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UNIFIED SEQUENT CALCULI AND NATURAL DEDUCTION SYSTEMS FOR UNTIL-FREE LINEAR-TIME TEMPORAL LOGICS[1]

Abstract

A unified Gentzen-style proof-theoretic framework for until-free propositional linear-time temporal logic and its intuitionistic variant is introduced. The framework unifies Gentzen-style single-succedent sequent calculi and natural deduction

[1]This paper extends the conference paper [28], which was published in the Proceedings of the International Conference on Non-Classical Logics: Theory and Applications 2024 (NCL'24). Compared to Sections 2 and 3 of [28], there is a significant difference: new G3-style sequent calculi with direct cut-elimination proofs are introduced, replacing the sequent calculi and their indirect cut-elimination proofs presented in [28]. Specifically, in the new calculi, sequents are treated as multisets rather than sets, allowing for the adoption of a distinct and direct cut-elimination strategy. Section 4 presents new findings on intuitionistic variants that were not included in [28]. Additionally, Section 5 provides new insights into the next-time fragments and related works, which were also not covered in [28].

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systems for both the classical and intuitionistic versions of these temporal logics. Theorems establishing the equivalence between the proposed sequent calculi and natural deduction systems are proved. Furthermore, the cut-elimination theorems for the proposed sequent calculi and the normalization theorems for the proposed natural deduction systems are established.

Keywords: linear-time temporal logic, intuitionistic linear-time temporal logic, sequent calculus, natural deduction, cut-elimination theorem, normalization theorem.

2020 Mathematical Subject Classification: 03B44, 03F05.

1. Introduction

1.1. Until-free LTL and its intuitionistic variant

Linear-time temporal logic (LTL) and its fragments and variants have been extensively studied [45, 33, 14, 3, 4, 6, 18, 15, 10, 8, 9, 32, 23, 17, 11].[2] The fragment of LTL without the until operator U is referred to as *until-free LTL*. Numerous Gentzen-style sequent calculi for LTL and until-free LTL have been introduced and investigated [33, 43, 44, 49, 4, 18, 15, 10, 23]. Several natural deduction systems for LTL and until-free LTL have also been explored [3, 6].

This study focuses on until-free LTL and its intuitionistic variant as the main target logics.[3] One reason for this focus is its high compatibility

[2]LTL was traditionally studied, for example, in [45, 14]. The fragment of LTL without the until operator U was investigated, for example, in [33, 3, 4, 18, 32, 23].

[3]The until operator U in LTL presents a certain difficulty in constructing a simple cut-free, two-sided, LK-compatible Gentzen-style sequent calculus. An extension of Kawai's Gentzen-style sequent calculus LT_ω, referred to as LT_ω^U, with the addition of U, was considered in [25], although it is unknown whether the cut-elimination and completeness theorems for LT_ω^U hold or not. A few alternative cut-free and complete sequent calculi extended by adding U were developed in [15, 10]. However, we cannot use these calculi in this study, as they are not compatible with the present approach, which treats both sequent calculi and natural deduction systems in a uniform manner.

with Gentzen's LK and NK for classical logic and Gentzen's LI and NI[4] for intuitionistic logic [48, 46]. Specifically, the proposed Gentzen-style single-succedent sequent calculus and Gentzen-style natural deduction system for until-free LTL can be seen as natural extensions of LI and NI, respectively.

In addition, the intuitionistic variant of until-free LTL is considered because of its strong compatibility with Gentzen's LI and NI. Specifically, the proposed Gentzen-style single-succedent sequent calculus and Gentzen-style natural deduction system for this intuitionistic variant are subsystems of the corresponding systems for until-free LTL. These subsystems are obtained in a modular way from the systems of until-free LTL by removing the temporal versions of the rules of excluded middle.[5]

1.2. Sequent calculi and natural deduction systems

Gentzen-style sequent calculi for LTL have been previously explored in the literature. Kawai introduced the sequent calculus LT_ω for first-order until-free LTL, proving both cut elimination and completeness [33].[6] Baratella and Masini developed the 2-sequent calculus $2\mathrm{S}\omega$ for first-order until-free LTL, and established the cut-elimination and completeness theorems [4]. Kamide demonstrated an equivalence theorem between the propositional fragments of LT_ω and $2\mathrm{S}\omega$, providing alternative proofs of cut elimination as a consequence of this equivalence [18]. Further, Kamide presented embedding-based proofs of the cut-elimination and completeness theorems for LT_ω and its propositional fragment [23]. This study introduces a single-succedent version, $\mathrm{G3cLT}_\omega$, of LT_ω and its intuitionistic variant, $\mathrm{G3iLT}_\omega$.

[4]We remark that Gentzen designated his intuitionistic calculus by NI, however, his handwriting for capital I was in the old Sütterlin handwriting, that, as explained by von Plato in [55], p. 83, has been rendered by capital J in printing. This practice has been followed, with rare exceptions, to these days, but recent literature shows a return to the originally intended nomenclature, see e.g. [51]. We shall follow the original notation also for the intuitionistic sequent calculus, with LI instead of LJ.

[5]The proposed sequent calculus for until-free LTL includes the sequent calculus version of the rule of excluded middle, referred to as (ex-middle), and the proposed natural deduction system for until-free LTL includes the natural deduction version of the rule of excluded middle, referred to as (EXM).

[6]We have not yet obtained a cut-free and complete extension of LT_ω with the until operator.

The Gentzen-style natural deduction systems PNK and PNJ were introduced by Baratella and Masini [3] for classical and intuitionistic until-free LTLs, respectively. These systems, PNK and PNJ, are regarded as extensions of Gentzen's NK and NI, and were referred to by the authors as the "logics of positions". In a separate development, Bolotov et al. introduced the natural deduction system PLTL_{ND} [6] for full classical propositional LTL, including the until operator U. The system PLTL_{ND} employs labelled formulas of the form $i : \alpha$ and a temporal induction rule that deals with the next-time operator X and the "globally in the future" operator G. It is notable that PNK, PNJ, and PLTL_{ND} utilize an induction rule and do not incorporate infinite premiss rules for handling temporal operators. In contrast, the natural deduction systems proposed in this study employ infinite premiss rules and do not rely on an induction rule. This alternative approach provides a novel natural deduction system, NLT_ω, of LT_ω and its intuitionistic variant, NILT_ω.

1.3. The approach of this study

In this study, we introduce a unified Gentzen-style framework for until-free propositional LTL and its intuitionistic variant. This framework seamlessly integrates both Gentzen-style single-succedent sequent calculi and Gentzen-style natural deduction systems. Specifically, it allows us to establish an equivalence between these systems and to demonstrate that the cut-elimination theorems for the single-succedent sequent calculi imply the normalization theorems for the natural deduction systems.

The primary aim and original contribution of this study lie in providing a unified treatment for both sequent calculi and natural deduction systems within the context of until-free LTL.[7] Until now, these systems have been

[7] The unified treatment or approach also means that we can obtain a natural correspondence between the sequent calculus and the natural deduction system. More specifically, a natural correspondence refers to a correspondence between the cut-elimination theorem for the sequent calculus and the normalization theorem for the natural deduction system. Thus, the unified approach implies that we can handle these fundamental theorems in a uniform way.

studied separately for LTL and its fragments, rather than in a unified manner. This is in contrast to recent work where a unified treatment of sequent calculus and tableaux calculus for branching-time temporal logics has been explored by Abuin et al. [1]. Our unified approach not only bridges the gap (i.e., non-uniformity) between sequent calculi and natural deduction systems in until-free LTL but also facilitates the transfer of meta-results between these formalisms, providing a significant theoretical advantage for their applications.

To address the issue of the correspondence between cut elimination and normalization, we require a Gentzen-style single-succedent sequent calculus. This necessity arises because the cut-elimination theorem for the typical Gentzen-style multiple-succedent sequent calculi in standard classical LTL does not imply the normalization theorem for the corresponding natural deduction system. A similar situation is observed in classical logic when comparing Gentzen's LK and NK. In contrast, it is well-established that the cut-elimination theorem for the single-succedent calculus LI directly implies the normalization theorem for NI in intuitionistic logic.[8] Therefore, our approach involves developing an LI-like single-succedent sequent calculus tailored for our target logic.

1.4. The proposed single-succedent sequent calculi

To obtain a classical single-succedent sequent calculus, we use the following temporal (single-succedent) excluded middle rule:

$$\frac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \gamma \quad \mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ \text{(ex-middle)}$$

where X^i is an i-times nested next-time operator. By employing this rule, we can prove the law of excluded middle, $\alpha\vee\neg\alpha$, for arbitrary formulas α. The non-temporal version of this rule, without X^i, was originally introduced by von Plato [53, 41]. Pursuing the idea of correspondence between cut elimination and normalization, von Plato developed a single-succedent

[8] See, for example, [41] and the references therein.

sequent calculus for classical logic and proved cut elimination and normalization for the corresponding sequent calculus and natural deduction systems. Building on this approach, we aim to extend these concepts to the target temporal logic. In fact, the G3-style single-succedent sequent calculus $\mathrm{G3cLT}_\omega$ proposed in this study can be seen as a temporal extension of von Plato's calculus. Moreover, the cut-elimination result for $\mathrm{G3cLT}_\omega$ serves as an extension of his cut-elimination results for classical logic. Additionally, an important advantage of this approach is that a single-succedent sequent calculus for an intuitionistic variant of the target logic can be easily derived from $\mathrm{G3cLT}_\omega$ by just removing (ex-middle).

1.5. The proposed natural deduction systems

To obtain the corresponding natural deduction system for until-free LTL, we use rules of the form:

$$\dfrac{\begin{matrix}[\mathrm{X}^i\neg\alpha] & [\mathrm{X}^i\alpha]\\ \vdots & \vdots\\ \gamma & \gamma\end{matrix}}{\gamma}\ (\mathrm{EXM}) \qquad \dfrac{\mathrm{X}^i\neg\alpha \quad \mathrm{X}^i\alpha}{\gamma}\ (\mathrm{EXP}) \qquad \dfrac{\begin{matrix}[\mathrm{X}^i\alpha] & [\mathrm{X}^i\alpha]\\ \vdots & \vdots\\ \mathrm{X}^j\neg\gamma & \mathrm{X}^j\gamma\end{matrix}}{\mathrm{X}^i\neg\alpha}\ (\neg\mathrm{I})$$

where (EXM) corresponds to (ex-middle). As mentioned above, the non-temporal version of (EXM) was originally introduced by von Plato [53, 41]; the non-temporal versions of (EXP) and (¬I) were instead originally introduced by Gentzen. For detailed information on these rules, see [54, 55].

Rule (EXP) has been applied in various contexts: Bolotov and Shangin [7] utilized it to construct the paracomplete logic PCont; Kürbis and Petrukhin [36] developed natural deduction systems for a family of many-valued logics, including N3, using this rule; Kamide and Negri [26, 30] employed it to formalize Gurevich logic [16] and Nelson logic [42, 2]. Additionally, Priest [47] proposed rules similar to (EXP) for creating natural deduction systems for logics in the FDE (First Degree Entailment) family.

Rule (EXP) is considered a counterpart to (EXM) and is particularly useful for handling natural deduction systems where negation is treated as a primitive connective, rather than being defined through implication and the falsity constant. In this study, the proposed natural deduction system NLT_ω can be seen as a modified temporal extension of von Plato's classical system, enhanced by incorporating (EXP) and ($\neg$I). The normalization result for NLT_ω extends von Plato's normalization result for classical logic. Moreover, a significant advantage of this approach is that a natural deduction system for an intuitionistic variant of until-free LTL can be easily obtained from NLT_ω by omitting rule (EXM).

We address some remarks on previous results presented in the papers [26, 30] concerning natural deduction systems for logics of strong negation. The paper [26] introduced natural deduction systems for Gurevich logic, intuitionistic logic, and classical logic using (EXP) and/or (EXM) and proved normalization theorems for the natural deduction systems of Gurevich logic and intuitionistic logic. However, [26] contained errors related to these normalization theorems. These errors arose from inappropriate definitions in the natural deduction system $\mathrm{NI}^\star$ for intuitionistic logic.

These issues were identified by Arnon Avron during the 1^{st} Workshop on Contradictory Logics, held in Bochum on December 6-8, 2023. He pointed out a gap in an earlier version of $\mathrm{NI}^\star$, specifically the absence of the non-temporal version of ($\neg$I). These errors have been corrected in the subsequent papers [30, 29]. The results of this study reflect the corrected findings in [30, 29].

1.6. Paper structure

The paper is structured as follows: In Section 2, we explore Gentzen-style sequent calculi and their cut-elimination theorems for until-free propositional LTL. We begin by presenting Kawai's Gentzen-style sequent calculus LT_ω and introducing the newly proposed Gentzen-style single-succedent sequent calculus $\mathrm{G3cLT}_\omega$. Subsequently, we prove the cut-elimination theorem for $\mathrm{G3cLT}_\omega$ using an extension of the standard methodology for G3-style sequent calculi, namely, by first proving invertibility of (most of) the rules and admissibility of weakening and contraction.

In Section 3, we begin by introducing the newly proposed Gentzen-style natural deduction system NLT_ω for until-free propositional LTL. We also define the reduction relation for NLT_ω. Subsequently, we establish the normalization theorem for NLT_ω by exploiting the equivalence theorem between NLT_ω and $\mathrm{G3cLT}_\omega$.

In Section 4, we introduce and investigate a Gentzen-style sequent calculus $\mathrm{G3iLT}_\omega$ and a Gentzen-style natural deduction system NILT_ω for an intuitionistic variant of the until-free propositional LTL. The systems $\mathrm{G3iLT}_\omega$ and NILT_ω are derived from $\mathrm{G3cLT}_\omega$ and NLT_ω, respectively, by removing (ex-middle), and its corresponding rule (EXM). All the structural results and the cut-elimination theorem for $\mathrm{G3iLT}_\omega$ follow directly from the corresponding results for $\mathrm{G3cLT}_\omega$. Then, we establish the normalization theorem for NILT_ω using the translation that gives the equivalence between NILT_ω and $\mathrm{G3iLT}_\omega$.

Section 5 concludes this study with some additional remarks.

2. Sequent calculus and cut elimination

In this study, we assume standard notions and terminologies regarding Gentzen-style sequent calculus and Gentzen-style natural deduction system, and do not provide precise definitions for some of these familiar notions and terminologies.

Formulas of the logic discussed in this study are constructed using countably many propositional variables, the logical connectives $\to$ (implication), $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), G (globally in the future), F (eventually in the future), and X (next-time). We use small letters $p, q, ...$ to denote propositional variables and Greek small letters $\alpha, \beta, ...$ to denote formulas.

We use Greek capital letters $\Gamma, \Delta, ...$ to denote finite (possibly empty) multisets[9] of formulas. For any $\sharp \in \{\mathrm{G}, \mathrm{F}, \mathrm{X}\}$ and any multisets Γ of formulas, we use an expression $\sharp\Gamma$ to denote the multisets $\{\sharp\gamma \mid \gamma \in \Gamma\}$. The symbol $\equiv$ is used to denote the equality of multisets of symbols. The

[9]For the newly introduced G3-style calculi we shall consider a definition based on multisets. This will also be the case for natural deduction although the practice of multiple-discharge makes the choice less visible. Additionally, for the sake of comparison between our calculus and Kawai's calculus LT_ω, LT_ω is also presented in this study using a multiset-based formulation. This modification is not essential to the results of this study.

symbol ω is used to represent the set of natural numbers. An expression $\mathrm{X}^i\alpha$ for any $i \in \omega$ is defined inductively by $\mathrm{X}^0\alpha \equiv \alpha$ and $\mathrm{X}^{n+1}\alpha \equiv \mathrm{X}^n\mathrm{X}\alpha$. We use lower-case letters i, j and k to denote any natural numbers.

We will define Kawai's Gentzen-style sequent calculus LT_ω [33] and a new alternative Gentzten-style single-succedent sequent calculus $\mathrm{G3cLT}_\omega$. Prior to defining these sequent calculi, we need to define some additional notions and notations.

DEFINITION 2.1. A *sequent* for LT_ω is an expression of the form $\Gamma \Rightarrow \Delta$, and a sequent for $\mathrm{G3cLT}_\omega$ is an expression of the form $\Gamma \Rightarrow \gamma$ where γ is a formula or the empty set. We use the expression $L \vdash S$ to represent the fact that a sequent S is derivable in a sequent calculus L. We say that "a rule R is *admissible* in a sequent calculus L" if the following condition is satisfied: For any instance $\frac{S_1 \cdots S_n}{S}$ of R, if $L \vdash S_i$ for all i, then $L \vdash S$.

Additionally, we have to define the notion of height of a derivation. This notion is formulated in a general way and is applicable to all the sequent calculi here presented. Derivations built using these rules are thus (in general) infinite trees, with countable branching but where (as may be proved by induction on the definition of derivation) each branch has finite length. The *leaves* of the trees are the initial sequents. To make this precise, we give a formal definition of the notion of *derivation* $\mathcal{D}$ and the associated notions of its *height* $ht(\mathcal{D})$ and its *end-sequent*.

DEFINITION 2.2.

1. Any sequent $\Gamma \Rightarrow \Delta$, where some formula $\mathrm{X}^i p$ occurs in both Γ and Δ, is a derivation of *height* 0 and with *end-sequent* $\Gamma \Rightarrow \Delta$.

2. If each $\mathcal{D}_n$ is a derivation of height α_n, with end-sequent $\Gamma_n \Rightarrow \Delta_n$ and

$$\frac{\ldots \quad \Gamma_n \Rightarrow \Delta_n \quad \ldots}{\Gamma \Rightarrow \Delta}\ R$$

is an inference (i.e. an instance of a rule), then

$$\dfrac{\cdots \quad \dfrac{\mathcal{D}_n}{\Gamma_n \Rightarrow \Delta_n} \quad \cdots}{\Gamma \Rightarrow \Delta}\ R$$

is a derivation, its height is the countable ordinal $sup_n(\alpha_n) + 1$, and its end-sequent is $\Gamma \Rightarrow \Delta$.

3. Thus, each derivation $\mathcal{D}$ has a countable ordinal height, denoted $ht(\mathcal{D})$, which is the successor of the supremum of the heights of its immediate subderivations. It follows that, if $\mathcal{D}'$ is a sub-derivation of $\mathcal{D}$, then $ht(\mathcal{D}') < ht(\mathcal{D})$.

4. We say that "a rule R of the form $\frac{S_1 \cdots S_n}{S}$ is *height-preserving admissible* in a sequent calculus L" if the following condition is satisfied: If the premisses are derivable with height at most n then also the conclusion is derivable with the same bound on the derivation height. Furthermore, we say that "R is *derivable* in L" if there is a derivation from $S_1, \cdots, S_n$ to S in L.

First, we introduce LT_ω.

Definition 2.3 (LT_ω). In the following definitions, i and k represent arbitrary natural numbers (i.e., $i, k \in \omega$).

The initial sequents of LT_ω are of the following form for any propositional variable p:

$$\mathrm{X}^i p \Rightarrow \mathrm{X}^i p \text{ (init)}$$

The structural rules of LT_ω are of the form:

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \text{(cut)}$$

$$\dfrac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ \text{(we-left)} \quad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ \text{(we-right)}$$

$$\dfrac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ \text{(co-left)} \quad \dfrac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha}\ \text{(co-right).}$$

The logical rules of LT_ω are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta}\ (\rightarrow\text{left}) \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha{\rightarrow}\beta)}\ (\rightarrow\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}\ (\neg\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha\wedge\beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha\wedge\beta)}\ (\wedge\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha\vee\beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha\vee\beta)}\ (\vee\text{right})$$

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Gleft}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright})$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Fleft}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{F}\alpha}\ (\text{Fright}).$$

Remark 2.4.

1. The calculus LT_ω introduced here is a modified propositional version of Kawai's sequent calculus [33] for until-free first-order linear-time temporal logic. Kawai's original sequent calculus was developed as a first-order system incorporating the Barcan formula. In that system, the next-time operator was not used as a modal operator but rather as a special symbol.

2. Note that (Gright) and (Fleft) have infinitely many premises and can also be represented as:

$$\frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha \mid j\in\omega\ \}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright}) \qquad \frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \Delta \mid j\in\omega\ \}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Fleft}).$$

3. The following cut-elimination theorem holds for LT_ω. Rule (cut) is admissible in cut-free LT_ω. For the details, see [33, 18, 23].

4. The cut-elimination theorem for the original first-order LT_ω was proved by Kawai in [33] and was indirectly re-proved for a slightly modified version of LT_ω by Kamide in [18], via the cut-free equivalence between LT_ω and Baratella-Masini's cut-free 2-sequent calculus $2\mathrm{S}\omega$ [4]. Additionally, Kamide provided an alternative proof for the cut-elimination theorem for the slightly modified LT_ω in [23]. This alternative proof was based on a theorem that embeds LT_ω into a Gentzen-style sequent calculus for infinitary logic. For more information on the cut-elimination theorem for LT_ω, see [33, 18, 23].

Next, we introduce $\mathrm{G3cLT}_\omega$. We use the same names for the rules of $\mathrm{G3cLT}_\omega$ as those of LT_ω.

Definition 2.5 ($\mathrm{G3cLT}_\omega$). In the following definitions, i and k denote arbitrary natural numbers and γ denotes either a formula or the empty multiset.

The initial sequents of $\mathrm{G3cLT}_\omega$ are of the following form for any propositional variable p:

$$\mathrm{X}^i p, \Gamma \Rightarrow \mathrm{X}^i p \text{ (init).}$$ [10]

The structural rules of $\mathrm{G3cLT}_\omega$ are of the form: [11]

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha} \text{ (we-right).}$$

The logical rules of $\mathrm{G3cLT}_\omega$ are of the form:

$$\frac{\mathrm{X}^i(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \gamma} \text{ }({\rightarrow}\text{left}) \quad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha{\rightarrow}\beta)} \text{ }({\rightarrow}\text{right})$$

[10] The context Γ is required in (init), which distinguishes it from LT_ω.

[11] As will be shown, weakening-left, contraction-left, and cut rules are admissible in $\mathrm{G3cLT}_\omega$.

$$\frac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow} \ (\neg\text{left}) \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \mathrm{X}^i\neg\alpha} \ (\neg\text{right})$$

$$\frac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \gamma \quad \mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma} \ (\text{ex-middle})$$

$$\frac{\mathrm{X}^i\alpha, \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \gamma} \ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \wedge \beta)} \ (\wedge\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \gamma} \ (\vee\text{left})$$

$$\frac{\Gamma \Rightarrow \mathrm{X}^i\alpha}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \vee \beta)} \ (\vee\text{right1}) \qquad \frac{\Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \vee \beta)} \ (\vee\text{right2})$$

$$\frac{\mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \gamma}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \gamma} \ (\text{Gleft}) \qquad \frac{\{\ \Gamma \Rightarrow \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha} \ (\text{Gright})$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \gamma\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \gamma} \ (\text{Fleft}) \qquad \frac{\Gamma \Rightarrow \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha} \ (\text{Fright}).$$

Remark 2.6.

1. Similar to Gentzen's LK and the single-succedent sequent calculus for classical logic, a theorem can be established to demonstrate the equivalence between LT_ω and $\mathrm{G3cLT}_\omega$, assuming the admissibility of structural rules, including (cut). However, a proof is omitted here, as it was presented in [28] for similar systems. For details on the proof of such an equivalence theorem, see [28], where the equivalence between a slightly modified version of LT_ω and a non-G3-style version, SLT_ω, of $\mathrm{G3cLT}_\omega$ was established.

2. Alternatively, the equivalence between LT_ω and $\mathrm{G3cLT}_\omega$ can be established through the equivalence among LT_ω, SLT_ω, and NLT_ω. Here, SLT_ω refers to the non-G3-style single-succedent sequent

calculus introduced in [28], and NLT_ω is introduced both in the present study and in [28]. Since NLT_ω and (a slightly modified version of) LT_ω are common to both studies, the required equivalence theorem can be obtained.

3. Rule (ex-middle) is a characteristic feature of $\mathrm{G3cLT}_\omega$. By utilizing this rule, we can formalize a single-succedent sequent calculus. This rule is a temporal generalization of the original rule introduced by von Plato [53, 41]. Von Plato originally developed a single-succedent sequent calculus for classical logic using this rule and proved the cut-elimination theorem for it. Therefore, $\mathrm{G3cLT}_\omega$ can be viewed as a temporal extension of his calculus, and the cut-elimination result for $\mathrm{G3cLT}_\omega$ extends his cut-elimination result for classical logic.

4. In [53, 41], (ex-middle) and the following rule were introduced:

$$\frac{\neg p, \Gamma \Rightarrow \gamma \quad p, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{ex-middle-at})$$

where p is a propositional variable. In [53, 41], the following results were presented. The cut rule and rule (ex-middle) are admissible in certain versions of cut-free LI that include (ex-middle-at). As a consequence of these results, these versions possess a weak subformula property that allows for propositional variables and their negations.

Proposition 2.7. Let L be LT_ω or $\mathrm{G3cLT}_\omega$. The sequents of the form $\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\alpha$ for any formula α, any multiset Γ of formulas, and any natural number i are derivable in L.

Proof: By induction on α. We show some cases.

1. Case $\alpha \equiv \neg\beta$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \mathrm{X}^i\neg\beta, \mathrm{X}^i\beta, \Gamma \Rightarrow \mathrm{X}^i\beta\end{array}}{\mathrm{X}^i\beta, \mathrm{X}^i\neg\beta, \Gamma \Rightarrow}\ (\neg\text{left})}{\mathrm{X}^i\neg\beta, \Gamma \Rightarrow \mathrm{X}^i\neg\beta}\ (\neg\text{right}).$$

2. Case $\alpha \equiv \beta{\rightarrow}\gamma$:

$$\dfrac{\dfrac{\dfrac{\vdots\ Ind.\,hyp.}{\mathrm{X}^i(\beta{\rightarrow}\gamma), \mathrm{X}^i\beta \Rightarrow \mathrm{X}^i\beta} \quad \dfrac{\vdots\ Ind.\,hyp.}{\mathrm{X}^i\gamma, \Gamma \Rightarrow \mathrm{X}^i\gamma}}{\mathrm{X}^i\beta, \mathrm{X}^i(\beta{\rightarrow}\gamma), \Gamma \Rightarrow \mathrm{X}^i\gamma}\ (\rightarrow\text{left})}{\mathrm{X}^i(\beta{\rightarrow}\gamma), \Gamma \Rightarrow \mathrm{X}^i(\beta{\rightarrow}\gamma)}\ (\rightarrow\text{right}).$$

3. Case $\alpha \equiv \mathrm{G}\beta$:

$$\dfrac{\dfrac{\dfrac{\vdots\ Ind.\,hyp.}{\{\ \mathrm{X}^i\mathrm{G}\beta, \mathrm{X}^{i+j}\beta, \Gamma \Rightarrow \mathrm{X}^{i+j}\beta\ \}_{j\in\omega}}}{\{\ \mathrm{X}^i\mathrm{G}\beta, \Gamma \Rightarrow \mathrm{X}^{i+j}\ \}_{j\in\omega}}\ (\text{Gleft})}{\mathrm{X}^i\mathrm{G}\beta, \Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\beta}\ (\text{Gright}).$$

□

PROPOSITION 2.8. The rule of left weakening

$$\dfrac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{we-left})$$

is height-preserving admissible in $\mathrm{G3cLT}_\omega$.

PROOF: By straightforward induction on the height of the derivation since weakening is in-built in initial sequents and all the rules have an arbitrary context on the left. □

LEMMA 2.9. *All the logical rules of* $\mathrm{G3cLT}_\omega$ *with the exception of* ($\rightarrow$left), ($\vee$right1), ($\vee$right2), (Fright) *are hp-invertible. Rule* ($\rightarrow$left) *is hp-invertible with respect to the right premiss.*

PROOF: By induction on the height of the derivation of the conclusion of each rule. We show the case of invertibility of ($\rightarrow$left) with respect to the right premiss, all the other cases being similar. Assume $\vdash_0 X^i(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \gamma$. Then we have an initial sequent with $\gamma \in \Gamma$ and γ is of the form X^jp, i.e. $\vdash_0 X^i(\alpha{\rightarrow}\beta), X^jp, \Gamma' \Rightarrow X^jp$. Then also $X^i\beta, X^jp, \Gamma' \Rightarrow X^jp$ is an initial sequent, i.e., $\vdash_0 X^i\beta, \Gamma \Rightarrow \gamma$.

Let us assume the statement to be true for n and prove it for $n+1$. We consider the last rule applied in the derivation. If it is ($\rightarrow$left) with $X^i(\alpha\rightarrow\beta)$ principal, the right premiss gives the desired conclusion. If some other formula is principal in the last step, assume for example that the last step is ($\rightarrow$left) with another principal formula. So we have $\vdash_{n+1} X^i(\alpha\rightarrow\beta), X^j(\epsilon\rightarrow\delta), \Gamma' \Rightarrow \gamma$ and the premisses give $\vdash_n X^i(\alpha\rightarrow\beta), \Gamma' \Rightarrow X^j\epsilon$ and $\vdash_n X^i(\alpha\rightarrow\beta), X^j\delta, \Gamma' \Rightarrow \gamma$. By inductive hypothesis we thus obtain $\vdash_n X^i\beta, \Gamma' \Rightarrow X^j\epsilon$ and $\vdash_n X^i\beta, X^j\delta, \Gamma' \Rightarrow \gamma$ and a step of ($\rightarrow$left) gives the conclusion $\vdash_{n+1} X^i\beta, X^j(\epsilon\rightarrow\delta), \Gamma' \Rightarrow \gamma$. □

LEMMA 2.10. *The rule of left contraction*

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{co-left})$$

is hp-admissible in G3cLT$_\omega$.

PROOF: By induction on the height of the derivation of the premiss. In the base case, with height zero, the premiss is an initial sequent and clearly also the conclusion is an initial sequent. Otherwise, we assume that the premiss has derivation height $n+1$ and assume the statement true for derivation height n. We proceed by cases on the rule used to derive $\alpha, \alpha, \Gamma \Rightarrow \gamma$. If it is derived by (we-right) we have $\vdash_n \alpha, \alpha, \Gamma \Rightarrow$, so by induction hypothesis $\vdash_n \alpha, \Gamma \Rightarrow$ and by (we-right) $\vdash_{n+1} \alpha, \Gamma \Rightarrow \gamma$. We proceed in a similar way if $\alpha, \alpha, \Gamma \Rightarrow \gamma$ is the conclusion of a right rule: we apply the induction hypothesis to the premiss(es) of the rule, and then obtain the required fact. If instead $\alpha, \alpha, \Gamma \Rightarrow \gamma$ is the conclusion of a left rule, we distinguish two cases: either α is principal in the last rule, or it is not. In this latter case, we apply the induction hypothesis to the premiss(es) of the last rule and then obtain the required fact. Otherwise, in the former case, we consider the last rule applied and distinguish two sub-cases, depending on whether the last rule is invertible. We observe that all the left rules with the exception of ($\rightarrow$left) are invertible. Additionally, ($\rightarrow$left), ($\neg$left), and (Gleft) have been made invertible with the repetition in the premiss of the principal formula. These cases are straightforward because we find the duplication in the premiss of the rule and the induction hypothesis applies. In the case of an invertible rule, say we have a derivation of $\vdash_{n+1} \mathrm{X}^i(\alpha \vee \beta), \mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \gamma$ and the premisses of the last rule give $\vdash_n \mathrm{X}^i\alpha, \mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \gamma$ and $\vdash_n \mathrm{X}^i\beta, \mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \gamma$. By hp-invertibility of ($\vee$left) we obtain $\vdash_n \mathrm{X}^i\alpha, \mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma$ and $\vdash_n \mathrm{X}^i\beta, \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma$, so by induction hypothesis we have $\vdash_n \mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma$ and $\vdash_n \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma$. Application of ($\vee$left) gives the conclusion $\vdash_{n+1} \mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \gamma$. If the last rule is ($\rightarrow$left), we have $\vdash_{n+1} \mathrm{X}^i(\alpha \rightarrow \beta), \mathrm{X}^i(\alpha \rightarrow \beta), \Gamma \Rightarrow \gamma$, and the premisses of the rule give $\vdash_n \mathrm{X}^i(\alpha \rightarrow \beta), \mathrm{X}^i(\alpha \rightarrow \beta), \Gamma \Rightarrow \mathrm{X}^i\alpha$ and $\vdash_n \mathrm{X}^i(\alpha \rightarrow \beta), \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma$. By inductive hypothesis the former gives $\vdash_n \mathrm{X}^i(\alpha \rightarrow \beta), \Gamma \Rightarrow \mathrm{X}^i\alpha$ and the letter by (partial) hp-invertibility of ($\rightarrow$left) gives $\vdash_n \mathrm{X}^i\beta, \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma$, so again by inductive hypothesis we obtain $\vdash_n \mathrm{X}^i\beta, \Gamma \Rightarrow \gamma$, and thus by ($\rightarrow$left), $\vdash_{n+1} \mathrm{X}^i(\alpha \rightarrow \beta), \Gamma \Rightarrow \gamma$. □

For G3c [41], the proof of cut elimination eliminates a topmost cut by induction on the complexity of the cut formula and subinduction on the sum of the heights of the derivations of the premisses of cuts. To adapt the proof to a G3-style calculus with infinitary rules, we proceed as in [39]: heights are given by ordinals, and we shall employ the standard notion of (natural or Hessenberg) addition $\alpha \# \beta$ for countable ordinals α and β (cf. e.g. 10.1.2B in [52] for the definition). We recall that # is commutative and that if $\alpha < \alpha'$ then $\alpha \# \beta < \alpha' \# \beta$.

The *rank* $\pi(I)$ of an instance I of (cut) with cut-free premisses $\mathcal{D}$ and $\mathcal{D}'$ is the pair comprising the depth $d(A)$ of the cut formula and the natural sum $h(\mathcal{D}) \# h(\mathcal{D}')$ of the heights of the premisses. We will call the second component the *total height* of the cut. Pairs are ordered lexicographically.

Ordinals are well-ordered, so we can reason by (transfinite) induction; since we actually do it for pairs, we call this *transfinite lexicographic induction.* It can be converted to ordinary transfinite induction by turning pairs into ordinals, e.g. the pair (δ, σ) can be converted to $\delta \cdot \epsilon_0 + \sigma$, where ϵ_0 has the useful property of being greater than any possible value of σ; but pairs are conceptually clearer.

Lemma 2.11. *In*

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma}\ \text{(cut)}$$

if the premisses admit cut-free derivations in $\mathrm{G3cLT}_\omega$*, then the conclusion also admits a cut-free derivation.*

Proof: By transfinite lexicographic induction on the rank of instances of *cut* and case analysis. We first show the reduction steps for cuts with cut formula principal in both premisses, i.e. *principal cuts*. Then we show how non-principal cuts are reduced by permutation, maintaining the cut formula but reducing the sum of heights. We give the details only of the permutations of cuts into the first premiss; permutations into the second premiss are covered generically.

1. If the cut formula is principal in each premiss for instances of initial sequents, then the conclusion is already an initial sequent, so the cut can be eliminated.

2. If the first premiss is an instance of an initial sequent with the atom $\mathrm{X}^i p$ principal and $\mathrm{X}^i p$ is the cut formula, then the conclusion may be obtained by admissible (we-left) from the second premiss, regardless of the rule used in the second premiss, as in

$$\frac{\Gamma, \mathrm{X}^i p \Rightarrow \mathrm{X}^i p \quad \mathrm{X}^i p, \Gamma' \Rightarrow \gamma}{\Gamma, \mathrm{X}^i p, \Gamma' \Rightarrow \gamma}\ \text{(cut)}$$

3. If the cut formula $\mathrm{X}^i\mathrm{F}\alpha$ is principal in each premiss, then we consider the cut

$$\frac{\dfrac{\Gamma \Rightarrow \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha}\ \text{(Fright)} \quad \dfrac{\{\mathrm{X}^{i+j}\alpha, \Gamma' \Rightarrow \gamma\}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma' \Rightarrow \gamma}\ \text{(Fleft)}}{\Gamma, \Gamma' \Rightarrow \gamma}\ \text{(cut)}$$

which we transform into

$$\frac{\Gamma \Rightarrow \mathrm{X}^{i+k}\alpha \quad \mathrm{X}^{i+k}\alpha, \Gamma' \Rightarrow \gamma}{\Gamma, \Gamma' \Rightarrow \gamma} \text{ (cut)}$$

a cut on a smaller formula. The case with cut formula $\mathrm{X}^i\mathrm{G}\alpha$ is treated in a dual way.

4. Principal cuts with formulas with binary connectives and quantifiers as outermost logical constant are reduced as in the standard proof for G3c (cf. [41]).

5. If the second premiss is an instance of an initial sequent with the atom $\mathrm{X}^i p$ principal and $\mathrm{X}^i p$ is the cut formula, then the conclusion may be obtained by (we-left) from the first premiss, regardless of the rule used in the first premiss.

6. If the second premiss is an instance of an initial sequent with the formula $X^i p$ principal but $X^i p$ not the cut formula, then the conclusion is already an initial sequent, regardless of the rule used in the first premiss.

7. If the cut formula α is not principal in the left premiss, we reason by cases on the last rule used to derive it. Since the calculus is single succedent, the rule cannot be a right rule.

8. It is (Fleft), we have

$$\frac{\dfrac{\{\Gamma, \mathrm{X}^{i+j}\beta \Rightarrow \alpha\}_{j\in\omega}}{\Gamma, \mathrm{X}^i\mathrm{F}\beta \Rightarrow \alpha} \text{ (Fleft)} \qquad \alpha, \Gamma' \Rightarrow \gamma}{\Gamma, \Gamma', \mathrm{X}^i\mathrm{F}\beta \Rightarrow \gamma} \text{ (cut)}$$

which can be transformed to

$$\frac{\dots \qquad \dfrac{\Gamma, \mathrm{X}^{i+j}\beta \Rightarrow \Delta, \alpha \quad \alpha, \Gamma' \Rightarrow \gamma}{\Gamma, \Gamma', \mathrm{X}^{i+j}\beta \Rightarrow \gamma} \text{ (cut)} \qquad \dots}{\Gamma, \Gamma', \mathrm{X}^i\mathrm{F}\beta \Rightarrow \gamma} \text{ (Fleft)}$$

i.e., the cut is "permuted upwards" to each of the premisses of (Fleft), with unchanged cut formula γ and reduced total height. All the other cases of non-principal cuts with finitary rules are treated in a similar way.

9. If the cut formula α is not principal in the second premiss, and that premiss is not an initial sequents, then a standard permutation into the second premiss is applicable, with resulting cut(s) of reduced height.

Observe that in each case we have reduced the rank of the cut. □

THEOREM 2.12. *Rule* (cut) *is admissible in* G3cLT$_\omega$.

PROOF: It remains to show that an arbitrary derivation using instances (possibly infinite in number) of rule (cut) can be transformed to a cut-free derivation. Since this number may be infinite, we argue by transfinite induction on the height of the derivation. Consider a derivation $\mathcal{D}$; if it does not end with (cut), but with a step by rule R, then, by inductive hypothesis, each premiss (which has height less than $ht(\mathcal{D})$) can be transformed to a cut-free derivation (with conclusion unchanged), and thus so, by adding an R-step, can $\mathcal{D}$. Otherwise, if $\mathcal{D}$ ends with a cut, the derivations of its premisses both have height less than $ht(\mathcal{D})$; by inductive hypothesis, each can be transformed to a cut-free derivation (with conclusion unchanged). We now use the Lemma to obtain a cut-free derivation of the conclusion of $\mathcal{D}$. □

3. Natural deduction and normalization

3.1. Natural deduction

Next, we define a Gentzen-style natural deduction system NLT$_\omega$ for until-free propositional linear-time temporal logic. As usual in a definition of a natural deduction system we use the notation $[\alpha]$ to denote a discharged assumption (i.e., the formula α is a discharged assumption by the underlying logical rule).

Definition 3.1 (NLT_ω). Let i and k be arbitrary natural numbers.[12] The logical rules of NLT_ω are of the following form, where the discharge in rules that discharge assumptions can be simple, vacuous, or multiple:[13]

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\\ \mathrm{X}^i\beta\end{array}}{\mathrm{X}^i(\alpha\to\beta)}\ (\to\mathrm{I}) \qquad \dfrac{\mathrm{X}^i(\alpha\to\beta) \quad \mathrm{X}^i\alpha}{\mathrm{X}^i\beta}\ (\to\mathrm{E})$$

$$\dfrac{\mathrm{X}^i\neg\alpha \quad \mathrm{X}^i\alpha}{\gamma}\ (\mathrm{EXP}) \qquad \dfrac{\begin{array}{c}[\mathrm{X}^i\neg\alpha]\\ \vdots\\ \gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\\ \gamma\end{array}}{\gamma}\ (\mathrm{EXM}) \qquad \dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\\ \mathrm{X}^j\neg\gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\\ \mathrm{X}^j\gamma\end{array}}{\mathrm{X}^i\neg\alpha}\ (\neg\mathrm{I})$$

$$\dfrac{\mathrm{X}^i\alpha \quad \mathrm{X}^i\beta}{\mathrm{X}^i(\alpha\wedge\beta)}\ (\wedge\mathrm{I}) \qquad \dfrac{\mathrm{X}^i(\alpha\wedge\beta)}{\mathrm{X}^i\alpha}\ (\wedge\mathrm{E1}) \qquad \dfrac{\mathrm{X}^i(\alpha\wedge\beta)}{\mathrm{X}^i\beta}\ (\wedge\mathrm{E2})$$

$$\dfrac{\mathrm{X}^i\alpha}{\mathrm{X}^i(\alpha\vee\beta)}\ (\vee\mathrm{I1}) \qquad \dfrac{\mathrm{X}^i\beta}{\mathrm{X}^i(\alpha\vee\beta)}\ (\vee\mathrm{I2}) \qquad \dfrac{\mathrm{X}^i(\alpha\vee\beta) \quad \begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\\ \gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\beta]\\ \vdots\\ \gamma\end{array}}{\gamma}\ (\vee\mathrm{E})$$

$$\dfrac{\{\ \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{GI}) \qquad \dfrac{\mathrm{X}^i\mathrm{G}\alpha}{\mathrm{X}^{i+k}\alpha}\ (\mathrm{GE})$$

---

[12] In this definition, α, β, and γ represent arbitrary formulas. In particular, compared with the definition in the sequent calculus, γ is treated as a formula rather than as a formula or the empty multiset.

[13] Observe that, for example, as an instance of ($\to$I) we include a rule of the form:

$$\dfrac{\mathrm{X}^i\beta}{\mathrm{X}^i(\alpha\to\beta)}\ (\mathrm{Wk}).$$

$$\frac{\mathrm{X}^{i+k}\alpha}{\mathrm{X}^i\mathrm{F}\alpha}\ (\mathrm{FI}) \qquad \frac{\mathrm{X}^i\mathrm{F}\alpha \quad \{\begin{array}{c}[\mathrm{X}^{i+j}\alpha]\\ \vdots\\ \gamma\end{array}\}_{j\in\omega}}{\gamma}\ (\mathrm{FE}).$$

Remark 3.2.

1. Rules (EXP), (EXM), and (¬I) are characteristic features of NLT_ω. Rules (EXP) and (¬I) are temporal generalizations of the original rules introduced by Gentzen. Rule (EXM) is a temporal generalization of the original rule introduced by von Plato [53, 41]. The non-temporal versions of (EXP), (EXM), and (¬I) were also used by Kamide and Negri in [30] for constructing natural deduction systems for logics of strong negation.

2. An extended intuitionistic natural deduction system with the following restricted version (EXM-at) of (the original non-temporal version of) (EXM) was introduced by von Plato who also proved a normalization theorem for this system [53]:

$$\frac{\begin{array}{c}[\neg p]\\ \vdots\\ \gamma\end{array} \quad \begin{array}{c}[p]\\ \vdots\\ \gamma\end{array}}{\gamma}\ (\mathrm{EXM\text{-}at})$$

   where p is a propositional variable. This system was introduced as a natural deduction system for classical propositional logic. It was thus shown in [53] that (EXM) can be restricted to (EXM-at) without changing the provability in classical propositional logic.

3. Using (EXP) and (EXM), we can prove the formulas of the form $(\neg\alpha\wedge\alpha)\to\gamma$ and $\neg\alpha\vee\alpha$, respectively:

$$\dfrac{\dfrac{\dfrac{[\neg\alpha\wedge\alpha]^1}{\neg\alpha}\ (\wedge\mathrm{E1}) \quad \dfrac{[\neg\alpha\wedge\alpha]^1}{\alpha}\ (\wedge\mathrm{E2})}{\gamma}\ (\mathrm{EXP})}{(\neg\alpha\wedge\alpha)\to\gamma}\ (\to\mathrm{I})^1 \qquad \dfrac{\dfrac{[\neg\alpha]^1}{\neg\alpha\vee\alpha}\ (\vee\mathrm{I1}) \quad \dfrac{[\alpha]^1}{\neg\alpha\vee\alpha}\ (\vee\mathrm{I2})}{\neg\alpha\vee\alpha}\ (\mathrm{EXM})^1.$$

4. Using (¬I) and (EXP), we can prove the formulas of the form $\alpha \to \neg\neg\alpha$ and $\neg\neg(\alpha \to \alpha)$:

$$
\dfrac{\dfrac{\dfrac{[\neg\alpha]^2 \quad [\alpha]^1}{\neg\alpha}\,(\mathrm{EXP}) \qquad \dfrac{[\neg\alpha]^2 \quad [\alpha]^1}{\alpha}\,(\mathrm{EXP})}{\neg\neg\alpha}\,(\neg\mathrm{I})^2}{\alpha \to \neg\neg\alpha}\,(\to\mathrm{I})^1
$$

and

$$
\dfrac{\dfrac{\dfrac{[\alpha]^3}{\alpha\to\alpha}\,(\to\mathrm{I})^3 \quad [\neg(\alpha\to\alpha)]^1}{\alpha\to\alpha}\,(\mathrm{EXP}) \qquad \dfrac{\dfrac{[\alpha]^2}{\alpha\to\alpha}\,(\to\mathrm{I})^2 \quad [\neg(\alpha\to\alpha)]^1}{\neg(\alpha\to\alpha)}\,(\mathrm{EXP})}{\neg\neg(\alpha\to\alpha)}\,(\neg\mathrm{I})^1
$$

5. (GI) has infinitely many premisses and is also represented as:

$$
\dfrac{\mathrm{X}^i\alpha \quad \mathrm{X}^{i+1}\alpha \quad \mathrm{X}^{i+2}\alpha \quad \cdots \quad \mathrm{X}^{i+n}\alpha \quad \cdots}{\mathrm{X}^i\mathrm{G}\alpha}\,(\mathrm{GI}).
$$

6. (FE) has infinitely many premisses and is also represented as:

$$
\dfrac{\mathrm{X}^i\mathrm{F}\alpha \quad \begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\mathrm{X}^{i+1}\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\mathrm{X}^{i+2}\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \cdots \quad \begin{matrix}[\mathrm{X}^{i+n}\alpha]\\ \vdots \\ \gamma\end{matrix} \quad \cdots}{\gamma}\,(\mathrm{FE}).
$$

Next, we define some notions for NLT_ω.

DEFINITION 3.3. Rules (→I), (∧I), (∨I1), (∨I2), (¬I), (GI), (FI), and (EXM) are called *introduction rules*, and rules (→E), (∧E1), (∧E2), (∨E), (GE), (FE), and (EXP) are called *elimination rules*. The notions of *major* and minor premisses of the rules without (EXM) and (EXP) are defined as usual. If $X^i\neg\alpha$ and $X^i\alpha$ are both premisses of (EXP), then $X^i\neg\alpha$ and $X^i\alpha$ are called the major and minor premisses of (EXP), respectively. The notions of *derivation*, *(open and discharged) assumptions* of a derivation, and *end-formula* of a derivation are also defined as usual. For a derivation $\mathcal{D}$, we use the expression $\mathrm{oa}(\mathcal{D})$ to denote the multiset of open assumptions of $\mathcal{D}$ and the expression $\mathrm{end}(\mathcal{D})$ to denote the end-formula of $\mathcal{D}$. A formula α is said to be *provable* in a natural deduction system L if there exists a derivation of L with no open assumption whose end-formula is α.

Remark 3.4. There are no notions of the major and minor premisses of (EXM) and ($\neg$I). Namely, the premisses of (EXM) and ($\neg$I) are neither major nor minor premisses. In this study, (EXP) is treated as an elimination rule, and (EXM) is treated as an introduction rule.

In order to define the notion of normal derivations in NLT_ω, we define a reduction relation $\rhd$ on the set of derivations in NLT_ω. Before defining $\rhd$, we introduce some notions related to $\rhd$.

Definition 3.5. Let α be a formula occurring in a derivation $\mathcal{D}$ in NLT_ω. Then, α is called a *maximum formula* in $\mathcal{D}$ if α satisfies the following conditions:

1. α is the conclusion of an introduction rule, ($\vee$E), or (EXP),

2. α is the major premiss of an elimination rule.

A derivation is said to be *normal* if it contains no maximum formula. The notion of substitution of derivations for assumptions is defined as usual. We assume that the set of derivations is closed under substitution.

Definition 3.6 (Reduction relation). Let γ be a maximum formula in a derivation that is the conclusion of a rule R.

The definition of the reduction relation $\rhd$ at γ in NLT_ω is obtained by the following conditions.

1. R is ($\to$I) and γ is $\mathrm{X}^i(\alpha\to\beta)$:

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\ \mathcal{D}\\ \mathrm{X}^i\beta\end{matrix}}{\mathrm{X}^i(\alpha\to\beta)}\,(\to\mathrm{I})\qquad \begin{matrix}\vdots\ \mathcal{E}\\ \mathrm{X}^i\alpha\end{matrix}}{\mathrm{X}^i\beta}\,(\to\mathrm{E}) \qquad\rhd\qquad \begin{matrix}\vdots\ \mathcal{E}\\ \mathrm{X}^i\alpha\\ \vdots\ \mathcal{D}\\ \mathrm{X}^i\beta.\end{matrix}$$

2. R is (EXP):

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \mathrm{X}^i\neg\delta\end{matrix}\qquad\begin{matrix}\vdots\ \mathcal{D}_2\\ \mathrm{X}^i\delta\end{matrix}}{\gamma}\,(\mathrm{EXP})\qquad\begin{matrix}\vdots\ \mathcal{E}_1\\ \pi_1\end{matrix}\qquad\begin{matrix}\vdots\ \mathcal{E}_2\\ \pi_2\end{matrix}}{\pi}\,R' \qquad\rhd\qquad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \mathrm{X}^i\neg\delta\end{matrix}\qquad\begin{matrix}\vdots\ \mathcal{D}_2\\ \mathrm{X}^i\delta\end{matrix}}{\pi}\,(\mathrm{EXP})$$

where R' is an arbitrary rule, and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premisses of R' if they exist.

3. R is ($\neg$I), γ is $\mathrm{X}^i\neg\alpha$, and β is the conclusion of (EXP):

$$\dfrac{\dfrac{\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\ \mathcal{D}_1\\ \mathrm{X}^j\neg\delta\end{matrix}\qquad\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\ \mathcal{D}_2\\ \mathrm{X}^j\delta\end{matrix}}{\mathrm{X}^i\neg\alpha}\,(\neg\mathrm{I})\qquad\begin{matrix}\vdots\ \mathcal{E}\\ \mathrm{X}^i\alpha\end{matrix}}{\beta}\,(\mathrm{EXP}) \qquad\rhd\qquad \dfrac{\begin{matrix}\vdots\ \mathcal{E}\\ \mathrm{X}^i\alpha\\ \vdots\ \mathcal{D}_1\\ \mathrm{X}^j\neg\delta\end{matrix}\qquad\begin{matrix}\vdots\ \mathcal{E}\\ \mathrm{X}^i\alpha\\ \vdots\ \mathcal{D}_2\\ \mathrm{X}^j\delta\end{matrix}}{\beta}\,(\mathrm{EXP}).$$

4. R is $(\neg\mathrm{I})$, γ is $\mathrm{X}^i\neg\delta$, and $\mathrm{X}^i\delta$ is the conclusion of (EXP):

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{X}^i\delta]\\ \vdots\ \mathcal{D}_1\\ \mathrm{X}^j\neg\beta\end{array} \quad \begin{array}{c}[\mathrm{X}^i\delta]\\ \vdots\ \mathcal{D}_2\\ \mathrm{X}^j\beta\end{array}}{\mathrm{X}^i\neg\delta}\,(\neg\mathrm{I}) \quad \begin{array}{c}\vdots\ \mathcal{E}\\ \mathrm{X}^i\delta\end{array}}{\mathrm{X}^i\delta}\,(\mathrm{EXP}) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{E}\\ \mathrm{X}^i\delta.\end{array}$$

5. R is (EXM) and γ is $\mathrm{X}^i(\gamma_1{\to}\gamma_2)$, $\mathrm{X}^i(\gamma_1{\wedge}\gamma_2)$, or $\mathrm{X}^i(\gamma_1{\vee}\gamma_2)$:

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{X}^i\neg\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array}}{\gamma}\,(\mathrm{EXM}) \quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'$$

$$\rhd \quad \dfrac{\dfrac{\begin{array}{c}[\mathrm{X}^i\neg\alpha]\\ \vdots\ \mathcal{D}_1\\ \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R' \quad \dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\ \mathcal{D}_2\\ \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{E}_1\\ \delta_1\end{array} \quad \begin{array}{c}\vdots\ \mathcal{E}_2\\ \delta_2\end{array}}{\delta}\,R'}{\delta}\,(\mathrm{EXM})$$

where R' is $(\to\mathrm{E})$, $(\wedge\mathrm{E1})$, $(\wedge\mathrm{E2})$, or $(\vee\mathrm{E})$, and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are derivations of the minor premisses of R' if they exist.

6. R is (EXM), γ is $\mathrm{X}^i\neg\delta$, and $\mathrm{X}^i\delta$ is the conclusion of (EXP):

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{X}^i\neg\alpha]\\ \vdots\ \mathcal{D}_1\\ \mathrm{X}^i\neg\delta\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\\ \vdots\ \mathcal{D}_2\\ \mathrm{X}^i\neg\delta\end{array}}{\mathrm{X}^i\neg\delta}\,(\mathrm{EXM}) \quad \begin{array}{c}\vdots\ \mathcal{E}\\ \mathrm{X}^i\delta\end{array}}{\mathrm{X}^i\delta}\,(\mathrm{EXP}) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{E}\\ \mathrm{X}^i\delta.\end{array}$$

7. R is $(\wedge\mathrm{I})$ and γ is $\mathrm{X}^i(\alpha_1\wedge\alpha_2)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \mathrm{X}^i\alpha_1\end{matrix}\qquad\begin{matrix}\vdots\,\mathcal{D}_2\\ \mathrm{X}^i\alpha_2\end{matrix}}{\mathrm{X}^i(\alpha_1\wedge\alpha_2)}\,(\wedge\mathrm{I})}{\mathrm{X}^i\alpha_i}\,(\wedge\mathrm{E}i) \qquad\triangleright\qquad \begin{matrix}\vdots\,\mathcal{D}_i\\ \mathrm{X}^i\alpha_i\end{matrix}$$

where i is 1 or 2.

8. R is $(\vee\mathrm{I1})$ or $(\vee\mathrm{I2})$ and γ is $\mathrm{X}^i(\alpha_1\vee\alpha_2)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ \mathrm{X}^i\alpha_i\end{matrix}}{\mathrm{X}^i(\alpha_1\vee\alpha_2)}\,(\vee\mathrm{I}i)\qquad\begin{matrix}[\mathrm{X}^i\alpha_1]\\ \vdots\,\mathcal{E}_1\\ \delta\end{matrix}\qquad\begin{matrix}[\mathrm{X}^i\alpha_2]\\ \vdots\,\mathcal{E}_2\\ \delta\end{matrix}}{\delta}\,(\vee\mathrm{E}) \qquad\triangleright\qquad \begin{matrix}\vdots\,\mathcal{D}\\ \mathrm{X}^i\alpha_i\\ \vdots\,\mathcal{E}_i\\ \delta\end{matrix}$$

where i is 1 or 2.

9. R is $(\vee\mathrm{E})$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \mathrm{X}^i(\alpha\vee\beta)\end{matrix}\qquad\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\,\mathcal{D}_2\\ \pi\end{matrix}\qquad\begin{matrix}[\mathrm{X}^i\beta]\\ \vdots\,\mathcal{D}_3\\ \pi\end{matrix}}{\pi}\,(\vee\mathrm{E})\qquad\begin{matrix}\vdots\,\mathcal{E}_n\\ \{\ \delta_n\ \}\end{matrix}}{\delta}\,R'$$

$$\triangleright\qquad\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \mathrm{X}^i(\alpha\vee\beta)\end{matrix}\qquad\dfrac{\begin{matrix}[\mathrm{X}^i\alpha]\\ \vdots\,\mathcal{D}_2\\ \pi\end{matrix}\qquad\begin{matrix}\vdots\,\mathcal{E}_n\\ \{\ \delta_n\ \}\end{matrix}}{\delta}\,R'\qquad\dfrac{\begin{matrix}[\mathrm{X}^i\beta]\\ \vdots\,\mathcal{D}_3\\ \pi\end{matrix}\qquad\begin{matrix}\vdots\,\mathcal{E}_n\\ \{\ \delta_n\ \}\end{matrix}}{\delta}\,R'}{\delta}\,(\vee\mathrm{E})$$

where R' is an arbitrary rule, and $\mathcal{E}_1$, $\mathcal{E}_2$, ... , $\mathcal{E}_n$, ... are derivations of the minor premisses of R' if they exist.

10. R is (GI) and γ is $\mathrm{X}^i\mathrm{G}\alpha$:

$$\dfrac{\dfrac{\{\ \begin{array}{c}\vdots\ \mathcal{D}_j\\ \mathrm{X}^{i+j}\alpha\end{array}\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{GI})}{\mathrm{X}^{i+k}\alpha}\ (\mathrm{GE}) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{D}_k\\ \mathrm{X}^{i+k}\alpha\end{array}$$

where $k \in \omega$.

11. R is (FI) and γ is $\mathrm{X}^i\mathrm{F}\alpha$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_k\\ \mathrm{X}^{i+k}\alpha\end{array}}{\mathrm{X}^i\mathrm{F}\alpha}\ (\mathrm{FI}) \qquad \begin{array}{c}[\mathrm{X}^{i+j}\alpha]\\ \vdots\ \mathcal{E}_j\\ \{\ \delta\ \}_{j\in\omega}\end{array}}{\delta}\ (\mathrm{FE}) \qquad \rhd \qquad \begin{array}{c}\vdots\ \mathcal{D}_k\\ \mathrm{X}^{i+k}\alpha\\ \vdots\ \mathcal{E}_k\\ \delta\end{array}$$

where $k \in \omega$.

12. R is (FE):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ \mathrm{X}^i\mathrm{F}\alpha\end{array} \qquad \begin{array}{c}[\mathrm{X}^{i+j}\alpha]\\ \vdots\ \mathcal{D}_j\\ \{\ \pi\ \}_{j\in\omega}\end{array}}{\pi}\ (\mathrm{FE}) \qquad \begin{array}{c}\vdots\ \mathcal{E}_n\\ \{\ \delta_n\ \}\end{array}}{\delta}\ R'$$

$$\rhd \qquad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ \mathrm{X}^i\mathrm{F}\alpha\end{array} \qquad \dfrac{\begin{array}{c}[\mathrm{X}^{i+j}\alpha]\\ \vdots\ \mathcal{D}_j\\ \pi\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{E}_n\\ \{\ \delta_n\ \}\end{array}}{\{\ \delta\ \}_{j\in\omega}}\ R'}{\delta}\ (\mathrm{FE})$$

where R' is an arbitrary rule, and $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n, \ldots$ are derivations of the minor premisses of R' if they exist.

13. The set of derivations is closed under $\triangleright$.

Remark 3.7. We could consider some other reduction conditions that can reduce a redundant derivation to a simpler derivation. The following are examples of such conditions. However, in this study, we do not introduce these conditions. If we did, we would have to appropriately change the notion of normal form according to these additional reduction conditions.

1. R is (EXP) and γ is in the premisses of (EXM):

$$\dfrac{\dfrac{[\mathrm{X}^i\neg\alpha]^1 \quad \overset{\vdots\;\mathcal{D}_1}{\mathrm{X}^i\alpha}}{\gamma}\,(\mathrm{EXP}) \qquad \dfrac{\overset{\vdots\;\mathcal{D}_2}{\mathrm{X}^i\neg\alpha} \quad [\mathrm{X}^i\alpha]^1}{\gamma}\,(\mathrm{EXP})}{\gamma}\,(\mathrm{EXM})^1$$

$$\triangleright \qquad \dfrac{\overset{\vdots\;\mathcal{D}_2}{\mathrm{X}^i\neg\alpha} \quad \overset{\vdots\;\mathcal{D}_1}{\mathrm{X}^i\alpha}}{\gamma}\,(\mathrm{EXP}).$$

2. R is (EXM), γ is the minor premiss of (EXP), and $\neg\gamma$ is the conclusion of (EXP):

$$\dfrac{\dfrac{\overset{[\mathrm{X}^i\neg\alpha]}{\overset{\vdots\;\mathcal{D}_1}{\gamma}} \quad \overset{[\mathrm{X}^i\alpha]}{\overset{\vdots\;\mathcal{D}_2}{\gamma}}}{\gamma}\,(\mathrm{EXM}) \qquad \overset{\vdots\;\mathcal{E}}{\neg\gamma}}{\neg\gamma}\,(\mathrm{EXP}) \qquad \triangleright \qquad \overset{\vdots\;\mathcal{E}}{\neg\gamma}.$$

Definition 3.8. If $\mathcal{D}'$ is obtained from $\mathcal{D}$ by the reduction relation defined in Definition 3.6 then this fact is denoted by $\mathcal{D} \triangleright \mathcal{D}'$. A sequence $\mathcal{D}_0, \mathcal{D}_1, \ldots$ of derivations is called a *reduction sequence* if it satisfies the following conditions:

1. $\mathcal{D}_i \triangleright \mathcal{D}_{i+1}$ for all $i \geq 0$,
2. the last derivation in the sequence is normal if the sequence is finite.

A derivation $\mathcal{D}$ is called *normalizable* if there is a finite reduction sequence starting from $\mathcal{D}$.

3.2. Equivalence and normalization

In the following discussion, a derivation of $\Gamma \Rightarrow$ in $\mathrm{G3cLT}_\omega$ is interpreted as a derivation $\mathcal{D}$ in NLT_ω such that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \neg p \wedge p$.

Definition 3.9. A multiset Δ of formulas is called a *multiset reduct* of a multiset Γ of formulas if Δ is obtained from Γ by multiplying formulas in Γ, where zero multiplicity is also permitted.[14] For example, $\{\alpha, \alpha, \alpha, \beta\}$ is a multiset reduct of $\{\alpha, \beta, \gamma\}$. Note that the relation of being a multiset reduct is reflexive and transitive.

We use an expression Γ^* to denote a multiset reduct of Γ. We also use an expression $\Gamma \subseteq^* \Delta$ to denote the fact that Γ is a multiset reduct of Δ.

Lemma 3.10. *The following hold:*

1. *If $\mathcal{D}$ is a derivation in NLT_ω such that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \beta$, then $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \beta$,*
2. *If $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \beta$, then we can obtain a derivation $\mathcal{D}'$ in NLT_ω such that*
   (a) $\mathrm{oa}(\mathcal{D}') \subseteq^* \Gamma$,
   (b) $\mathrm{end}(\mathcal{D}') = \beta$,
   (c) *$\mathcal{D}'$ is normal.*

Proof:

1. We prove this by induction on the height of the derivations $\mathcal{D}$ of NLT_ω such that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \beta$. We distinguish the cases according to the last rule of $\mathcal{D}$. We show some cases.

   (a) Case ($\rightarrow$I): This case is divided into three cases.

[14] The notion of multiset reduct was introduced in [40, pp. 1805, Definition 1]. It is sometimes thought that natural deduction would not be able to express the rule of weakening and therefore derivability in natural deduction is defined as: γ is derivable from Γ if there is a derivation with open assumptions contained in Γ. Instead of this, the notion of multiset reduct was introduced.

i. $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\ \Gamma\\ \vdots\ \mathcal{E}\\ \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i(\alpha{\rightarrow}\gamma)}\ ({\rightarrow}\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \{\mathrm{X}^i\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypothesis, we have $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma$. Then, we obtain that $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i(\alpha{\rightarrow}\gamma)$:

$$\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\,hyp.}\\ \mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma\end{array}}{\Gamma \Rightarrow \mathrm{X}^i(\alpha{\rightarrow}\gamma)}\ ({\rightarrow}\mathrm{right}).$$

ii. $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\ \mathcal{E}\\ \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i(\alpha{\rightarrow}\gamma)}\ ({\rightarrow}\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypothesis, we have $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i\gamma$. Then, we obtain that $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i(\alpha{\rightarrow}\gamma)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\,hyp.}\\ \Gamma \Rightarrow \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma}\ (\text{we-left})}{\Gamma \Rightarrow \mathrm{X}^i(\alpha{\rightarrow}\gamma)}\ ({\rightarrow}\mathrm{right})$$

where (we-left) is admissible in $\mathrm{G3cLT}_\omega$ by Proposition 2.8.

iii. $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\alpha, \mathrm{X}^i\alpha]\ \Gamma \\ \vdots\ \mathcal{E} \\ \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i(\alpha{\to}\gamma)}\ ({\to}\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \{\mathrm{X}^i\alpha, \mathrm{X}^i\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypothesis, we have $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\alpha, \mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma$. Then, by applying an admissible step of contraction we obtain that $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i(\alpha{\to}\gamma)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\ hyp.} \\ \mathrm{X}^i\alpha, \mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma\end{array}}{\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\gamma}\ (\text{co-left})}{\Gamma \Rightarrow \mathrm{X}^i(\alpha{\to}\gamma)}\ ({\to}\text{right})$$

where (co-left) is admissible in $\mathrm{G3cLT}_\omega$ by Lemma 2.10. All the other cases where multiple discharge of assumptions is used in natural deduction are handled in a similar way via admissible contraction steps and we shall not detail them further.

(b) Case $(\neg\mathrm{I})$: $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]\Gamma_1 \\ \vdots\ \mathcal{D}_1 \\ \mathrm{X}^j\neg\gamma\end{array} \qquad \begin{array}{c}[\mathrm{X}^i\alpha]\Gamma_2 \\ \vdots\ \mathcal{D}_2 \\ \mathrm{X}^j\gamma\end{array}}{\mathrm{X}^i\neg\alpha}\ (\neg\mathrm{I})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \mathrm{X}^i\neg\alpha$. By induction hypotheses, we have $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma_1 \Rightarrow \mathrm{X}^j\neg\gamma$ and $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \mathrm{X}^j\gamma$. Then, we obtain that $\mathrm{G3cLT}_\omega \vdash \Gamma_1, \Gamma_2 \Rightarrow \mathrm{X}^i\neg\alpha$:

$$
\dfrac{
\begin{array}{c} \vdots\ \textit{Ind. hyp.} \\ \mathrm{X}^i\alpha, \Gamma_1 \Rightarrow \mathrm{X}^j\neg\gamma \end{array}
\qquad
\dfrac{\dfrac{\begin{array}{c} \vdots\ \textit{Ind. hyp.} \\ \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \mathrm{X}^j\gamma \end{array}}{\mathrm{X}^i\neg\gamma, \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \mathrm{X}^j\gamma}\ (\text{we-left})}{\mathrm{X}^j\neg\gamma, \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow}\ (\neg\text{left})
}{
\dfrac{\dfrac{\mathrm{X}^i\alpha, \mathrm{X}^i\alpha, \Gamma_1, \Gamma_2 \Rightarrow}{\mathrm{X}^i\alpha, \Gamma_1, \Gamma_2 \Rightarrow}\ (\text{co-left})}{\Gamma_1, \Gamma_2 \Rightarrow \mathrm{X}^i\neg\alpha}\ (\neg\text{right})
}\ (\text{cut})
$$

where (we-left) and (co-left) are admissible in $\mathrm{G3cLT}_\omega$ by Proposition 2.8 and Lemma 2.10, respectively.

(c) Case (EXP): $\mathcal{D}$ is of the form:

$$
\dfrac{\begin{array}{c} \Gamma_1 \\ \vdots\ \mathcal{E}_1 \\ \mathrm{X}^i\neg\alpha \end{array} \qquad \begin{array}{c} \Gamma_2 \\ \vdots\ \mathcal{E}_2 \\ \mathrm{X}^i\alpha \end{array}}{\beta}\ (\text{EXP})
$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \beta$. By induction hypotheses, we have $\mathrm{G3cLT}_\omega \vdash \Gamma_1 \Rightarrow \mathrm{X}^i\neg\alpha$ and $\mathrm{G3cLT}_\omega \vdash \Gamma_2 \Rightarrow \mathrm{X}^i\alpha$. Then, we obtain the required fact that $\mathrm{G3cLT}_\omega \vdash \Gamma_1, \Gamma_2 \Rightarrow \beta$:

$$
\dfrac{\dfrac{
\begin{array}{c} \vdots\ \textit{Ind. hyp.} \\ \Gamma_2 \Rightarrow \mathrm{X}^i\alpha \end{array}
\qquad
\dfrac{
\begin{array}{c} \vdots\ \textit{Ind. hyp.} \\ \Gamma_1 \Rightarrow \mathrm{X}^i\neg\alpha \end{array}
\qquad
\dfrac{\begin{array}{c} \vdots\ \textit{Prop. } 2.7 \\ \mathrm{X}^i\neg\alpha, \mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha \end{array}}{\mathrm{X}^i\neg\alpha, \mathrm{X}^i\alpha \Rightarrow}\ (\neg\text{left})
}{\mathrm{X}^i\alpha, \Gamma_1 \Rightarrow}\ (\text{cut})
}{\Gamma_1, \Gamma_2 \Rightarrow}\ (\text{cut})}{\Gamma_1, \Gamma_2 \Rightarrow \beta}\ (\text{we-right})
$$

where (cut) is admissible in $\mathrm{G3cLT}_\omega$ by Theorem 2.12.

(d) Case (EXM): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\neg\alpha]\Gamma_1\\ \vdots\ \mathcal{E}_1\\ \gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\Gamma_2\\ \vdots\ \mathcal{E}_2\\ \gamma\end{array}}{\gamma}\ (\mathrm{EXM})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma_1 \cup \Gamma_2$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypotheses, we have $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\neg\alpha, \Gamma_1 \Rightarrow \gamma$ and $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \gamma$. Then, we obtain the required fact that $\mathrm{G3cLT}_\omega \vdash \Gamma_1, \Gamma_2 \Rightarrow \gamma$:

$$\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\,hyp.}\\ \mathrm{X}^i\neg\alpha, \Gamma_1 \Rightarrow \gamma\\ \vdots\ (\text{we-left})\\ \mathrm{X}^i\neg\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathit{Ind.\,hyp.}\\ \mathrm{X}^i\alpha, \Gamma_2 \Rightarrow \gamma\\ \vdots\ (\text{we-left})\\ \mathrm{X}^i\alpha, \Gamma_1, \Gamma_2 \Rightarrow \gamma\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \gamma}\ (\text{ex-middle})$$

where (we-left) is admissible in $\mathrm{G3cLT}_\omega$ by Proposition 2.8.

(e) Case (GI): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma_j\\ \vdots\ P_j\\ \{\ \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}\end{array}}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{GI})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma = \bigcup_{j\in\omega} \Gamma_j$ and $\mathrm{end}(\mathcal{D}) = \mathrm{X}^i\mathrm{G}\alpha$. By induction hypotheses, we have $\mathrm{G3cLT}_\omega \vdash \Gamma_j \Rightarrow \mathrm{X}^{i+j}\alpha$ for all $j \in \omega$. Then, we obtain the required fact $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha$:

$$\dfrac{\dfrac{\dfrac{\vdots\ \textit{Ind. hyp.}}{\Gamma_j \Rightarrow \mathrm{X}^{i+j}\alpha}}{\vdots\ \text{(we-left)}}\quad \{\ \Gamma \Rightarrow \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha}\ \text{(Gright)}$$

where (we-left) is admissible in $\mathrm{G3cLT}_\omega$ by Proposition 2.8. Note that the induction hypothesis is applied for each of the denumerable set of premisses.

(f) Case (GE): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\ \mathcal{D}'\\ \mathrm{X}^i\mathrm{G}\alpha\end{matrix}}{\mathrm{X}^{i+k}\alpha}\ \text{(GE)}$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \mathrm{X}^{i+k}\alpha$. By induction hypothesis, we have $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha$. Then, we obtain the required fact that $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^{i+k}\alpha$:

$$\dfrac{\begin{matrix}\vdots\ \textit{Ind. hyp.}\\ \Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\ \textit{Prop. 2.7}\\ \mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^{i+k}\alpha \Rightarrow \mathrm{X}^{i+k}\alpha\end{matrix}}{\mathrm{X}^i\mathrm{G}\alpha \Rightarrow \mathrm{X}^{i+k}\alpha}\ \text{(Gleft)}}{\Gamma \Rightarrow \mathrm{X}^{i+k}\alpha}\ \text{(cut)}$$

where (cut) is admissible in $\mathrm{G3cLT}_\omega$ by Theorem 2.12.

(g) Case (FI): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\ \mathcal{D}'\\ \mathrm{X}^{i+k}\alpha\end{matrix}}{\mathrm{X}^i\mathrm{F}\alpha}\ \text{(FI)}$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \mathrm{X}^i\mathrm{F}\alpha$. By induction hypothesis, we have $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^{i+k}\alpha$. Then, we obtain the required fact that $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha$:

$$\dfrac{\vdots\ \mathit{Ind.\,hyp.} \atop \Gamma \Rightarrow \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha}\ (\mathrm{Fright}).$$

(h) Case (FE):
$\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\Gamma' \\ \vdots\ \mathcal{D}' \\ \mathrm{X}^i\mathrm{F}\alpha\end{array} \quad \left\{ \begin{array}{c}[\mathrm{X}^{i+j}\alpha]\Gamma_j \\ \vdots\ \mathcal{D}_j \\ \gamma\end{array} \right\}_{j\in\omega}}{\gamma}\ (\mathrm{FE})$$

where $\mathrm{oa}(\mathcal{D}) = \Gamma' \cup \Gamma$ with $\Gamma = \bigcup_{j\in\omega} \Gamma_j$ and $\mathrm{end}(\mathcal{D}) = \gamma$. By induction hypotheses, we have $\mathrm{G3cLT}_\omega \vdash \Gamma' \Rightarrow \mathrm{X}^i\mathrm{F}\alpha$ and $\mathrm{G3cLT}_\omega \vdash \mathrm{X}^{i+j}\alpha, \Gamma_j \Rightarrow \gamma$ for all $j \in \omega$. Then, we obtain the required fact, that $\mathrm{G3cLT}_\omega \vdash \Gamma', \Gamma \Rightarrow \gamma$ by the following derivation where the induction hypothesis is applied for each of the denumerable set of premisses:

$$\dfrac{\begin{array}{c}\vdots\ \mathit{Ind.\,hyp.} \\ \Gamma' \Rightarrow \mathrm{X}^i\mathrm{F}\alpha\end{array} \quad \dfrac{\left\{ \begin{array}{c}\vdots\ \mathit{Ind.\,hyp.} \\ \mathrm{X}^{i+j}\alpha, \Gamma_j \Rightarrow \gamma \\ \vdots\ (\text{we-left}) \\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \gamma\end{array} \right\}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \gamma}\ (\mathrm{Fleft})}{\Gamma', \Gamma \Rightarrow \gamma}\ (\mathrm{cut})$$

where (cut) and (we-left) are admissible in $\mathrm{G3cLT}_\omega$ by Theorem 2.12 and Proposition 2.8, respectively.

2. We prove this by induction on the derivations $\mathcal{D}$ of $\Gamma \Rightarrow \beta$ in $\mathrm{G3cLT}_\omega$.

We distinguish the cases according to the last rule of $\mathcal{D}$. We show some cases.

(a) Case (init): $\mathcal{D}$ is of the form:

$$\begin{array}{c} \vdots\ \mathcal{D} \\ \mathrm{X}^i p, \Gamma \Rightarrow \mathrm{X}^i p. \end{array}$$

Then, we have a normal derivation $\mathcal{E}$ in NLT_ω of the form:

$$\begin{array}{c} \vdots\ \mathcal{E} \\ \mathrm{X}^i p \end{array}$$

where $\mathrm{oa}(\mathcal{E}) = \{\mathrm{X}^i p\} \subseteq^* \{\mathrm{X}^i p\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}) = \mathrm{X}^i p$.

(b) Case (we-right): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \vdots\ \mathcal{D}' \\ \Gamma \Rightarrow \end{array}}{\Gamma \Rightarrow \alpha}\ (\text{we-right})$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in NLT_ω of the form:

$$\begin{array}{c} \Gamma^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p \end{array}$$

where $\mathrm{oa}(\mathcal{E}') = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}') = \neg p \wedge p$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\dfrac{\begin{array}{c} \Gamma^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p \end{array}}{\neg p}\ (\wedge\text{E1}) \quad \dfrac{\begin{array}{c} \Gamma^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p \end{array}}{p}\ (\wedge\text{E2})}{\alpha}\ (\text{Exp})$$

where $\mathrm{oa}(\mathcal{E}) = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}) = \alpha$.

(c) Case (¬left): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \mathrm{X}^i\alpha\end{array}}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow}\ (\neg\text{left}).$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in NLT_ω of the form:

$$\begin{array}{c}(\mathrm{X}^i\neg\alpha, \Gamma)^*\\ \vdots\ \mathcal{E}'\\ \mathrm{X}^i\alpha\end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\mathrm{X}^i\neg\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\neg\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \mathrm{X}^i\alpha$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\mathrm{X}^i\neg\alpha \qquad \begin{array}{c}(\mathrm{X}^i\neg\alpha, \Gamma)^*\\ \vdots\ \mathcal{E}'\\ \mathrm{X}^i\alpha\end{array}}{\neg p\wedge p}\ (\text{EXP})$$

where $\mathrm{oa}(\mathcal{E}) = (\mathrm{X}^i\neg\alpha, \mathrm{X}^i\neg\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\neg\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}) = \neg p\wedge p$ (i.e., $\bot$). We remark that $\{\mathrm{X}^i\neg\alpha, \mathrm{X}^i\neg\alpha\} \cup \Gamma$ is a multiset reduct of $\{\mathrm{X}^i\neg\alpha\} \cup \Gamma$. We also remark that the last rule (EXP) in $\mathcal{E}$ cannot be replaced with (→E), because using (→E) entails a possibility of developing a non-normal derivation. Namely, there is a possibility of the case that the last rule of $\mathcal{E}'$ is (→I).

(d) Case (¬right): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \mathrm{X}^i\alpha, \Gamma \Rightarrow\end{array}}{\Gamma \Rightarrow \mathrm{X}^i\neg\alpha}\ (\neg\text{right}).$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in NLT_ω of the form:

$$\begin{array}{c}(\mathrm{X}^i\alpha,\Gamma)^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p\end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\mathrm{X}^i\alpha,\Gamma)^* \subseteq^* \{\mathrm{X}^i\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \neg p \wedge p$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]^1\ \Gamma^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p\end{array}}{\neg p}\ (\wedge\mathrm{E}1) \qquad \dfrac{\begin{array}{c}[\mathrm{X}^i\alpha]^1\ \Gamma^* \\ \vdots\ \mathcal{E}' \\ \neg p \wedge p\end{array}}{p}\ (\wedge\mathrm{E}2)}{\mathrm{X}^i\neg\alpha}\ (\neg\mathrm{I})^1$$

where $\mathrm{oa}(\mathcal{E}) = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}) = \mathrm{X}^i\neg\alpha$.

(e) Case (ex-middle): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1 \\ \mathrm{X}^i\neg\alpha,\Gamma \Rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2 \\ \mathrm{X}^i\alpha,\Gamma \Rightarrow \gamma\end{array}}{\Gamma \Rightarrow \gamma}\ (\text{ex-middle}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_1$ and $\mathcal{E}_2$ in NLT_ω of the form:

$$\begin{array}{c}(\mathrm{X}^i\neg\alpha,\Gamma)^* \\ \vdots\ \mathcal{E}_1 \\ \gamma\end{array} \qquad \begin{array}{c}(\mathrm{X}^i\alpha,\Gamma)^* \\ \vdots\ \mathcal{E}_2 \\ \gamma\end{array}$$

where $\mathrm{oa}(\mathcal{E}_1) = (\mathrm{X}^i\neg\alpha,\Gamma)^* \subseteq^* \{\mathrm{X}^i\neg\alpha\} \cup \Gamma$, $\mathrm{oa}(\mathcal{E}_2) = (\mathrm{X}^i\alpha,\Gamma)^* \subseteq^* \{\mathrm{X}^i\alpha\} \cup \Gamma$, $\mathrm{end}(\mathcal{E}_1) = \gamma$, and $\mathrm{end}(\mathcal{E}_2) = \gamma$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\begin{array}{c}[\mathrm{X}^i\neg\alpha]\Gamma^* \\ \vdots\ \mathcal{E}_1 \\ \gamma\end{array} \quad \begin{array}{c}[\mathrm{X}^i\alpha]\Gamma^* \\ \vdots\ \mathcal{E}_2 \\ \gamma\end{array}}{\gamma}\ (\mathrm{EXM})$$

where $\mathrm{oa}(\mathcal{E}) = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}) = \gamma$.

(f) Case (Gleft): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \gamma\end{array}}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \gamma}\ \text{(Gleft)}.$$

By induction hypothesis, we have a normal derivation $\mathcal{E}'$ in NLT_ω of the form:

$$\begin{array}{c}(\mathrm{X}^i\mathrm{G}\alpha \quad \mathrm{X}^{i+k}\alpha \quad \Gamma)^*\\ \vdots\ \mathcal{E}'\\ \gamma\end{array}$$

where $\mathrm{oa}(\mathcal{E}') = (\mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^{i+k}\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^{i+k}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}') = \gamma$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\begin{array}{ccc}(\mathrm{X}^i\mathrm{G}\alpha & \dfrac{\mathrm{X}^i\mathrm{G}\alpha}{\mathrm{X}^{i+k}\alpha}\ \text{(GE)} & \Gamma)^*\\ \vdots & & \vdots\\ & \vdots\ \mathcal{E}' & \\ & \gamma & \end{array}$$

where $\mathrm{oa}(\mathcal{E}) = (\mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}^i\mathrm{G}\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\mathrm{G}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}) = \gamma$.

(g) Case (Gright): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \{\ \Gamma \Rightarrow \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}\end{array}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha}\ \text{(Gright)}.$$

By induction hypotheses, we have normal derivations $\mathcal{E}_j$ for all $j \in \omega$ in NLT$_\omega$ of the form:

$$\begin{array}{c} \Gamma_j^* \\ \vdots\ \mathcal{E}_j \\ \mathrm{X}^{i+j}\alpha \end{array}$$

where $\mathrm{oa}(\mathcal{E}_j) = \Gamma_j^* \subseteq^* \Gamma_j$ with $\Gamma^* = \bigcup_{j\in\omega} \Gamma_j^* \subseteq^* \Gamma$, and $\mathrm{end}(\mathcal{E}_j) = \mathrm{X}^{i+j}\alpha$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\begin{array}{c} \Gamma_j^* \\ \vdots\ \mathcal{E}_j \\ \{\ \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega} \end{array}}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{GI})$$

where $\mathrm{oa}(\mathcal{E}) = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}) = \mathrm{X}^i\mathrm{G}\alpha$.

(h) Case (Fleft): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c} \vdots\ \mathcal{D}' \\ \{\ \mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \gamma\ \}_{j\in\omega} \end{array}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \gamma}\ (\mathrm{Fleft}).$$

By induction hypotheses, we have normal derivations $\mathcal{E}_j$ for all $j \in \omega$ in NLT$_\omega$ of the form:

$$\begin{array}{c} (\mathrm{X}^{i+j}\alpha \quad \Gamma_j)^* \\ \vdots\ \mathcal{E}_j \\ \gamma \end{array}$$

where $\mathrm{oa}(\mathcal{E}_j) = (\mathrm{X}^{i+j}\alpha, \Gamma_j)^* \subseteq^* \{\mathrm{X}^{i+j}\alpha\} \cup \Gamma_j$ with $\Gamma^* = \bigcup_{j\in\omega} \Gamma_j^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}_j) = \gamma$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\mathrm{X}^i\mathrm{F}\alpha \quad \left\{ \begin{matrix} [\mathrm{X}^{i+j}\alpha]^1 \quad \Gamma_j^* \\ \vdots\ \mathcal{E}_j \\ \gamma \end{matrix} \right\}_{j\in\omega}}{\gamma}\ (\mathrm{FE})^1$$

where $\mathrm{oa}(\mathcal{E}) = (\mathrm{X}^i\mathrm{F}\alpha, \Gamma)^* \subseteq^* \{\mathrm{X}^i\mathrm{F}\alpha\} \cup \Gamma$ and $\mathrm{end}(\mathcal{E}) = \gamma$.

(i) Case (Fright): $\mathcal{D}$ is of the form:

$$\dfrac{\begin{matrix} \vdots\ \mathcal{D}' \\ \Gamma \Rightarrow \mathrm{X}^{i+k}\alpha \end{matrix}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{F}\alpha}\ (\mathrm{Fright}).$$

By induction hypothesis, we have a normal derivations $\mathcal{E}'$ in NLT_ω of the form:

$$\begin{matrix} \Gamma^* \\ \vdots\ \mathcal{E}' \\ \mathrm{X}^{i+k}\alpha \end{matrix}$$

where $\mathrm{oa}(\mathcal{E}') = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}') = \mathrm{X}^{i+k}\alpha$. Then, we obtain a required normal derivation $\mathcal{E}$ by:

$$\dfrac{\begin{matrix} \Gamma^* \\ \vdots\ \mathcal{E}' \\ \mathrm{X}^{i+k}\alpha \end{matrix}}{\mathrm{X}^i\mathrm{F}\alpha}\ (\mathrm{FI})$$

where $\mathrm{oa}(\mathcal{E}) = \Gamma^* \subseteq^* \Gamma$ and $\mathrm{end}(\mathcal{E}) = \mathrm{X}^i\mathrm{F}\alpha$. □

We obtain the following required theorems.

Theorem 3.11 (Equivalence between NLT_ω and $\mathrm{G3cLT}_\omega$). *For any formula α, $\mathrm{G3cLT}_\omega \vdash\ \Rightarrow \alpha$ iff α is derivable in NLT_ω.*

PROOF: Taking $\emptyset$ as Γ in Lemma 3.10, we obtain the required fact. □

THEOREM 3.12 (Normalization for NLT_ω). *All derivations in* NLT_ω *are normalizable. More precisely, if a derivation* $\mathcal{D}$ *in* NLT_ω *is given, then we can obtain a normal derivation* $\mathcal{E}$ *in* NLT_ω *such that* $\mathrm{oa}(\mathcal{E}) \subseteq^* \mathrm{oa}(\mathcal{D})$ *and* $\mathrm{end}(\mathcal{E}) = \mathrm{end}(\mathcal{D})$.

PROOF: Suppose that a derivation $\mathcal{D}$ in NLT_ω is given, and suppose that $\mathrm{oa}(\mathcal{D}) = \Gamma$ and $\mathrm{end}(\mathcal{D}) = \beta$. Then, by Lemma 3.10 (1), we obtain $\mathrm{G3cLT}_\omega \vdash \Gamma \Rightarrow \beta$. Then, by Lemma 3.10 (2), we can obtain a normal derivation $\mathcal{E}$ in NLT_ω such that $\mathrm{oa}(\mathcal{E}) \subseteq^* \mathrm{oa}(\mathcal{D})$ and $\mathrm{end}(\mathcal{E}) = \mathrm{end}(\mathcal{D})$. □

4. Intuitionistic variant

4.1. Sequent calculus and cut elimination

The language of $\mathrm{G3iLT}_\omega$ is the same as defined in Section 2 for $\mathrm{G3cLT}_\omega$. A sequent for $\mathrm{G3iLT}_\omega$ is an expression of the form $\Gamma \Rightarrow \gamma$ where Γ is a (possibly empty) multiset of formulas and γ is a formula or the empty multiset. The same notions and notations as introduced and presented in Section 2 are used for $\mathrm{G3iLT}_\omega$.

We now define $\mathrm{G3iLT}_\omega$.

DEFINITION 4.1 ($\mathrm{G3iLT}_\omega$). $\mathrm{G3iLT}_\omega$ is obtained from $\mathrm{G3cLT}_\omega$ by deleting (ex-middle).

We have the following propositions.

PROPOSITION 4.2. The sequents of the form $\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\alpha$ for any formula α, any set Γ of formulas, and any natural number i are derivable in $\mathrm{G3iLT}_\omega$.

PROOF: Similar to the proof of Proposition 2.7. By induction on α. □

PROPOSITION 4.3. Rule (we-left) is height-preserving admissible in $\mathrm{G3iLT}_\omega$.

PROOF: Similar to the proof of Proposition 2.8. □

PROPOSITION 4.4. Rule (co-left) is height-preserving admissible in $\mathrm{G3iLT}_\omega$.

PROOF: Similar to the proof of Proposition 2.10. □

Theorem 4.5. *The rule* (cut) *is admissible in* $\mathrm{G3iLT}_\omega$.

Proof: Similar to the proof of Theorem 2.12. □

We also obtain the following constructive property for $\mathrm{G3iLT}_\omega$.

Theorem 4.6 (Timed disjunction property for $\mathrm{G3iLT}_\omega$). *For any formulas* α *and* β *and any* $i \in \omega$, *if* $\mathrm{G3iLT}_\omega \vdash\ \Rightarrow \mathrm{X}^i(\alpha \vee \beta)$, *then either* $\mathrm{G3iLT}_\omega \vdash\ \Rightarrow \mathrm{X}^i\alpha$ *or* $\mathrm{G3iLT}_\omega \vdash\ \Rightarrow \mathrm{X}^i\beta$.

Proof: Immediate by Theorem 4.5. □

4.2. Natural deduction and normalization

The same notions and notations as introduced and presented in Section 3 are used for NILT_ω.

First, we define NILT_ω.

Definition 4.7 (NILT_ω). NILT_ω is obtained from NLT_ω by deleting (EXM).

Next, we define the reduction relation on NILT_ω.

Definition 4.8 (Reduction relation). The definition of the reduction relation on NILT_ω is obtained form Definition 3.6 in NLT_ω by deleting the conditions concerning (EXM).

We then obtain the following lemma.

Lemma 4.9. *We have the following statements.*

1. *If* $\mathcal{D}$ *is a derivation in* NILT_ω *such that* $\mathrm{oa}(\mathcal{D}) = \Gamma$ *and* $\mathrm{end}(\mathcal{D}) = \beta$, *then* $\mathrm{G3iLT}_\omega \vdash \Gamma \Rightarrow \beta$,
2. *If* $\mathrm{G3iLT}_\omega \vdash \Gamma \Rightarrow \beta$, *then we can obtain a derivation* $\mathcal{D}'$ *in* NILT_ω *such that*
   (a) $\mathrm{oa}(\mathcal{D}') \subseteq^* \Gamma$,
   (b) $\mathrm{end}(\mathcal{D}') = \beta$,
   (c) $\mathcal{D}'$ *is normal.*

Proof: Similar to the proof of Lemma 3.10. □

We then obtain the following required theorems.

THEOREM 4.10 (Equivalence between NILT_ω and $\mathrm{G3iLT}_\omega$). *For any formula α, $\mathrm{G3iLT}_\omega \vdash \Rightarrow \alpha$ iff α is derivable in NILT_ω.*

PROOF: Similar to the proof of Theorem 3.11. We use Lemma 4.9. □

THEOREM 4.11 (Normalization for NILT_ω). *All derivations in NILT_ω are normalizable. More precisely, if a derivation $\mathcal{D}$ in NILT_ω is given, then we can obtain a normal derivation $\mathcal{E}$ in NILT_ω such that $\mathrm{oa}(\mathcal{E}) \subseteq^* \mathrm{oa}(\mathcal{D})$ and $\mathrm{end}(\mathcal{E}) = \mathrm{end}(\mathcal{D})$.*

PROOF: Similar to the proof of Theorem 3.12. We use Lemma 4.9. □

5. Conclusion and remarks

5.1. Conclusion

In this study, we introduced a unified Gentzen-style proof-theoretic framework for until-free propositional linear-time temporal logic (LTL) and its intuitionistic variant.

First, we proposed the Gentzen-style single-succedent sequent calculus $\mathrm{G3cLT}_\omega$ for until-free propositional LTL. Subsequently, we proved the cut-elimination theorem for $\mathrm{G3cLT}_\omega$ following the methodology for G3-style sequent calculi with infinitary rules, as in [39].

Second, we introduced the Gentzen-style natural deduction system NLT_ω for until-free propositional LTL, along with a reduction relation for NLT_ω. Following this, we established the normalization theorem for NLT_ω by utilizing the equivalence theorem between NLT_ω and $\mathrm{G3cLT}_\omega$.

Third, we introduced and investigated a Gentzen-style sequent calculus, $\mathrm{G3iLT}_\omega$, and a Gentzen-style natural deduction system, NILT_ω, for an intuitionistic variant of the until-free propositional LTL. The systems $\mathrm{G3iLT}_\omega$ and NILT_ω are derived from $\mathrm{G3cLT}_\omega$ and NLT_ω by omitting the rules (ex-middle) and (EXM), respectively. The cut-elimination theorem for $\mathrm{G3iLT}_\omega$ is then immediate as a subcase of the cut-elimination theorem for $\mathrm{G3cLT}_\omega$. Subsequently, we established the normalization theorem for NILT_ω by utilizing the equivalence theorem between NILT_ω and $\mathrm{G3iLT}_\omega$.

5.2. Remarks on the merits of our approach

We now highlight the merits of our approach. In particular, we emphasize the advantages of the proposed infinitary systems, which incorporate logical inference rules with infinitely many premises. These systems exhibit three key features: uniformity, modularity, and compatibility.

Regarding uniformity, by employing inference rules with infinitely many premises, we can treat both Gentzen-style sequent calculi and Gentzen-style natural deduction systems in a uniform manner. In particular, we establish a natural correspondence between the Gentzen-style single-succedent sequent calculi G3cLT$_\omega$ and G3iLT$_\omega$ and the Gentzen-style natural deduction systems NLT$_\omega$ and NILT$_\omega$, respectively.

Regarding modularity, the systems with infinitary rules can be extended in a modular way. In particular, G3cLT$_\omega$ and NLT$_\omega$ are obtained from G3iLT$_\omega$ and NILT$_\omega$ simply by adding the rules *(ex-middle)* and *(EXM)*, respectively. This modularity, together with uniformity, is a distinctive advantage not available in previously proposed systems.

By using rules with infinitely many premises, we also gain an advantage in establishing smoothly a Glivenko theorem for G3cLT$_\omega$ and G3iLT$_\omega$. This result is an analogue of the Glivenko theorem for Gentzen's LK and LI in classical and intuitionistic logics. This theorem is formally presented as follows: For any formula α, G3cLT$_\omega$ $\vdash\ \Rightarrow \alpha$ if and only if G3iLT$_\omega$ $\vdash\ \Rightarrow \neg\neg\alpha$. The proof of this theorem can be given in a similar way as presented in [22].

In addition to these merits, by using rules with infinitely many premises, we can obtain certain theorems for embedding G3cLT$_\omega$ and G3iLT$_\omega$ into a Gentzen-style sequent calculus LK$_\omega$ for infinitary classical logic and a Gentzen-style sequent calculus LI$_\omega$ for infinitary intuitionistic logic, respectively. These theorems can be proved in the same way as presented in [19, 23].

Further, we can construct finite fragments of G3cLT$_\omega$ and G3iLT$_\omega$, in which the infinite domain ω of rules with infinitely many premises is restricted to a finite domain $\omega_l = \{n \in \omega \mid n \leq l\}$ for a fixed positive integer l. A system based on LT$_\omega$ of this kind was studied, for example, in [21]. These systems have been shown to be embeddable into LK or LI, and hence are decidable.

The above-mentioned merits imply that our framework is highly compatible with the traditional frameworks of classical logic, intuitionistic logic, infinitary classical logic, and infinitary intuitionistic logic. This naturally extends the traditional proof theory for these standard logics. This was the basic aim of this study.

5.3. Remarks on next-time fragments

The next-time fragments (i.e., the $\{G, F\}$-less fragments) of the proposed systems possess several desirable properties. To fix the terminology, let XT_ω, SXT_ω, $SIXT_\omega$, NXT_ω, and $NIXT_\omega$ denote the next-time fragments of LT_ω, $G3cLT_\omega$, $G3iLT_\omega$, NLT_ω, and $NILT_\omega$, respectively.

Then, the cut-elimination theorems for XT_ω, SXT_ω, and $SIXT_\omega$ hold by virtue of the cut-elimination theorems for XT_ω, $G3cLT_\omega$, and $G3iLT_\omega$ and their conservativeness. We can demonstrate theorems for embedding XT_ω and $G3iLT_\omega$ into Gentzen-style sequent calculi for classical logic and intuitionistic logic, respectively. Such a Gentzen-style sequent calculus, referred to here as LK, for classical logic is the X-less fragment of XT_ω (i.e., LK is obtained from XT_ω by deleting all occurrences of X^i). Similarly, such a Gentzen-style sequent calculus, referred to here as LI, for intuitionistic logic is the X-less fragment of $SIXT_\omega$ (i.e., LI is obtained from $SIXT_\omega$ by deleting all occurrences of X^i).

The equivalence between NXT_ω (or $NIXT_\omega$) and SXT_ω (or $SIXT_\omega$, respectively) can also be established. The normalization theorems for NXT_ω and $NIXT_\omega$ can be demonstrated similarly to those for NLT_ω and $NILT_\omega$, since NXT_ω and $NIXT_\omega$ are proper subsystems of NLT_ω and $NILT_\omega$, respectively.

As demonstrated above, we can derive the theorems for embedding XT_ω and $SIXT_\omega$ into LK and LI, respectively. By virtue of these theorems, we can also establish the decidability of XT_ω and $SIXT_\omega$, as well as the Craig interpolation theorems for XT_ω and $SIXT_\omega$. These results, based on the embedding theorems into LK and LI, cannot be obtained for LT_ω and $G3iLT_\omega$ because these systems are not embeddable into LK and LI, respectively. Instead, they can be embedded into Gentzen-style sequent calculi

LK_ω and LI_ω for infinitary logic and infinitary intuitionistic logic, respectively, which are known to be undecidable. Additionally, it is well-known that the Craig interpolation theorem does not hold for LTL. For more information on Craig interpolation theorem for the next-time fragment of LTL, see [24]. In conclusion, XT_ω and $SIXT_\omega$ are analogous to classical logic and intuitionistic logic, respectively, while LT_ω and $G3iLT_\omega$ are analogous to infinitary logic and infinitary intuitionistic logic, respectively.

5.4. Related and future works

Gentzen-style sequent calculi and natural deduction systems for some extended intuitionistic variants of until-free propositional LTL with paraconsistent negation were examined by Kamide and Wansing in [31], where the corresponding display sequent calculi were also discussed. Kamide clarified the relationship among until-free propositional LTL, first-order monadic omega-logic, propositional generalized definitional reflection logic, and propositional infinitary logic in [25], using Gentzen-style sequent calculi for the investigation. Recently, Kamide proposed and investigated refutation-aware Gentzen-style sequent calculi for until-free propositional LTL in [27], although their intuitionistic variants and Gentzen-style natural deduction systems were not studied.

Gentzen-style natural deduction systems and related typed λ-calculi for various fragments of LTL and related modal logics have been extensively studied [3, 5, 6, 12, 13, 34, 37, 38, 35, 50, 56, 20] to establish a foundation for staged computation in multi-level programming. Gentzen-style natural deduction systems and sequent calculi for variants of the next-time fragment of LTL were surveyed and investigated in [20], where Davies' logic for binding-time analysis was also discussed.

From an application perspective, Davies [12] proposed a typed λ-calculus λ° (including a next-time operator $\bigcirc$ insted of X) for a fragment of intuitionistic LTL to discuss multi-level binding-time analysis. Taha et al. [50] introduced an extension of λ° called MetaML, which incorporates properties like run-time generation and persistent code. Moggi et al. [37] further developed an extension of MetaML called AIM (an idealized MetaML), and Benaissa et al. [5] proposed a refinement of AIM known as λ^{BN}.

Davies and Pfenning [13] introduced an alternative typed λ-calculus $\lambda^{[]}$ (incorporating an S4-type modal operator $[]$) for intuitionistic S4-modal logic, aimed at analyzing staged computation. Nanevski [38] and Kim et al. [34] explored various type systems based on $\lambda^{[]}$, while Yuse and Igarashi [56] introduced $\lambda^{\circ[]}$, a type system combining λ° and $\lambda^{[]}$, designed to manage both persistent code (using $[]$) and ephemeral code (using $\bigcirc$).

In future work, we aim to prove the strong normalization and Church-Rosser theorems for NLT_{ω} and NILT_{ω}, as well as for their first-order extensions. Additionally, we plan to introduce the corresponding typed λ-calculi for NLT_{ω} and NILT_{ω} with the Curry-Howard correspondence, and to apply these calculi to the analysis of staged computation in multi-level programming.

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