and the non-linear advertising response function (1) influence the advertising campaigns and the company's profits.

We consider two different durable products or services sold on a monopolistic market. It is assumed that the companies recognise the need for segmentation of the market where they sell their goods and, therefore, the market is divided in terms of the user experience of consumers.

Both products or services are 'experience goods', i.e., goods whose particular attributes are recognised by consumers after some time of use (see [Nelson, 1974]). We assume that the first good is of low quality. Hence the number of consumers who negatively evaluate this good increases with

time, and as time goes by, more consumers cease to use it. Therefore, for this product, one can observe an increasing depreciation rate of goodwill δ and a decreasing recommendation function R . The second good is of high quality. In this case, the situation is reversed. The longer the usage experience is, the more positive attributes are discovered by the consumers. This implies that more consumers become attached to this good. In the model, this is reflected by the decreasing depreciation rate of goodwill δ and an increasing number of positive recommendations of the product. Moreover, we assume that firms influence the levels of goodwill by their marketing activities, such as a loyalty program. A loyalty program may impact the level of the goodwill depreciation rate and consumer recommendations. A good loyalty program, a program that is well received by consumers, encourages them to remain customers of the company and hence the depreciation rate decreases. It can also stimulate the number of product recommendations.

To summarise, we consider four experiments varied due to the product quality and the occurrence of a loyalty program. Table 1 contains the shapes of δ and R , and their average values.

type of good	depreciation rate function	consumer recommendation function
no loyalty program and low quality good	$\delta(a) = 1.4 \left(1 - \frac{0.5}{1-e^{-1}} e^{-a}\right)$, $\int_0^1 \delta(a) da = 0.7$	$R(a) = 0.2 \left(\frac{3}{5} - \frac{3}{21} \sqrt{a}\right)$, $\int_0^1 R(a) da = 0.1$
well received loyalty program and low quality good	$\delta(a) = 1 - \frac{0.5}{1-e^{-1}} e^{-a}$, $\int_0^1 \delta(a) da = 0.4$	$R(a) = \frac{3}{5} - \frac{3}{21} \sqrt{a}$, $\int_0^1 R(a) da = 0.4$
no loyalty program and high quality good	$\delta(a) = 0.75\sqrt{1-a}$, $\int_0^1 \delta(a) da = 0.4$	$R(a) = 0.1 + \frac{0.4}{e^{-1}} e^a$, $\int_0^1 R(a) da = 0.4$
well received loyalty program and high quality good	$\delta(a) = 0.2 \left(0.75\sqrt{1-a}\right)$, $\int_0^1 \delta(a) da = 0.1$	$R(a) = 2 \left(0.1 + \frac{0.4}{e^{-1}} e^a\right)$, $\int_0^1 R(a) da = 1$

TABLE 1. *Shapes of the consumer recommendation function and depreciation rate function*

Moreover, we divide consumers into those who have a low goodwill elasticity of demand $\epsilon_g = 0.1$, and those who have a high value of $\epsilon_g = 1$. A low goodwill elasticity of demand may happen in a situation where the consumer is blocked by other commitments or obligations to use a substitute product, thus being willing to purchase only a small amount of the good. A good example is provided by mobile operators: post-paid service requires that the consumers sign a contract. Hence the users of the mobiles post-paid service are unwilling to break off their contracts, and so they may only purchase additional services. Thus, their contribution to demand is relatively small. Therefore, the goodwill elasticity of demand (ϵ_g) for this group of consumers should be low. In contrast, pre-paid customers are free of any obligation to the mobile operator and may at any time purchase the services they desire. Therefore, the goodwill elasticity of demand for this group of users is high.

Next, we examine how the non-linear shape of the advertising response function (cf. (1)) affects the optimal advertising strategies, optimal goodwill paths, and the firm's profit. For these reasons, we analyse the linear $\rho = 1$ and concave-downward $\rho = 0.5$ responses.

The rest of the model parameters are calibrated relying on Polish macro- and micro-economic data. We calibrate the model parameters ϵ_p , α based on the information and communication sector of the Polish economy in the year 2010. Based on the Polish National Accounts by institutional sectors report for the years 2007–2010 published by the Central Statistical Office and using the large proportions, we calculate the mark-up m as the ratio of operating surplus gross and the difference between gross output and operating surplus gross in 2010 for the information and communication sector, thus $m = 1.39$ (hence $\epsilon_p = 3.56$). The elasticity of variable cost with respect to output α is calculated as the proportion of the average rate of change in revenues from the sale of products and the average rate of change in the prime cost of sold products (services) in the years 2005–2012. Thus we obtain $\alpha = 1.85$.

On the basis of the annual financial reports, one can discover that the proportion of advertising expenditures to revenues from the sale of products in this sector is equal to 11.5%. This allows us to calculate the unit advertising cost $\frac{\beta}{2}$, using the equation $11.5\% = \frac{\beta}{m}$. Hence the unit advertising cost is $\beta = 0.16$. We set a continuous interest rate $r = 2.8\%$.

The results of 16 simulations are shown in Figures 1–16. Each graphical presentation consist of the following four plots (from left to right): contour plot of optimal advertising strategy, 3D plot of optimal advertising strategy, contour plot of optimal goodwill path, and 3D plot of optimal goodwill path.

We find two types of optimal advertising strategies, which we will call ‘supportive’ and ‘strengthening’. The first maintains the level of goodwill at most at its initial level, while the latter causes a significant increase in the level of goodwill from its initial value.

Strengthening strategies occur if the good is high quality and has a well received loyalty program, for both low and high goodwill elasticity of demand as well as for both linear and concave advertising response functions, i.e. for $\rho = 1$ and $\rho = 0.5$, respectively (see Figures 1–4).

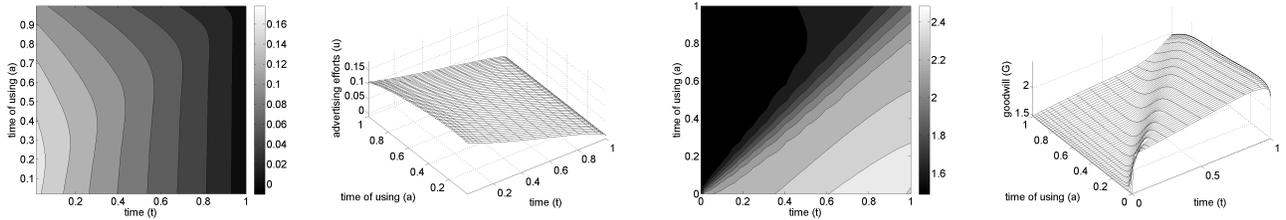


FIGURE 1. *The experiment for a well received loyalty program and high quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 0.1$; $\rho = 0.5$, $\int_0^1 \delta(a)da = 0.1$, $\int_0^1 R(a)da = 1$.*

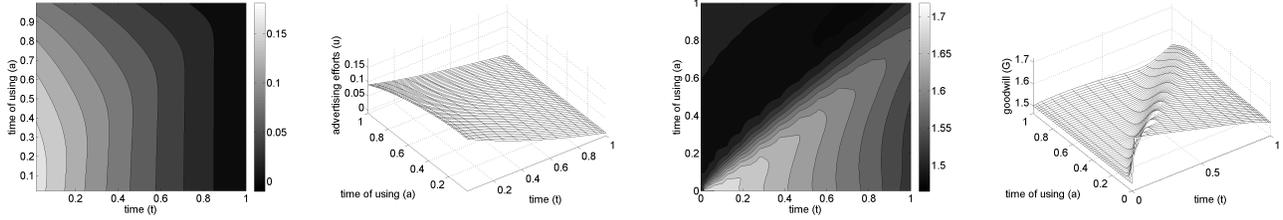


FIGURE 2. *The experiment for a well received loyalty program and high quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 0.1$; $\rho = 1$, $\int_0^1 \delta(a)da = 0.1$, $\int_0^1 R(a)da = 1$*

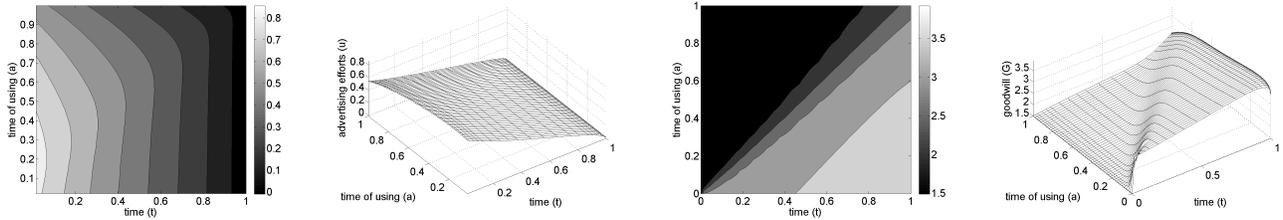


FIGURE 3. *The experiment for a well received loyalty program and high quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 1$; $\rho = 0.5$, $\int_0^1 \delta(a)da = 0.1$, $\int_0^1 R(a)da = 1$*

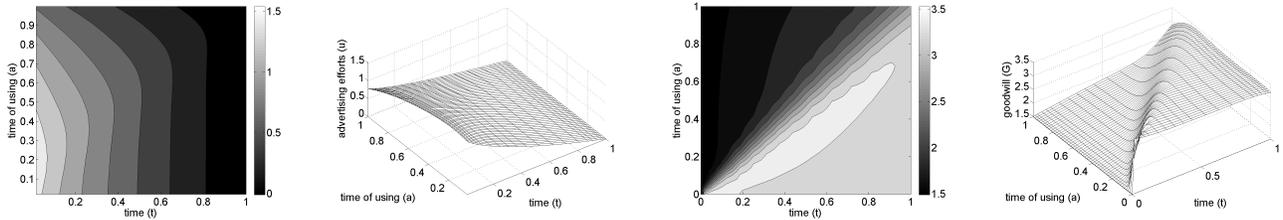


FIGURE 4. *The experiment for a well received loyalty program and high quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 1$; $\rho = 1$, $\int_0^1 \delta(a)da = 0.1$, $\int_0^1 R(a)da = 1$*

The strengthening strategy is also observed in the case of a high quality good when the purchase is not accompanied by any loyalty program (see Figures 5 and 6). A similar situation may be observed for a low quality product with well received loyalty program (Figures 7 and 8). In the above four scenarios the level of goodwill elasticity is high $\epsilon_g = 1$ and the advertising response function is both linear and concave ($\rho = 1$ and $\rho = 0.5$).

All optimal strengthening strategies have a concave decreasing shape. This means that in each market segment the maximum level of optimal advertising intensity is reached at the beginning

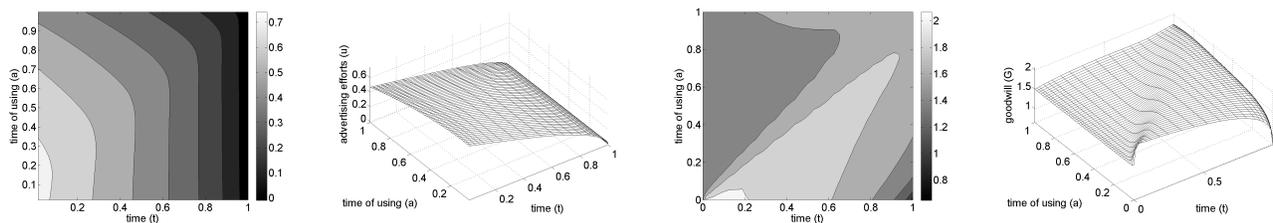


FIGURE 5. *The experiment for no loyalty program and high quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 1$; $\rho = 0.5$, $\int_0^1 \delta(a)da = 0.4$, $\int_0^1 R(a)da = 0.4$*

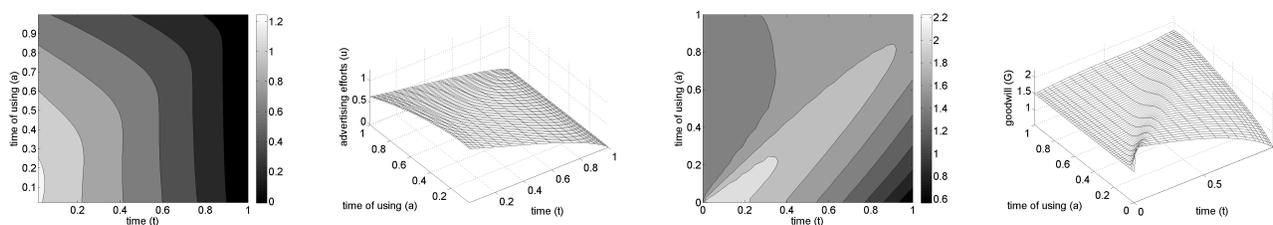


FIGURE 6. *The experiment for no loyalty program and high quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 1$; $\rho = 1$, $\int_0^1 \delta(a)da = 0.4$, $\int_0^1 R(a)da = 0.4$*

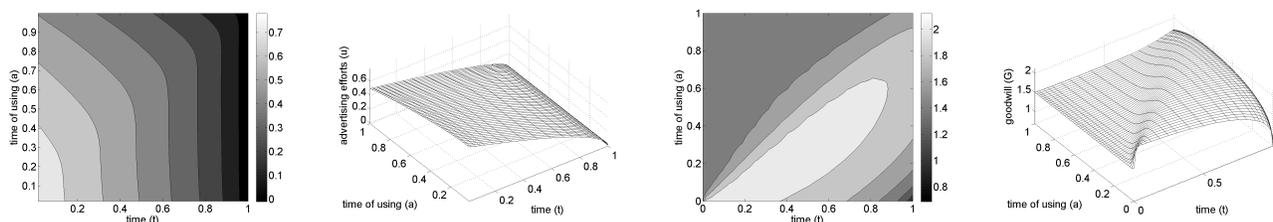


FIGURE 7. *The experiment for a well received loyalty program and low quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 1$; $\rho = 0.5$, $\int_0^1 \delta(a)da = 0.4$, $\int_0^1 R(a)da = 0.4$*

of the period considered. The main differences are in the value of the maximum level of optimal strategies. Consumers with shorter usage experiences require higher values of advertising intensity compared to more experienced users.

Supporting strategies are found in those scenarios involving low quality goods without any loyalty program, whereby the optimal policies in each market segment are divided into two shapes: decreasing concave (Figures 9 and 10) and parabolic with a maximum (Figures 11 and 12).

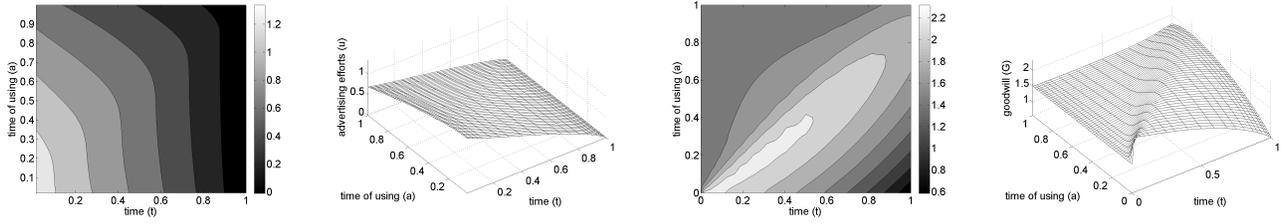


FIGURE 8. *The experiment for a well received loyalty program and low quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 1$; $\rho = 1$, $\int_0^1 \delta(a)da = 0.4$, $\int_0^1 R(a)da = 0.4$*

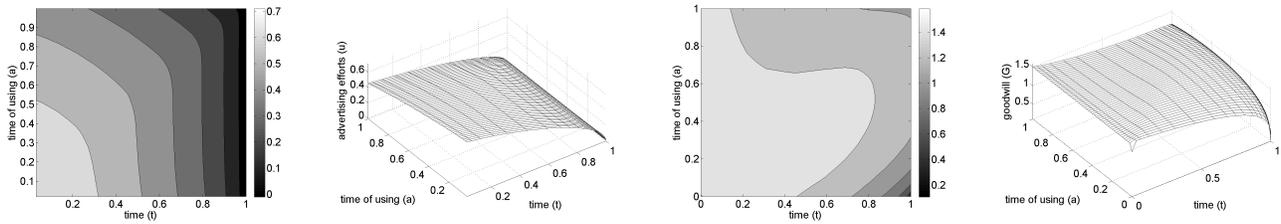


FIGURE 9. *The experiment for no loyalty program and low quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 1$; $\rho = 0.5$, $\int_0^1 \delta(a)da = 0.7$, $\int_0^1 R(a)da = 0.1$*

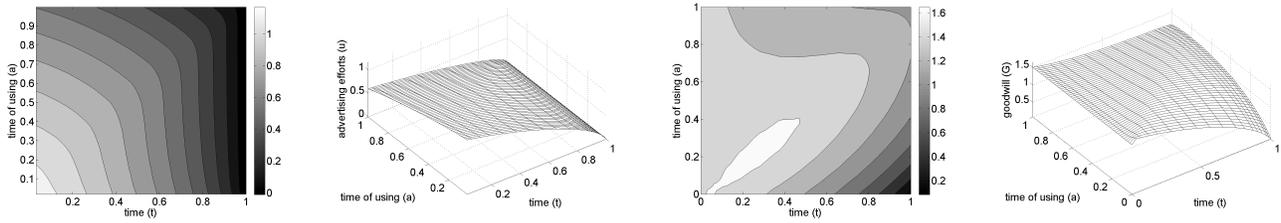


FIGURE 10. *The experiment for no loyalty program and low quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 1$; $\rho = 1$, $\int_0^1 \delta(a)da = 0.7$, $\int_0^1 R(a)da = 0.1$*

The optimal strategies of a parabolic shape with a maximum occur in cases when the goodwill elasticity of demand is small ($\epsilon_g = 0.1$). For a higher level of goodwill elasticity of demand, the optimal advertising strategies take a decreasing concave shape. This implies that a low goodwill elasticity of demand causes the maximum level of advertising intensity to occur much later. Still, the values of the maximum level of u^* are different in different market segments.

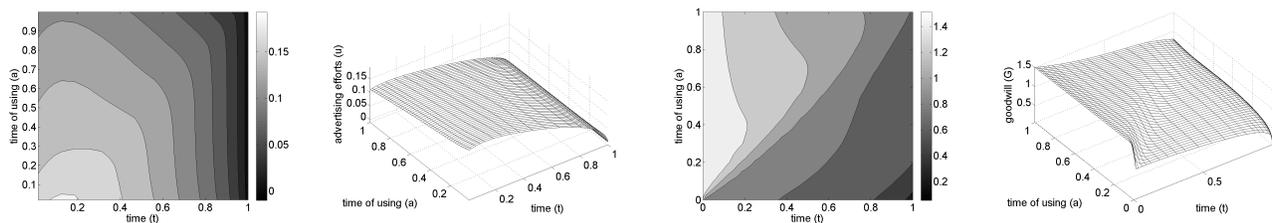


FIGURE 11. *The experiment for no loyalty program and low quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 0.1$; $\rho = 0.5$, $\int_0^1 \delta(a)da = 0.7$, $\int_0^1 R(a)da = 0.1$*

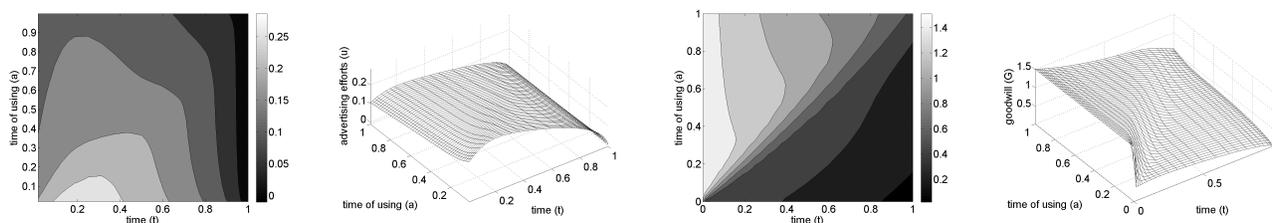


FIGURE 12. *The experiment for no loyalty program and low quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 0.1$; $\rho = 1$, $\int_0^1 \delta(a)da = 0.7$, $\int_0^1 R(a)da = 0.1$*

Supporting strategies are also found in experiments with a low quality product and well received loyalty program (see Figures 13 and 14), as well as with high quality goods without any loyalty program (see Figures 15 and 16).

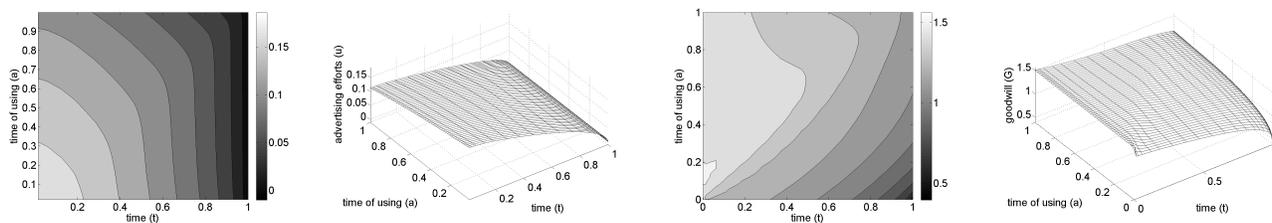


FIGURE 13. *The experiment for a well received loyalty program and low quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 0.1$; $\rho = 0.5$, $\int_0^1 \delta(a)da = 0.4$, $\int_0^1 R(a)da = 0.4$*

In both cases the goodwill elasticity of demand is low, whereby the value of ρ changes the shape of the optimal policies. For $\rho = 0.5$, in both cases, u^* has a concave-downward shape; but for $\rho = 1$, it is parabolic with a maximum. In addition to changes in the shape of the u^* in some

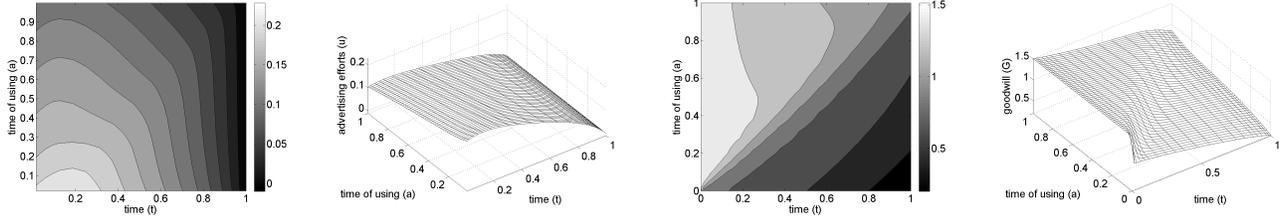


FIGURE 14. *The experiment for a well received loyalty program and low quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 0.1$; $\rho = 1$, $\int_0^1 \delta(a)da = 0.4$, $\int_0^1 R(a)da = 0.4$*

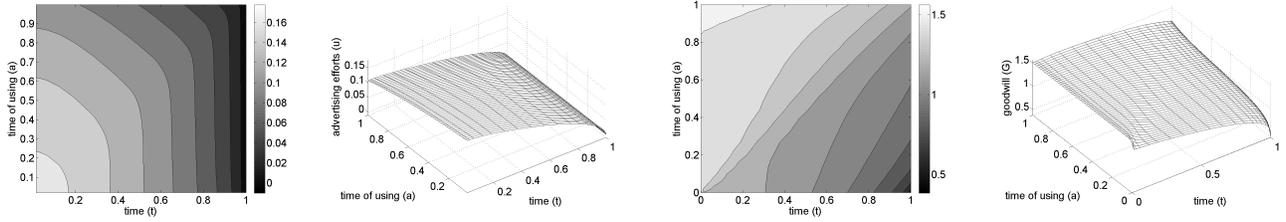


FIGURE 15. *The experiment for no loyalty program and high quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 0.1$; $\rho = 0.5$, $\int_0^1 \delta(a)da = 0.4$, $\int_0^1 R(a)da = 0.4$*

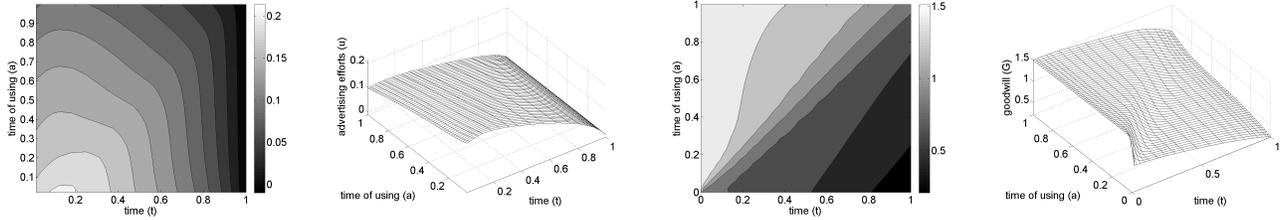


FIGURE 16. *The experiment for no loyalty program and high quality good with $G_0 = 1.5$; $T = 1$; $\epsilon_g = 0.1$; $\rho = 1$, $\int_0^1 \delta(a)da = 0.4$, $\int_0^1 R(a)da = 0.4$*

experiments, ρ also significantly affects the value of the total profit as well as the maximum levels of u^* (see Tables 2 and 3).

In Tables 2 and 3, the ratio $\frac{\Delta J}{J_0}$ is a measure of the benefit from an advertising campaign and is equal to the percentage change of the firm's profits caused by introducing the optimal advertising campaign.

Surprisingly, the non-linear-concave effect of advertising efforts ($\rho = \frac{1}{2}$) increases the benefits from advertising by up to 12% compared with the linear advertising response function ($\rho = 1$).

$\int_0^1 \delta(a) da$	$\int_0^1 R(a) da$	ρ	ϵ_g	J_0	J	$\Delta J/J_0$	$\max u$	$\max u_0$	$\max G$
0.4	0.4	0.5	0.1	0.31	0.318	3%	0.167	0.105	1.55
0.4	0.4	0.5	1	0.275	0.379	38%	0.73	0.458	2.055
0.4	0.4	1	0.1	0.31	0.312	1%	0.203	0.101	1.5
0.4	0.4	1	1	0.275	0.352	28%	1.235	0.612	2.213
0.1	1	0.5	0.1	0.316	0.324	3%	0.168	0.103	2.474
0.1	1	0.5	1	0.363	0.461	27%	0.846	0.522	3.908
0.1	1	1	0.1	0.316	0.321	2%	0.171	0.085	1.708
0.1	1	1	1	0.363	0.439	21%	1.528	0.738	3.518

TABLE 2. Results for experiments for goods with high quality. J_0 is the firm's profit without advertising investment.

$\int_0^1 \delta(a) da$	$\int_0^1 R(a) da$	ρ	ϵ_g	J_0	J	$\Delta J/J_0$	$\max u$	$\max u_0$	$\max G$
0.7	0.1	0.5	0.1	0.298	0.314	5%	0.182	0.114	1.5
0.7	0.1	0.5	1	0.218	0.341	56%	0.699	0.44	1.574
0.7	0.1	1	0.1	0.298	0.307	3%	0.275	0.137	1.5
0.7	0.1	1	1	0.218	0.313	44%	1.156	0.578	1.638
0.4	0.4	0.5	0.1	0.31	0.318	3%	0.175	0.11	1.55
0.4	0.4	0.5	1	0.276	0.387	40%	0.769	0.485	2.111
0.4	0.4	1	0.1	0.31	0.313	1%	0.217	0.108	1.5
0.4	0.4	1	1	0.276	0.36	30%	1.325	0.662	2.307

TABLE 3. Experiments for goods with low quality. J_0 is the firm's profit without advertising investment.

Moreover, our simulations confirm that the low level of goodwill elasticity of demand causes a small percentage increase in the firm's profits $\frac{\Delta J}{J_0}$ and that the advertising intensities and goodwill paths have significantly lower magnitudes than in the case where $\epsilon_g = 1$.

Another enlightening observation is that the maximum advertising intensity is greater in the consumer segments with some usage experience than in the segment of new consumers. Furthermore, the advertising campaign turns out to be the most fruitful for the company which produces a low quality product and does not provide any loyalty program, where $\frac{\Delta J}{J_0}$ reaches 56% under the assumption that the goodwill elasticity ϵ_g is high and the advertising response function is concave ($\rho = 0.5$). For high quality products, the advertising is less effective and brings benefits up to $\frac{\Delta J}{J_0} = 38\%$ in the case of a concave advertising response function $\rho = 0.5$ and for the firm without the loyalty program and selling the good with high goodwill elasticity $\epsilon_g = 1$. In the case of high quality products equipped with additional marketing support in the form of a loyalty program, the

benefits from advertising investments are relatively modest (1% – 3%). This means that advertising can be an effective marketing tool, in the sense of generating the highest profits, for products with low quality and, above all, if the firm is not using any other marketing tools, e.g., loyalty programs.

5. CONCLUSIONS

We analysed a dynamical optimal control model of product goodwill whose dynamics is affected by non-linear advertising. We enhanced previous research by introducing the novel mechanism of recruiting new consumers. We also gave a mathematical representation of consumer recommendations on a segmented market using a partial differential equation of the Lotka–Sharpe–McKendrick type.

We formulated models that appear to be representations of four development strategies which are differentiated with respect to the quality of product and the existence of a loyalty program. We demonstrated, for a set of plausible model parameters, that the optimal advertising strategies can be either strengthening or supporting. Moreover, each of them has a different shape and achieves a specific level for a market segment. Hence, one should take into account consumer diversity, since by identifying those consumer segments in which the product is evaluated positively, one can significantly affect the company’s profit.

Clearly, there are some limitations to this model approach, and it is not possible with normative models to capture all the complexities of real market environments. However, our findings provide evidence that optimal advertising policies are not always able to increase the level of goodwill. This is closely related to numerous empirical facts which report inefficient advertising expenditures and overspending in particular (e.g. [Aaker and Carman, 1982]). It is surprising that in these cases, the company can achieve the greatest relative increase in profits. Therefore, it requires closer consideration by marketing managers and researchers alike.

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