Bulletin of the **S**ection of **L**ogic
Volume 55/1 (2026), pp. 1–47
https://doi.org/10.18778/0138-0680.2026.01

Antonio Ledda*
Hanamantagouda P. Sankappanavar
Gandolfo Vergottini

THE AMALGAMATION PROPERTY IN THE VARIETY OF REGULAR DOUBLE STONE ALGEBRAS: A CONSTRUCTIVE VIEW

Abstract

In this paper we give a constructive proof that the variety of Boolean algebras has the strong amalgamation property by describing constructively the strong amalgams in the variety. Then, capitalizing on this construction, we investigate several forms of amalgamation, such as the strong amalgamation property and Maksimova super-amalgamation for the varieties of regular double Stone algebras and centered regular double Stone algebras. In fact, we prove that the amalgamation property holds for the variety **RDS**. Then, we introduce the variety $\mathbf{RDS}^k$ of centered regular double Stone algebras and prove that $\mathbf{RDS}^k$ enjoys

*Corresponding author.

Presented by: Janusz Ciuciura
Received: February 28, 2025, **Received in revised form**: February 10, 2026,
Accepted: February 11, 2026, **Published online**: April 24, 2026

the strong amalgamation property. It is also proved that the varieties of Boolean algebras and centered regular double Stone algebras have the super-amalgamation property. We close the paper by providing a number of concrete examples and applications to illustrate the theory developed in the paper.

Keywords: Boolean algebras, regular double Stone algebras, Kleene algebras, amalgamation, strong amalgamation, super-amalgamation.

2020 Mathematical Subject Classification: 06A06, 06B05, 06B15, 06B20, 06B23, 06C15, 06D05, 06D15, 06D30, 06D35, 06E05, 06E75.

1. Introduction

In model theory and algebraic logic, there are properties of classes of structures that are "reflected" in the logics that are associated to those classes of structures, thus providing a useful interplay between those classes of algebras and their logics. One such property of paramount importance is the amalgamation property. Under standard algebraizability assumptions, the amalgamation property of a variety is reflected in the Craig's interpolation property of the associated logic; in other words, a variety $\mathbf{V}$ has an amalgamation property if and only if its corresponding logic has the Craig's interpolation property.

In this paper we mostly focus on amalgamation property, strong amalgamation property, and super-amalgamation property for the varieties of Boolean algebras, regular double Stone algebras and centered regular double Stone algebras.

Double Stone algebras are a natural generalization of Boolean algebras. Recall that, in a Boolean algebra, the complement of an element a is characterized both as the greatest element x such that $a \wedge x = 0$ and as the least element y such that $a \vee y = 1$. Dropping one of these two requirements leads to the notions of *pseudocomplement* $^{\sim}$ and *dual pseudocomplement* $^{+}$, which give rise, respectively, to the classes of p-algebras and dual p-algebras. From a logical perspective, this amounts to a splitting of classical negation into two unary operations: $^{\sim}$, which captures a form of negation enforcing non-contradiction, and $^{+}$, which captures a form of negation enforcing the law of excluded middle.

Following an important result of Ribenboim in 1949 [48], it was shown that the class of pseudocomplemented distributive lattices forms in fact an equational class of algebras of type $(2, 2, 1, 0)$ in the language $(\vee, \wedge, ^{\sim}, 0)$. Early on, it was also observed that the identity $x^{\sim} \vee x^{\sim\sim} = 1$, corresponding to the so-called *weak law of excluded middle*, holds in some p-algebras. In fact, already in the mid-1930s, Stone posed the problem of investigating the (sub)class of p-algebras satisfying this identity. In response to Stone's proposal, Grätzer and Schmidt initiated a systematic study of this subclass, which they termed the class of Stone algebras, with their dual counterparts naturally called dual Stone algebras.

As a consequence, the class of Stone algebras, being itself a variety, became the object of intensive investigation. It was then natural to ask what would arise from expanding a Stone algebra by equipping it with the additional structure of a dual Stone algebra. This line of inquiry led to the introduction and subsequent systematic study of the class of DS double Stone algebras. In this paper, we are actually interested in an important subvariety of **DS**, called the variety **RDS** of regular double Stone algebras (see Section 2 for the definition). The variety **RDS** is a subvariety of the variety of regular double p-algebras, the latter was first introduced as a quasi-variety, in 1972, by Varlet [52] in connection with the problem of characterizing the congruence-regular double p-algebras. A little later, Katriňák [25] proved, in 1973, that the regular double p-algebras, indeed, form a variety (see also [14]). Since then, there is a considerable amount of literature on the variety of regular double p-algebras and, in particular, on the variety **RDS**; see e.g. [1, 14] (and the references therein). It is shown in [1] that there are $2^{\aleph_0}$ subvarieties of the variety of regular double p-algebras. Our present paper is a further addition to the already existing rich literature on **RDS**.

It may be worth noticing that the study of regular double Stone algebras provides algebraic tools for modeling uncertainty and partial information. From a logical perspective, regular double Stone algebras generalize classical logic to a substructural logic [17] where the principles of non-contradiction and excluded middle are not valid.

Interestingly, there is a connection between the theory of rough sets due to Pawlak ([43] and [44]) and regular double Stone algebras; for example,[47] and [13] have shown that every regular double Stone algebra is isomorphic to the algebra arising from an approximation space. Thus, regular double Stone algebras provide the interplay between lower and upper approximations in rough set theory. This approach is central to artificial intelligence and cognitive science, with applications in machine learning, knowledge discovery, data mining, expert systems, approximate reasoning, and pattern recognition [43, 44]. Moreover, these structures can be regarded as the algebraic counterpart of three-valued Łukasiewicz logic[7], a paradigmatic system for reasoning under indeterminacy, which itself may be seen as a special case of fuzzy logic [20] with a three-element chain of truth values.

Regular double Stone algebras play a structural role as distributive "sharp" contexts within broader non-classical frameworks, just as Boolean algebras serve as classical blocks in orthomodular lattices. In this sense, they provide natural building blocks for unsharp quantum logics [18, 31, 32].

As mentioned earlier, our first goal in this paper is to provide a novel proof of the strong amalgamation property (AP) for the variety of Boolean algebras. As a second objective, we use it to provide a constructive proof for the amalgamation property for regular double Stone algebras, even though this result is already known (see [15]). Our constructive proof relies on a construction due to Johnstone [24].

The facts that the variety **BA** has the strong amalgamation property (SAP) and the variety **RDS** fails to have (SAP) led us to consider the variety of centered regular double Stone algebras, an expansion of **RDS** by a center. The notion of a center is not new; for example, already in 1940, Moisil [38] introduced it in the context of 3-valued Łukasiewicz algebras. Later, in 1972, Cignoli [11] used it to show that the variety of centered n-valued Łukasiewicz algebras is term equivalent to the variety of Post algebras of order n and Cignoli [12] used it to characterize injective 3-valued Łukasiewicz algebras.

Let $\mathbf{A} \in \mathbf{RDS}$ and let $k \in \mathbf{A}$. We say that k is called a *center* of $\mathbf{A}$ if it satisfies:

$$k^{\sim} = 0 \text{ and } k^{+} = 1. \tag{K}$$

We expand the language $L = (\wedge, \vee, ^{\sim}, ^{+}, 0, 1)$ of regular double Stone algebras, by adding a new constant k to obtain the language:

$$L^k = (\wedge, \vee, ^{\sim}, ^{+}, 0, k, 1),$$

of type $(2, 2, 1, 1, 0, 0, 0)$.

Our third objective is to introduce a new variety of algebras called "centered regular double Stone algebras" in the language L^k, and show that it has the (SAP). In fact, we prove a stronger result: the variety of centered regular double Stone algebras satisfies Maksimova super-amalgamation property.

The introduction of the constant k can be thought of as the explicit algebraic counterpart of an intermediate designated value, that separates the two extremal truth values 0 and 1.[1] This expansion provides a way to capture contexts of uncertainty or indeterminacy that are central in rough set theory, where k corresponds to the boundary region between lower and upper approximations, as well as in three-valued Łukasiewicz logic, where it represents the "undetermined" truth value. From a computational perspective, k can also be interpreted as a marker of error or inconsistency, thus offering a formal tool for distinguishing reliable from unreliable information states.

The paper is structured as follows. In Section 2 we provide all the specific notions from algebra and category theory, that may be expedient for a comprehensive reading of our discourse. In Section 3, we give a constructive proof of the strong amalgamation property for the variety of Boolean algebras by providing a constructive description of the strong amalgams. In Section 4, we use the construction of the strong amalgams of V-formations of Boolean algebras from Section 3 to prove the amalgamation property for the variety **RDS**. In Section 5 we introduce the variety $\mathbf{RDS}^k$ of centered regular double Stone algebras and prove that $\mathbf{RDS}^k$ enjoys the strong amalgamation property. In Section 6 it is shown that the varieties

[1] A paramount example of this approach traces back to the early works of Pavelka on multiple-valued logics [40, 41, 42].

of Boolean algebras and centered regular double Stone algebras have the super-amalgamation property. We conclude the paper by discussing several examples and applications to illustrate the theory we have developed in this paper.

2. Preliminaries

For standard facts about double Stone algebras we refer the reader to Grätzer [19] or Balbes and Dwinger [2].

2.1. Regular double Stone algebras

Definition 2.1 ([19, 2]). An algebra $\mathbf{L} = (L, \wedge, \vee, ^{\sim}, 0, 1)$ is a *p-algebra* if $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $^{\sim}$ satisfies:

$$x \wedge y = 0 \text{ if and only if } x \leq y^{\sim}.$$

Dually, an algebra $\mathbf{L} = (L, \wedge, \vee, ^{+}, 0, 1)$ is a *dual p-algebra* if $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $^{+}$ satisfies:

$$x \vee y = 1 \text{ if and only if } x \geq y^{+}.$$

An algebra $\mathbf{L} = (L, \wedge, \vee, ^{\sim}, ^{+}, 0, 1)$ is a *double p-algebra* if the following conditions are satisfied:

1. $\mathbf{L} = (L, \vee, \wedge, ^{\sim}, 0, 1)$ is a p-algebra.
2. $\mathbf{L} = (L, \vee, \wedge, ^{+}, 0, 1)$ is a dual p-algebra.

As mentioned in the introduction, the class of double p-algebras is a variety [2]. Let us now introduce the notions of a Stone algebra and a double Stone algebra.

Definition 2.2 ([19, 2]).

1. A p-algebra $\mathbf{L} = (L, \wedge, \vee, ^{\sim}, 0, 1)$ is a *Stone algebra* if it satisfies:

$$x^{\sim} \vee x^{\sim\sim} = 1. \qquad \text{(Stone Condition)}$$

2. A dual p-algebra $\mathbf{L} = (L, \wedge, \vee, ^+, 0, 1)$ is a *dual Stone algebra* if it satisfies:
$$x^+ \wedge x^{++} = 0. \tag{Dual Stone Condition}$$

3. An algebra $\mathbf{L} = (L, \wedge, \vee, ^\sim, ^+, 0, 1)$ is a double Stone algebra if $(L, \wedge, \vee, ^\sim, 0, 1)$ is a Stone algebra and $(L, \wedge, \vee, ^+, 0, 1)$ is a dual Stone algebra.

As mentioned earlier, the class of double Stone algebras form a variety. Varlet [52] investigated the following important condition on double Stone algebras:

$$\text{if } \ x^\sim = y^\sim \quad \text{and} \quad x^+ = y^+ \quad \text{then} \quad x = y. \tag{Regularity}$$

Following Varlet [52], we call a double Stone algebra *regular* if it satisfies the Condition (Regularity). In fact, "regularity" is an appropriate name. Actually, in [52], Varlet proved that Condition (Regularity) is equivalent to congruence regularity: if two congruences coincide on a congruence class, then they are in fact the same congruence. In 1973, Katriňák [25] proved that Condition (Regularity) is equivalent to the following identity:

$$x \wedge x^+ \leq y \vee y^\sim. \tag{M1}$$

Thus, the class of regular double p-algebras is a variety.

THEOREM 2.3. *[6] Let* $\mathbf{L}$ *be a p-algebra, then the following statements are equivalent:*

1. $\mathbf{L}$ *satisfies* (Stone Condition)*;*
2. $(x \vee y)^{\sim\sim} = x^{\sim\sim} \vee y^{\sim\sim}$*;*
3. $(x \wedge y)^\sim = x^\sim \vee y^\sim$*;*
4. $S_K(\mathbf{L}) = \{x^{\sim\sim} : x \in L\}$ *is a Boolean subalgebra.*

Given a regular double Stone algebra $\mathbf{L}$, three of its subsets, namely $S_K(\mathbf{L})$, $D^\sim(\mathbf{L})$ and $D^+(\mathbf{L})$, will play a significant role in what follows.

$S_K(\mathbf{L})$ is defined above in Theorem 2.3 and its elements are called *sharp elements of* $\mathbf{L}$. It is easy to see that

$$S_K(\mathbf{L}) = \{x \in L : x = x^{\sim\sim}\} = \{x^{\sim} : x \in L\} = \{x \in L : x^{+} = x^{\sim}\}.$$

Let us also notice that if $\mathbf{L}$ is a regular double Stone algebra, then $S_K(\mathbf{L})$ is the largest Boolean subalgebra of $\mathbf{L}$.

As observed in [39], it is possible to define two unary operations that behave as the modal operators *necessarily* "$\Box$" and *possibly* "$\Diamond$" as follows:

$$\Box x = x^{++} \quad \text{and} \quad \Diamond x = x^{\sim\sim}. \tag{2.1}$$

Moreover, *all elements in* $S_K(\mathbf{L})$ are *stable* under both $\Diamond$, and $\Box$:

$$\Diamond x = \Box x = x.$$

It is straightforward to verify that in $S_K(\mathbf{L})$ (Regularity) is nothing but a triviality. Indeed, as we mentioned earlier $S_K(\mathbf{L})$ is in fact a Boolean algebra.

Next, we define $D^{\sim}(\mathbf{L})$ by

$$D^{\sim}(\mathbf{L}) = \{x \in L : x^{\sim} = 0\}.$$

The elements of $D^{\sim}(\mathbf{L})$ are called *dense elements of* $\mathbf{L}$.

Note that $S_K(\mathbf{L}) \cap D^{\sim}(\mathbf{L}) = \{1\}$. Furthermore, for all $x \in L$,

$$x \vee x^{\sim} \in D^{\sim}(\mathbf{L}).$$

Lastly, we define the set $D^{+}(\mathbf{L})$ of *dually dense elements of* $\mathbf{L}$ by:

$$D^{+}(\mathbf{L}) = \{x \in L : x^{+} = 1\}.$$

Also, we have that $S_K(\mathbf{L}) \cap D^{+}(\mathbf{L}) = \{0\}$. Moreover, for all $x \in L$,

$$x \wedge x^{+} \in D^{+}(\mathbf{L}).$$

2.2. A representation theorem for RDS

We need the representation theorem proved in [26], see also [10]. So we recall it below.

Given a Boolean algebra $\mathbf{B} = (B, \wedge, \vee, ', 0, 1)$ and a filter F on $\mathbf{B}$, let us consider the following set:

$$[B, F] = \{(x, y) \in B^2 : (x \leq y) \text{ and } (x \vee y' \in F)\}. \tag{A}$$

We will turn the set $[B, F]$ into an algebra $[\mathbf{B}, F]$. We will define the algebra $[\mathbf{B}, F]$, in the language $(\wedge, \vee, ^{\sim}, ^{+}, 0, 1)$ as follows:

Let $[\mathbf{B}, F] = ([B, F], \wedge, \vee, ^{\sim}, ^{+}, 0, 1)$, where $\wedge, \vee$ are defined componentwise, and the unary operations $^{\sim}$ and $^{+}$ are defined as follows:

$$(x, y)^{\sim} = (y', y'); \quad (x, y)^{+} = (x', x'), \text{ where } (x, y) \in [B, F]. \tag{2.2}$$

It turns out that the algebra $[\mathbf{B}, F]$ is a regular double Stone algebra. Even more importantly, we have the following representation theorem proved in [26, 27] (see also [32]).

THEOREM 2.4. *Every regular double Stone algebra* $\mathbf{A}$ *is isomorphic to* $[S_K(\mathbf{A}), g(D^{\sim}(\mathbf{A}))]$, *where the isomorphism* f *is given, for each* a, *by*

$$f(a) = (\Box a, \Diamond a), \tag{2.3}$$

and $g : D^{\sim}(\mathbf{A}) \to S_K(\mathbf{A})$ *is such that for any* $a \in D^{\sim}(\mathbf{A})$*:*

$$g(a) = a^{++}.$$

Furthermore, any homomorphism h *between regular double Stone algebras* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *factorizes through a Boolean homomorphism acting componentwise:*

$$h(a) = f((\Box a, \Diamond a)) = (h \restriction_{S_K(\mathbf{A})} (\Box a), h \restriction_{S_K(\mathbf{A})} (\Diamond a)).$$

Additionally, in any regular double Stone algebra $\mathbf{A}$, a unary operation $'$ can be defined as given in Equation (2.4):

$$x' = x^{\sim} \vee (x \wedge x^{+}). \tag{2.4}$$

The operation $'$ turns out to be the Kleene negation, i.e., an antitone and involutive negation satisfying

$$x \wedge x' \leq y \vee y'. \tag{Kleene}$$

On the algebra $[\mathbf{B}, F]$, the Kleene operation $'$ is defined by:

$$(x, y)' = (y', x')$$

Let us now close the present section discussing an important subset of a regular double Stone algebra.

Definition 2.5. Let $\mathbf{A}$ be a regular double Stone algebra. The *core of* $\mathbf{A}$ is defined as the intersection of the set of dense elements and the set of dually dense elements; that is,

$$D^{\sim}(\mathbf{A}) \cap D^{+}(\mathbf{A}).$$

Let us notice that if a regular double Stone algebra $\mathbf{A}$ has a non-empty core, then it possesses rather remarkable properties. In fact, it can be seen that algebras of this sort are all of the form $[\mathbf{B}, B]$, for a certain Boolean algebra $\mathbf{B}$. Actually, the element $(0, 1)$ will be in the core of $[\mathbf{B}, B]$. Moreover, $(0, 1)' = (0, 1)$, see Equation (2.4). Lemma 2.6 summarizes these facts.

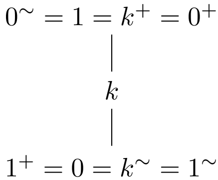
Lemma 2.6. *[32] Let $\mathbf{A}$ be a regular double Stone algebra and $x \in A$. We have that:*

1. *if $x \in D^{\sim}(\mathbf{A}) \cap D^{+}(\mathbf{A})$, then $x = (0, 1)$.*
2. *the cardinality of the core of $\mathbf{A}$ is at most 1 [51].*
3. *x belongs to the core of $\mathbf{A}$ if and only if $x = x'$, i.e. x is a fixpoint.*

In other words, Lemma 2.6 expresses the fact that a regular double Stone algebra $\mathbf{A}$ admits at most one fixpoint $k = k'$ which would be dense and dually dense, and

$$D^{\sim}(\mathbf{A}) \cap D^{+}(\mathbf{A}) = \{k\}.$$

Moreover, for a regular double Stone algebra, the conditions of having a non-empty core, satisfying Condition (K), and possessing a fixed point of $'$ are all equivalent. A prominent example of a regular double Stone algebra having a non-empty core is the three-element algebra **3** whose Hasse diagram is

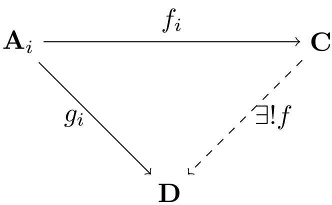


(**3**)

2.3. Basics of category theory

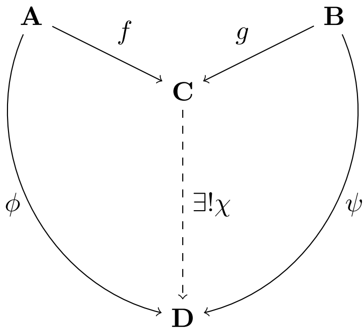
In order to fix the notation needed from category theory, we begin the present subsection by recalling a few categorial notions that will be needed in this paper. For our purposes we can restrict our attention to algebraic categories. For a comprehensive account we refer the reader to the classical textbooks [33, 21, 46].

Let $\mathcal{A}$ be a category. The *coproduct* of a family of objects $(\mathbf{A}_i : i \in I)$ in $\mathcal{A}$ is an object $\mathbf{C}$ in $\mathcal{A}$ equipped with the morphisms $f_i : \mathbf{A}_i \to \mathbf{C}$, $i \in I$, such that for any $\mathbf{D}$ in $\mathcal{A}$ and any collection of morphisms $g_i : \mathbf{A}_i \to \mathbf{D}$ there exists a unique $f : \mathbf{C} \to \mathbf{D}$ such that Diagram (2.5) commutes:

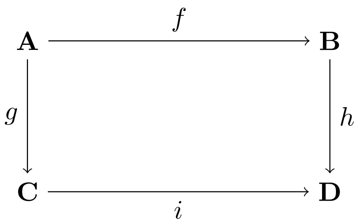


(2.5)

In other words there is a unique morphism f such that $f \circ f_i = g_i$. In case the family contains only two objects and the morphisms, there is a unique morphism χ such that the following diagram commutes:

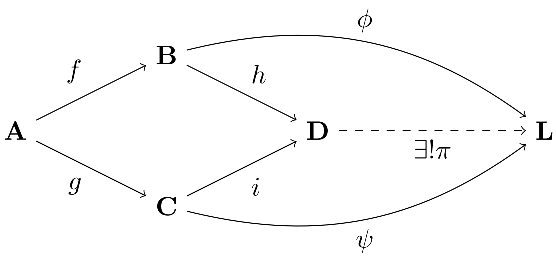


Given a pair $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$ of maps in $\mathcal{A}$ with a common domain $\mathbf{A}$, the *pushout* of (f, g) is a commutative square, such as given in the commutative Diagram (2.6)



(2.6)

and, furthermore, for any morphisms $\phi : \mathbf{B} \to \mathbf{L}$, and $\psi : \mathbf{C} \to \mathbf{L}$ there is a unique $\pi : \mathbf{D} \to \mathbf{L}$ such that Diagram (2.7) commutes:



(2.7)

that is,

$$\phi = \pi \circ f \text{ and } \psi = \pi \circ g.$$

We will conclude this section by defining the amalgamation property and the strong amalgamation property (for a brief historical account, the reader may refer to [34, 15]).

The category $\mathcal{A}$ possesses the *amalgamation property* (AP) if given $\mathbf{A}, \mathbf{B}$, $\mathbf{C}$ in $\mathcal{A}$ and the embeddings $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$ (referred to as V*-formation* or *amalgam*), then there is some $\mathbf{D}$ in $\mathcal{A}$ and embeddings $h : \mathbf{B} \to \mathbf{D}$, $i : \mathbf{C} \to \mathbf{D}$ with $h \circ f = i \circ g$.

A category $\mathcal{A}$ enjoys the *strong amalgamation property* (SAP), if for every V-formation with $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{A}$, and the embeddings $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$, there exist both an object $\mathbf{D}$ in $\mathcal{A}$ and embeddings $h : \mathbf{B} \to \mathbf{D}$, $i : \mathbf{C} \to \mathbf{D}$ such that $h \circ f = i \circ g$ and

$$h(\mathbf{B}) \cap i(\mathbf{C}) = h \circ f(\mathbf{A}) = i \circ g(\mathbf{A}).$$

3. Revisiting the strong amalgamation property for Boolean algebras: a constructive approach

In this section we present a novel proof of the well-known result that the variety $\mathbf{BA}$ has the strong amalgamation property. Our proof is constructive and uses a simplified version of an outstanding construction proposed by Banaschewski in [3], which partly relies on a previous work by Lagrange [30].

First, we introduce the notation $a * b$. Let $\mathbf{A} \coprod \mathbf{B}$ denote the coproduct of $\mathbf{A}, \mathbf{B} \in \mathbf{BA}$, with the canonical injections

$$f : \mathbf{A} \to \mathbf{A} \coprod \mathbf{B} \text{ and } g : \mathbf{B} \to \mathbf{A} \coprod \mathbf{B}.$$

For $a \in \mathbf{A}$ and $b \in \mathbf{B}$, we define $a * b$ by

$$a * b = f(a) \wedge g(b) \tag{3.1}$$

in $\mathbf{A} \coprod \mathbf{B}$.

We will now provide an explicit description of the coproduct. We follow below the construction by Johnstone from the context of frames [23, 24].

Let **A** and **B** be Boolean algebras. We consider the set

$$A \times B = \{(a, b) : a \in A,\ b \in B\}$$

and let **D** be the free bounded distributive lattice generated by the set $A \times B$. We think of the ordered pairs (a, b), with $a \in A$ and $b \in B$, as formal symbols. Observe that the following terms are (some of the) elements of **D**:

$$(a, 0), (0, b), (a, b) \wedge (c, d), (a \wedge c, b \wedge d), (a, b) \vee (c, b), (a \vee c, b),$$
$$(a, b) \vee (a, d), (a, b \vee d),$$

where $0, a, c \in A$ and $0, b, d \in B$. Let R be a binary relation on **D** consisting, precisely, of the following ordered pairs:

$$\begin{aligned} &((0, 0), (a, 0)), ((a, 0), (0, b));\\ &((a, b) \wedge (c, d), (a \wedge c, b \wedge d));\\ &((a, b) \vee (a, d), (a, b \vee d));\\ &((a, b) \vee (c, b), (a \vee c, b)). \end{aligned} \tag{3.2}$$

Let $\theta(R)$ (or simply, θ) be the congruence on **D** generated by R.

We denote by $\mathbf{C} = \mathbf{D}/\theta$. We will indicate the elements of **C** by x/θ, where $x \in \mathbf{D}$. Then the following lemma trivially holds.

Lemma 3.1. **C** *is a bounded distributive lattice satisfying the following conditions:*

$$\begin{aligned} &(1)\ 0^{\mathbf{C}} = (a, 0^{\mathbf{B}})/\theta = (0^{\mathbf{A}}, b)/\theta;\\ &(2)\ (a, b)/\theta\ \wedge\ (c, d)/\theta = (a\ \wedge^{\mathbf{A}} c,\ b\ \wedge^{\mathbf{B}} d)/\theta;\\ &(3)\ (a, b)/\theta\ \vee\ (a, d)/\theta = (a,\ b \vee^{\mathbf{B}} d)/\theta;\\ &(4)\ (a, b)/\theta\ \vee\ (c, b)/\theta = (a \vee^{\mathbf{A}} c,\ b)/\theta;\\ &(5)\ 1^{\mathbf{C}} = (1^{\mathbf{A}}, 1^{\mathbf{B}})/\theta. \end{aligned} \tag{Relations}$$

Let us recall that any element x in free distributive lattice $\mathbf{L}$ can be represented as

$$x = \bigvee_i \bigwedge_j y_{i,j}$$

where all $y_{i,j}$ are in the set of generators of $\mathbf{L}$ (see e.g. [5, Lemma III.3]). As a consequence,

Lemma 3.2. *Any element x of $\mathbf{D}$ is of the form*

$$x = \bigvee_i \bigwedge_j (y,z)_{i,j},$$

where all $(y,z)_{i,j}$ are in the set of generators of $\mathbf{D}$.

We say that two generators (a,b) and (c,d) of $\mathbf{D}$ are extremely distinct if $a \neq c$ and $b \neq d$.

By virtue of the fact that $\mathbf{C} = \mathbf{D}/\theta$, Lemma 3.2 applies to obtain a normal form lemma also for $\mathbf{C}$ in terms of congruence classes of extremely distinct generators of $\mathbf{D}$.

Lemma 3.3 (Normal Form). *Each element of $\mathbf{C}$ admits a canonical representation as a finite join of congruence classes of pairwise extremely distinct generators of $\mathbf{D}$. Moreover, any $x \in C$ is of the form*

$$\bigvee_i \left((\bigwedge_j a_{i,j}, 1)/\theta \wedge (1, \bigwedge_j b_{i,j})/\theta \right).$$

Proof: Let $x \in \mathbf{C}$. By Lemma 3.2,

$$\begin{aligned} x =& y/\theta \\ =& (\bigvee_i \bigwedge_j (a,b)_{i,j})/\theta \\ =& \bigvee_i (\bigwedge_j a_{i,j}, \bigwedge_j b_{i,j})/\theta \end{aligned}$$

where the third equality is by virtue of the definition of θ, (Relations)-(2).

Next, we wish to show that the generators can be chosen to be pairwise extremely distinct in **D**. Without loss of generality we can assume that x is a join of the congruence classes of two generators of **D**, say:

$$x = (a, b)/\theta \ \vee^{\mathbf{C}} (c, d)/\theta.$$

If $a = c$, then from (Relations)-(3), we get $x = (a, b \vee^{\mathbf{B}} d)/\theta$, which is a congruence class of a generator of **D**.
If $b = d$, then from (Relations)-(4), we get $x = (a \vee^{\mathbf{A}} c, b)/\theta$, which is also a congruence class of a generator of **D**. Thus we can conclude that x can be written as a finite join of congruence classes of pairwise extremely distinct generators of **D**. The moreover part is straightforward. □

This normal form will allow us to extend definitions given on congruence classes of generators, such as complementation, to arbitrary elements of **C** by recursion.

When there is no danger of confusion, to ease the notation we will omit unnecessary superscripts and subscripts.

Informally speaking, as we will see later, (Relations)-(3) and (Relations)-(4) are meant to "mimic" in **C** the operations of the factors **A**, **B** which are to be preserved in the embeddings. Next, we show that it is in fact possible to define a complement operation $'$ on **C** which renders the distributive lattice **C** into a Boolean algebra.

Remark 3.4. From now on, we will refer to an element of **C** by one of its representatives. In particular, if x is the congruence class of a generator of **D**, say (a, b), we will (by a mild abuse of language) denote x simply by (a, b) (instead of $(a, b)/\theta$) and call it a *generator* of **C**. By Lemma 3.3, every element of **C** can be expressed as a finite join of congruence classes of generators of **D**, that is, of elements of the form (a, b). Furthermore, the *moreover* part of Lemma 3.3 shows that each such generator admits a decomposition

$$(a, b) = (a, 1) \wedge (1, b).$$

As a consequence, the algebra $\mathbf{C}$ is generated by the set

$$\{(a, 1) : a \in A\} \ \cup \ \{(1, b) : b \in B\},$$

which is a generating set strictly contained in the set of all generators (a, b).

We will freely switch between generators of the form (a, b) and the more specific generators $(a, 1)$ and $(1, b)$, depending on the context. The former provide a uniform and symmetric description of elements of $\mathbf{C}$ (e.g. in the Definition 3.5 of complementation), while the latter form a smaller generating set, which is technically convenient in arguments involving universal properties and homomorphism extensions (see, e.g., the application of Sikorski's Criterion in Theorem 3.10).

Definition 3.5. Let $x \in C \setminus \{0, 1\}$. The unary operation $'$ on $\mathbf{C}$ is defined by recursion as follows:

Step 1. Let x be a generator, say $x = (a, b)$. Then x' is given by

$$x' = (a, b)' = (a', 1) \vee (1, b').$$

Step 2. Let x be a non-generator. Then by normal form Lemma 3.3, we have

$$x = \bigvee_{i=1}^{n} t_i, \text{ where } t_i \text{ is a generator.}$$

In this case, we define complementation as follows:

$$x' = \Big(\bigvee_{i=1}^{n} t_i\Big)' = \bigwedge_{i=1}^{n} t_i'.$$

Lemma 3.6. *The algebra* $\mathbf{C}$ *with the operation* $'$ *is a Boolean algebra.*

Proof: We already know that the algebra $\mathbf{C}$ is a bounded distributive lattice. So we only need to show that $'$ is an involution and satisfies the complementation laws. Let $t \in \mathbf{C}$.

(1) Involution. We rely on Lemma 3.3.

For a generator (a, b) we have

$$((a,b)')' = ((a',1) \vee (1,b'))' = (a',1)' \wedge (1,b')'$$

where the latter equation is from Definition 3.5-Step 2. By definition of complement on generators,

$$(a',1)' = (a'',1) \vee (1,1') = (a,1) \vee (1,0) = (a,1) \vee 0 = (a,1),$$
$$(1,b')' = (1',1) \vee (1,(b')') = (0,1) \vee (1,b) = 0 \vee (1,b) = (1,b).$$

Hence

$$\big((a,b)'\big)' = (a',1)' \wedge (1,b')' = (a,1) \wedge (1,b) = (a,b).$$

For compound terms, the claim follows by induction using De Morgan laws in Definition 3.5.

(2) Complementation law. Again we rely on Lemma 3.3.

(i) We first prove that $t \wedge t' = 0$.

Step 1. Let t be a generator, say $t = (a, b)$. We note that

$$\begin{aligned} t \wedge t' =& (a,b) \wedge (a,b)' \\ =& (a,b) \wedge ((a',1) \vee (1,b')) \\ =& ((a,b) \wedge (a',1)) \vee ((a,b) \wedge (1,b')) \\ =_{(2)}& (a \wedge a', b) \vee (a, b \wedge b') \\ =& (0,b) \vee (a,0) \\ =_{(1)}& 0 \vee 0 = 0. \end{aligned}$$

Step 2. Let t be a non-generator. Then, $t = \bigvee_{i=1}^{n}(a_i, b_i)$. We have that

$$t' = (\bigvee_{i=1}^{n}(a_i,b_i))' = \bigwedge_{i=1}^{n}(a_i,b_i)' = \bigwedge_{i=1}^{n}((a_i',1) \vee (1,b_i')).$$

Making use of distributivity,

$$t \wedge t' = \bigvee_{i=1}^{n} \bigwedge_{i=1}^{n} ((a_i, b_i) \wedge ((a_i', 1) \vee (1, b_i'))) = 0 \vee 0 = 0.$$

Next we prove that $t \vee t' = 1$. We confine our argument only to the generators, since the argument to the case of a non-generator is dual to the previous case. Let again $t = (a, b)$. Then

$$\begin{aligned} t \vee t' =& (a, b) \vee (a, b)' \\ =& (a, b) \vee (a', 1) \vee (1, b') \\ \geq_{(4)}& (a, b) \vee (a', b) \vee (1, b') \\ =_{(3)}& (a \vee a', b) \vee (1, b') \\ =& (1, b) \vee (1, b') \\ =_{(3)}& (1, b \vee b') \\ =& (1, 1) =_{(5)} 1. \end{aligned}$$

In conclusion, **C** is a Boolean algebra. □

Next we wish to show that **C** is the coproduct of **A** and **B**. We will achieve this through a few lemmas.

LEMMA 3.7. *Define the mappings* $f : \mathbf{A} \to \mathbf{C}$ *and* $g : \mathbf{B} \to \mathbf{C}$ *as in Display* (3.3)*:*

$$f(a) = (a, 1) \quad and \quad g(b) = (1, b). \tag{3.3}$$

Then, the maps f, g *are injective homomorphisms.*

PROOF: The fact that f, g are injective is due to the definition of the mappings and the fact that $\ker(f) = \ker(g) = \{0\}$, since

$$0/\theta = \{(x, y) : x = 0 \text{ or } y = 0\}$$

by construction of θ. To show that f, g are homomorphisms, we have

$$\begin{aligned} f(a_1 \wedge a_2) =& (a_1 \wedge a_2, 1) \\ =& (a_1, 1) \wedge (a_2, 1) \\ =& f(a_1) \wedge f(a_2). \end{aligned}$$

Also,

$$\begin{aligned} f(a_1 \vee a_2) =& (a_1 \vee a_2, 1) \\ =& (a_1, 1) \vee (a_2, 1) \\ =& f(a_1) \vee f(a_2). \end{aligned}$$

Finally,

$$f(a') = (a', 1) = (a', 1) \vee 0 = (a', 1) \vee (1, 0) = (a, 1)' = f(a)'$$

by Lemma 3.6. □

Corollary 3.8 (Independent Algebras). The algebras $f(\mathbf{A})$, $g(\mathbf{B})$ are independent in $\mathbf{C}$, i.e., for all $x \in f(A)$, $y \in g(B)$

$$x \wedge^{\mathbf{C}} y = 0 \text{ if and only if } x = 0 \text{ or } y = 0.$$

In other words, Corollary (3.8) entails that, for any finite set $F \subseteq f(A) \cup g(B)$ such that $0 \notin F$, whenever

$$\bigwedge_{x \in F} x = 0,$$

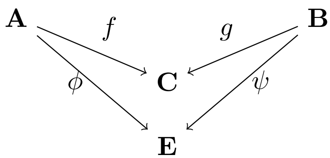
then

$$0 = \bigwedge_{x \in F} x = \bigwedge_{a \in f(A) \cap F} a \wedge \bigwedge_{b \in f(B) \cap F} b$$

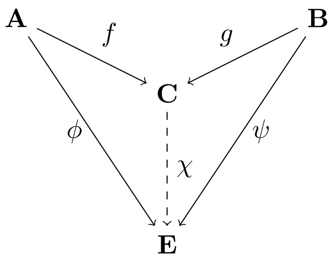
if and only if either $\bigwedge_{a \in f(A) \cap F} a = 0$ or $\bigwedge_{b \in f(B) \cap F} b = 0$, or both $\bigwedge_{a \in f(A) \cap F} a = \bigwedge_{b \in f(B) \cap F} b = 0$.

We are now ready to prove that $\mathbf{C}$ is the coproduct $\mathbf{A} \coprod \mathbf{B}$.

In fact, suppose that there are mappings ϕ, ψ so that



We wish to find a unique morphism χ such that Diagram (3.4) commutes:



(3.4)

To this aim, Sikorski Criterion will be expedient. Consider the mapping defined for any element x of a Boolean algebra $\mathbf{B}$, and a homomorphism $\varphi : \mathbf{B} \to \mathbf{2}$:

$$p(x, \varphi(x)) = \begin{cases} x, & \text{if } \varphi(x) = 1; \\ x', & \text{if } \varphi(x) = 0. \end{cases}$$

THEOREM 3.9 (Sikorski Criterion [50, 29]). *A mapping h from a generating set K of a Boolean algebra $\mathbf{B}$ into a Boolean algebra $\mathbf{A}$ can be extended to a homomorphism from $\mathbf{B}$ into $\mathbf{A}$ just in case for every 2-valued function φ on a finite subset F of K,*

$$\bigwedge_{a \in F} p(a, \varphi(a)) = 0 \textit{ implies } \bigwedge_{a \in F} p(h(a), \varphi(a)) = 0.$$

THEOREM 3.10. *$\mathbf{C}$ is the coproduct $\mathbf{A} \coprod \mathbf{B}$.*

PROOF: Recall that for $a \in A, b \in B$ we have by Display (3.3) in Lemma 3.7

$$f(a) = (a, 1) \text{ and } g(b) = (1, b).$$

Now consider a Boolean algebra $\mathbf{E}$ and homomorphisms $\phi : \mathbf{A} \to \mathbf{E}$, $\psi : \mathbf{B} \to \mathbf{E}$ (see Diagram (3.4)). By Lemma 3.3, it follows that the set

$$K = \{(x, 1) : x \in A\} \cup \{(1, y) : y \in B\}$$

generates $\mathbf{C}$. We define a map $\chi : K \to \mathbf{E}$ as follows:

$$\chi((x, y)) = \phi(x) \wedge^{\mathbf{E}} \psi(y).$$

We shall see that this assignment extends by Theorem 3.9 to a Boolean homomorphism. Moreover, we will also prove that this extension will be unique in making Diagram (3.4) commutative.

Suppose that $\bigwedge_{a \in F} p(a, \varphi(a)) = 0$ for any 2-valued function, and $F \subseteq_{fin} K$. By Corollary 3.8, without loss of generality we can assume all $a \in F$ of the form $(x, 1)$. This means,

$$\begin{aligned}
\bigwedge_{a \in F} p(a, \varphi(a)) = & \bigwedge_{a \in F} p((x, 1), \varphi(a)) \\
= & \bigwedge_{a \in F} p(f(x), \varphi(a)) \\
= & f(\bigwedge_{a \in F} p(x, \varphi(a))) \\
= & f(0) = 0.
\end{aligned}$$

Then,

$$\bigwedge_{a \in F} p(\chi((x, 1)), \varphi(a)) = \bigwedge_{a \in F} p(\phi(x), \varphi(a)) = \phi(\bigwedge_{a \in F} p(x, \varphi(a))) = \phi(0) = 0.$$

It is also easy to observe that for $a \in A$:

$$\begin{aligned}
\chi \circ f(a) = & \chi((a, 1)) \\
= & \phi(a) \wedge \psi(1) \\
= & \phi(a) \wedge 1 \\
= & \phi(a).
\end{aligned}$$

An analogous argument applies to any $b \in B$. Therefore, Diagram (3.4) commutes.

Finally, we prove that χ is unique in making Diagram (3.4) commutative. Suppose that there exists a $\delta : \mathbf{C} \to \mathbf{E}$ such that the diagram commutes. Hence, $\delta \circ f(a) = \phi(a)$ and $\delta \circ g(b) = \psi(b)$, for $a \in A, b \in B$. As a consequence of this observation, it readily follows that

$$\begin{aligned}\delta \circ f(a) =& \delta((a,1))\\ =&\phi(a)\\ =&\phi(a) \wedge \psi(1)\\ =&\chi((a,1))\end{aligned}$$

and dually for g. Therefore, the proof of the theorem is complete. □

Using our explicit description of $\mathbf{A} \coprod \mathbf{B}$, we can prove a technical lemma that will turn useful for our arguments that follow.

Lemma 3.11 (Comparison Lemma). *Let* $\mathbf{A}, \mathbf{B} \in \mathbf{BA}$*. For any* $a_1, a_2 \in A$*,* $b_1, b_2 \in B$*, if* $a_1 * b_1 \leq a_2 * b_2$*, then* $a_1 \leq a_2$ *and* $b_1 \leq b_2$*.*

Proof: Let $a_1, a_2 \in A$, $b_1, b_2 \in B$ be such that $a_1 * b_1 \leq a_2 * b_2$, where $*$ is defined as in Display (3.1): $x * y = f(x) \wedge g(y)$. Now, by hypothesis, we have

$$f(a_1) \wedge g(b_1) \leq f(a_2) \wedge g(b_2),$$

which implies that

$$(a_1, 1) \wedge (1, b_1) \leq (a_2, 1) \wedge (1, b_2),$$

whence

$$(a_1, b_1) \leq (a_2, b_2).$$

Therefore,

$$(a_1, b_1) = (a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2).$$

by (Relations)–(2), and by the injectivity of the mappings f, g (see Lemma 3.7) we have that $a_1 = a_1 \wedge a_2$ and $b_1 = b_1 \wedge b_2$. Thus $a_1 \leq a_2$ and $b_1 \leq b_2$. □

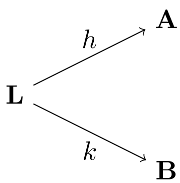
For a different perspective, we would refer the interested reader also to Banaschewski [3]. However, in [3] the result is mentioned without an explicit proof, and the author refers to it as the "Comparison Principle".

For a subset X of a Boolean algebra $\mathbf{B}$, we denote by $\langle X\rangle$ the ideal generated by X. Recall that for $x \in \mathbf{B}$, we have that

$$x \in \langle X\rangle \text{ if and only if } x \leq \bigvee_{i=1}^{n} x_i \qquad \text{for some } x_1, \ldots, x_n \in X. \tag{3.5}$$

Let us now use the coproduct $\mathbf{A} \coprod \mathbf{B}$ to construct a strong amalgam of any V-formation in $\mathbf{BA}$.

Given the V-formation in Display (3.6):



(3.6)

let us consider the coproduct $\mathbf{A} \coprod \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$. Define the subset H of the coproduct by

$$H = \langle h(a) * k(a)' : a \in L\rangle, \tag{3.7}$$

where "$*$" is defined in Display (3.1). Clearly, H is an ideal of the Boolean algebra $\mathbf{A} \coprod \mathbf{B}$ (this idea modifies an intuition due to Banaschewski [3]).

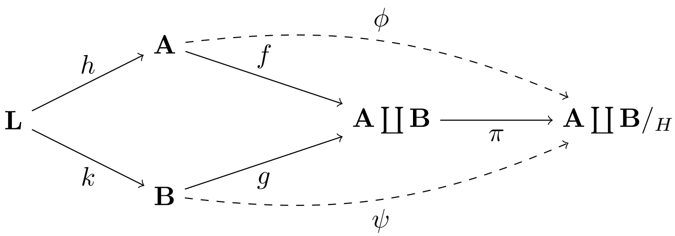
Lemma 3.12 will be fundamental to our discourse:

Lemma 3.12 (Lagrange [30], Banaschewski[3]). *For any $a \in A$, $b \in B$, if $a * b \in H$, then there already exists $l \in L$ such that $a * b \leq h(l) * k(l)'$.*

Let us now take into account the *pushout* diagram of the formation in Display (3.6), which will be the coproduct $\mathbf{A} \coprod \mathbf{B}$ modulo the ideal H in Display (3.7).[2]

[2]In fact, the interested reader may verify that H assumes the role of the coequalizer.

The corresponding diagram is the following:



(3.8)

where the maps ϕ and ψ are defined by:

$$\phi = \pi \circ f \text{ and } \psi = \pi \circ g.$$

THEOREM 3.13. *The maps ϕ and ψ are embeddings. Moreover, Diagram (3.8) commutes.*

PROOF: We show that $\ker(\phi) = \{0\}$, the case for ψ being similar. Suppose by contradiction that $0 \neq a \in A$ and $\phi(a) = 0$. Then $f(a) \in H$. By the definition of H in Display (3.7), this means that

$$f(a) \leq \bigvee_{i=1}^{n} \big(h(l_i) * k(l_i)'\big) \qquad \text{for some } l_1, \ldots, l_n \in L. \tag{3.9}$$

By Lemma 3.12, there exists $l \in L$ such that

$$f(a) = (a, 1) \leq h(l) * k(l)' = f \circ h(l) \wedge g \circ k(l)' = (h(l), k(l)').$$

This entails both $a \wedge h(l) \neq 0$ and $k(l)' = 1$, i.e. $k(l) = 0$. Since k is an embedding, $k(l) = 0$ implies $l = 0$; but then $h(l) = 0$, and thus $a \wedge h(l) = a \wedge 0 = 0$, contradicting $a \wedge h(l) \neq 0$.

Thus no nonzero $a \in A$ can belong to $\ker(\phi)$, and ϕ is injective. The same argument applies dually to the other cone of the diagram:

$$\begin{array}{ccc} & \mathbf{A}\coprod\mathbf{B} \xrightarrow{\ \pi\ } \mathbf{A}\coprod\mathbf{B}/_H \\ \mathbf{B} \overset{g}{\nearrow} & \overset{\psi}{\dashrightarrow} \end{array}$$

Finally, commutativity of Diagram (3.8) follows directly from the definition $\phi = \pi \circ f$, $\psi = \pi \circ g$. □

It is well known (see e.g. [28, Lemma 5.22]) that, in Boolean algebras, for any ideal I,

$$x/I = y/I \text{ if and only if } x\Delta y = (x \wedge y') \vee (x' \wedge y) \in I.$$[3]

Now note that, by definition of H, for any $a \in L$ we have

$$f \circ h(a) \wedge (g \circ k(a))' = (h(a), k(a)') \in H,$$

and moreover

$$(f \circ h(a))' \wedge g \circ k(a) = (h(a)', k(a)) \in H.$$

By closure of H under joins, it follows that

$$(f \circ h(a) \wedge (g \circ k(a))') \vee ((f \circ h(a))' \wedge g \circ k(a)) \in H.$$

Hence,

$$(f \circ h(a))\Delta(g \circ k(a)) \in H,$$

and therefore

$$(f \circ h(a))/_H = (g \circ k(a))/_H.$$

By virtue of Theorem 3.13, this is equivalent to

$$\phi \circ h(a) = \pi \circ f \circ h(a) = \pi \circ g \circ k(a) = \psi \circ k(a).$$

In other words, the algebra $\mathbf{A}\coprod\mathbf{B}/_H$ provides an amalgam of the V-formation $(\mathbf{L}, \mathbf{A}, \mathbf{B})$ in Diagram (3.6).

[3] In other words, the *symmetric difference* Δ of x, y is in I.

Our next step is to show that this merging is in fact *strong*. Namely, if $\phi(a) = \psi(b)$ in $\mathbf{A}\coprod\mathbf{B}/_H$, for any $a \in A$, $b \in B$, then $a \in L$, i.e. $h^{-1}(a) = k^{-1}(b) = \{l\}$ for a unique $l \in L$. In other words, $\mathbf{L}$ is indistinguishably mapped into $\mathbf{A}\coprod\mathbf{B}/_H$ by any cone of Diagram (3.8).

This is exactly the content of Theorem 3.14:

THEOREM 3.14. *The Boolean V-formation $(\mathbf{L}, \mathbf{A}, \mathbf{B})$ is strongly amalgamated into $\mathbf{A}\coprod\mathbf{B}/_H$.*

PROOF: Suppose $\phi(a) = \psi(b)$ in $\mathbf{A}\coprod\mathbf{B}/_H$. This means $f(a)/_H = g(b)/_H$, i.e.

$$f(a) \wedge g(b)' \in H.$$

By Lagrange Lemma 3.12, there is a $l \in L$ such that

$$f(a) \wedge g(b)' = (a, b') \leq (h(l), k(l)').$$

By the Comparison Lemma 3.11 this implies

$$a \leq h(l) \quad \text{and} \quad b' \leq k(l)'.$$

Equivalently $b \geq k(l)$.

Now, since $\mathbf{A}\coprod\mathbf{B}/_H$ is an amalgam, we have $\phi(h(l)) = \psi(k(l))$. Thus

$$\phi(a) \leq \phi(h(l)) = \psi(k(l)) \leq \psi(b).$$

But $\phi(a) = \psi(b)$ by assumption, so we must have equalities throughout:

$$\phi(a) = \phi(h(l)), \qquad \psi(b) = \psi(k(l)).$$

As ϕ, ψ are embeddings, it follows that $a = h(l)$ and $b = k(l)$. Therefore a, b both come from the same $l \in L$, proving that the amalgam is strong. □

4. The amalgamation property for regular double Stone algebras: a constructive view

The first studies on the amalgamation property are to be traced back to the early work of Fraïssé, and consequences of this property have found fruitful applications in contemporary model theory, logic, and algebra.

Capitalizing on the results from Section 3, we investigate in this section the amalgamation property for the variety of regular double Stone algebras. We provide, in Theorem 4.1, a constructive proof of the amalgamation property for regular double Stone algebras. While this result had already been proved in [15], our proof proceeds along different lines and is constructive.

THEOREM 4.1. *The variety of regular double Stone algebras enjoys the amalgamation property.*

PROOF: Consider the V-formation

$$\begin{array}{ccc} & & \mathbf{C}_1 \\ & \overset{f_1}{\nearrow} & \\ \mathbf{A} & & \\ & \underset{f_2}{\searrow} & \\ & & \mathbf{C}_2 \end{array} \tag{4.1}$$

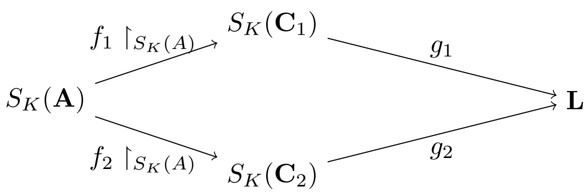
in the variety of regular double Stone algebras. By virtue of Theorem 2.4, we can describe any regular double Stone algebra $\mathbf{D}$ in the form $[\mathbf{B}, F]$, where $\mathbf{B}$ is a Boolean algebra isomorphic to $S_K(\mathbf{D})$, and F is a filter on $\mathbf{B}$ isomorphic to $D^{\sim}(\mathbf{D})$(see Theorem 2.4). Therefore, the formation in Display (4.1) can be rewritten as in Diagram (4.2).

$$\begin{array}{ccc} & & [S_K(\mathbf{C}_1), G_1] \\ & \overset{f_1}{\nearrow} & \\ [S_K(\mathbf{A}), F] & & \\ & \underset{f_2}{\searrow} & \\ & & [S_K(\mathbf{C}_2), G_2] \end{array} \tag{4.2}$$

Now by Theorem 2.4 (see Display (2.1) for the notation), any element a in A can be factored out as a pair $(\Box a, \Diamond a)$. As a consequence, by virtue of Theorem 2.4 we can factor the morphisms f_i, $i \in \{1, 2\}$, as

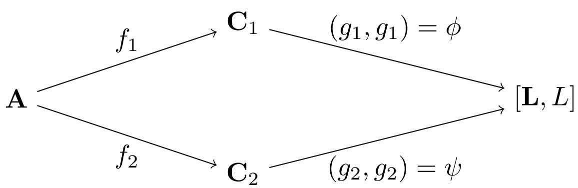
$$f_i(a) = f_i(\Box a, \Diamond a) = (f_i \restriction_{S_K(A)} (\Box a), f_i \restriction_{S_K(A)} (\Diamond a)). \tag{4.3}$$

Due to Theorem 3.14, which proves the strong amalgamation property for the case of Boolean algebras, we can construct a Boolean algebra $\mathbf{L}$ so that the V-formation finds a strong amalgam $\mathbf{L}$:



(4.4)

Then, we can close Diagram (4.2):



(4.5)

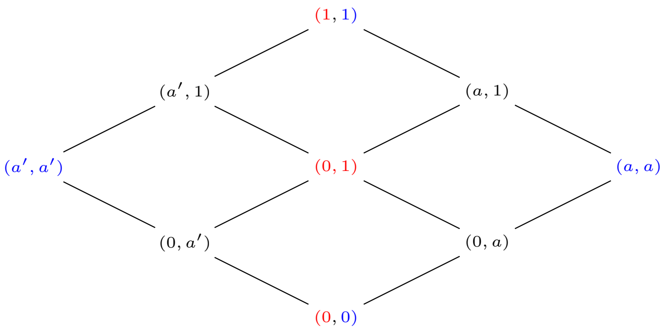
Furthermore, we have

$$\begin{aligned} (g_1, g_1) \circ f_1(x) = \quad & (g_1, g_1)(f_1 \restriction_{S_K(A)} (\Box x), f_1 \restriction_{S_K(A)} (\Diamond x)) \\ = \quad & (g_1(f_1 \restriction_{S_K(A)} (\Box x)), g_1(f_1 \restriction_{S_K(A)} (\Diamond x))) \\ = \quad & (g_2(f_2 \restriction_{S_K(A)} (\Box x)), g_2(f_2 \restriction_{S_K(A)} (\Diamond x))) \\ = \quad & (g_2, g_2) \circ f_2(x) \end{aligned} \tag{4.6}$$

which implies that the proof is complete. $\Box$

Example 4.2 below provides a concrete illustration of the construction of an amalgam for a V-formation in the variety of regular double Stone algebras.

Example 4.2. Consider the V-formation $(\mathbf{2}, \mathbf{3}, \mathbf{4})$, where $\mathbf{2}, \mathbf{4}$ are the two and four-element Boolean algebras respectively, and $\mathbf{3}$ is the three-element regular double Stone algebra. Then, the algebra $\mathbf{9} = [\mathbf{4}, 4]$:



is an *amalgam* for the V-formation $(\mathbf{2}, \mathbf{3}, \mathbf{4})$, where the images of $\mathbf{3}$ and $\mathbf{4}$ are coloured in red and blue, respectively. Indeed, $\mathbf{9}$ is a *strong amalgam*. In fact, the intersection of the images of $\mathbf{3}$, $\mathbf{4}$ is exactly the image of $\mathbf{2}$ in $\mathbf{9}$.

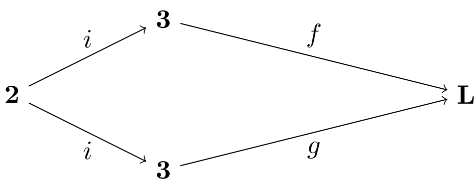
Let us remark that Example 4.2 is a clear case of an amalgam that closes a V-formation that presents Boolean and non-Boolean components, namely $\mathbf{2}, \mathbf{4}$ are Boolean algebras, whilst $\mathbf{3}$ is non-Boolean. It is worth noting that such an amalgam must necessarily have a non-empty core (or, equivalently, it must possess a fixpoint for Kleene negation). We shall later see that the presence or absence of this fixpoint plays a crucial role in the development of our arguments.

We have seen that Theorem 4.1 smoothly extends the construction of the amalgam that we have discussed in Section 3. However, we shall see that the theory of regular double Stone algebras diverges from that of Boolean algebras if we aim at stronger forms of amalgamation.

4.1. Failure of the strong amalgamation property for RDS

Example 4.3 below describes a V-formation of regular double Stone algebras which is not strongly amalgamable in **RDS**, thus showing that the variety **RDS** does not possess the strong amalgamation property.

Example 4.3. Consider the V-formation $(\mathbf{2}, \mathbf{3}, \mathbf{3})$, where $\mathbf{2}, \mathbf{3}$ are as in Example 4.2. Note that if the strong amalgamation property were true in the variety of regular double Stone algebras, i.e.



(4.7)

for some $\mathbf{L}$, then $f(\mathbf{3}) \cap g(\mathbf{3}) = \mathbf{2}$. However, this is impossible because $\mathbf{L}$ must possess a non-empty core, in particular $f(\mathbf{3}) \cap g(\mathbf{3}) = \mathbf{3}$.

In fact, Theorem 4.4 was first observed by Fussner.[4]

THEOREM 4.4. *The variety of regular double Stone algebras fails to have the strong amalgamation property.*

We will see in Section 5 that Fussner Theorem does not hold any longer if we expand the language by a special element k. The motivation of this failure will become evident in the proof of Theorem 5.6.

5. The variety of centered regular double Stone algebras

In this section we define and investigate a new variety, closely related to the variety **RDS**, called centered regular double Stone algebras.

Let $\mathbf{A} \in \mathbf{RDS}$. An element k in $\mathbf{A}$ is called a *center* of $\mathbf{A}$ if k satisfies:

[4]Personal communication.

$$k^{\sim} = 0 \text{ and } k^{+} = 1. \tag{K}$$

According to Lemma 2.6, Condition (K) is equivalent to $k = k'$, where the operation $'$ is defined as in Equation (2.4). In other words, k is a fixed point of the operation $'$.

We expand the language $L = (\wedge, \vee, ^{\sim}, ^{+}, 0, 1)$ of regular double Stone algebras by a new constant k to the language

$$L^k = (\wedge, \vee, ^{\sim}, ^{+}, 0, k, 1),$$

of type $(2, 2, 1, 1, 0, 0, 0)$.

We will now introduce the variety of centered regular double Stone algebras.

Definition 5.1. An algebra $\mathbf{A} = (A, \wedge, \vee, ^{\sim}, ^{+}, 0, k, 1)$ in the language L^k is called a *centered regular double Stone algebra* if and only if:

1. $(A, \wedge, \vee, ^{\sim}, ^{+}, 0, 1)$ is a regular double Stone algebra;
2. the constant k is a center of $\mathbf{A}$.

It is evident that the class of all centered regular double Stone algebras in the language L^k forms a variety. Let $\mathbf{RDS}^k$ denote the variety of all centered regular double Stone algebras.

It is proved in [49] that the variety **RDS** is a discriminator variety (see [8] for an extensive discussion on discriminator varieties). It immediately follows that the variety $\mathbf{RDS}^k$ is a discriminator variety.

Lemma 5.2. *[9, Corollary 6.9] Let* $\mathbf{V}$ *be a discriminator variety and let* S *be the class of simple algebras in* $\mathbf{V}$*. Then the following are equivalent:*

1. *Every* $\mathbf{V}$*-epimorphism is a surjection.*[5]
2. *Every* S*-epimorphism is a surjection.*

Lemma 5.3. *[22, Corollary 2.5.23, page 52] Let* $\mathbf{V}$ *be a variety. Then* $\mathbf{V}$ *has the strong amalgamation property if and only if* $\mathbf{V}$ *has the amalgamation property and every* $\mathbf{V}$*-epimorphism is a surjection.*

[5] A morphism f in a class of algebras is an *epimorphism* in case $k \circ f = h \circ f$ implies that $k = h$.

Theorem 5.4 will be useful to prove Theorem 5.6:

THEOREM 5.4 (Beazer [4]). *The unique non-Boolean subdirectly irreducible regular double Stone algebra is* $\mathbf{3}$.

Let $\mathbf{3}^k$ denote the 3-element centered regular double Stone algebra. Then the following corollary is immediate from 5.4.

COROLLARY 5.5. The only non-trivial subdirectly irreducible (simple) centered regular double Stone algebra is $\mathbf{3}^k$.

THEOREM 5.6. *The variety* $\mathbf{RDS}^k$ *(in the language* L^k*) enjoys the strong amalgamation property.*

PROOF: By virtue of Theorem 4.1 any V-formation in the variety of regular double Stone algebras has an amalgam, see Diagram (4.5). We are left with the task of showing that this amalgam is strong. We have already noted that the variety $\mathbf{RDS}^k$ is a discriminator variety. Moreover, since, by Corollary 5.5, $\mathbf{RDS}^k$ has $\mathbf{3}^k$ as the only simple algebra in which every epimorphism is trivially surjective, it follows from Lemma 5.2 that every $\mathbf{V}$-epimorphism is a surjection. Therefore, by Lemma 5.3 $\mathbf{RDS}^k$ enjoys the strong amalgamation property. □

The proof of Theorem 5.6 is, in our view, interesting also because it explains, from a different perspective the motivation for which the strong amalgamation property fails if we consider the variety of regular double Stone algebras in the language without the constant k.[6] In fact, in such a case the simple members of the variety are $\mathbf{2}, \mathbf{3}$, and $i : \mathbf{2} \to \mathbf{3}$ is the obvious inclusion, and the sole possible morphism. Now, for $f : \mathbf{A} \to \mathbf{2}$ and $g : \mathbf{A} \to \mathbf{2}$, if $f \circ i = g \circ i$, clearly $f = g$. However, i is not a surjection, i.e. in the variety of regular double Stone algebras without fixpoint epimorphisms are not surjective. So by Lemma 5.3 the strong amalgamation property need not hold in general, as stated in Theorem 4.4.

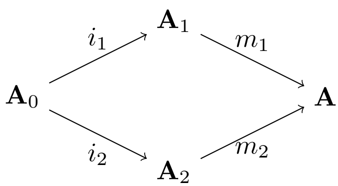
[6]This was also observed by Düntsch [16, Corollary 3] in a different context as a direct corollary of Katriňák's Theorem [27]. Actually, the variety of regular double Stone algebras contains epimorphisms that are not surjective. We sketch here a new constructive and direct argument that proves this fact.

6. The super-amalgamation property

The super-amalgamation property is a rather strong algebraic property that entails quite important logical properties, such as the Craig interpolation property and the Maehara property [17], if the class of algebras under consideration is the equivalent algebraic semantics of a certain logic. The investigation of these connections traces back to the seminal works of Maximova [35, 36, 37], and Pitt [45] from the mid seventies.

Let us begin with the definition of the super-amalgamation property for a class of partially ordered algebras.

Definition 6.1 (Maksimova). Let K be a class of partially ordered algebras. We say that K has the *super-amalgamation property* (*SUPAP* for short) if for any V-formation $(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2)$ in K, there is an algebra $\mathbf{A}$ in K and embeddings m_1, m_2 such that Diagram (6.1) commutes:



(6.1)

where, for embeddings $i_1 : \mathbf{A}_0 \to \mathbf{A}_1$ and $i_2 : \mathbf{A}_0 \to \mathbf{A}_2$, there exist an $\mathbf{A} \in K$ and embeddings $m_1 : \mathbf{A}_1 \to \mathbf{A}$ and $m_2 : \mathbf{A}_2 \to \mathbf{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$, and the following *Maksimova property* holds:

$$(\forall x \in \mathbf{A}_j)(\forall y \in \mathbf{A}_k)\,(m_j(x) \leq m_k(y) \Rightarrow (\exists z \in \mathbf{A}_0)(x \leq i_j(z) \,\&\, i_k(z) \leq y))\,, \tag{MP}$$

where $\{j, k\} = \{1, 2\}$.

Capitalizing on the results in Section 3, we firstly provide a novel proof of the fact that the variety $\mathbf{BA}$ of Boolean algebras enjoy the SUPAP.

Theorem 6.2. *The variety* $\mathbf{BA}$ *enjoys the super-amalgamation property.*

Proof: Let $(\mathbf{L}, \mathbf{A}, \mathbf{B})$ be a V-formation of Boolean algebras with embeddings $h : L \to A$, $k : L \to B$. Let $\mathbf{A} \coprod \mathbf{B}/_H$ be the amalgam constructed in Theorem 3.14. Suppose $\phi(a) \leq \psi(b)$ in $\mathbf{A} \coprod \mathbf{B}/_H$, with $a \in A$ and $b \in B$. Then

$$\pi(f(a)) \leq \pi(g(b)).$$

By definition of the order in the quotient, this means

$$\pi(f(a) \wedge g(b)') = 0,$$

hence

$$f(a) \wedge g(b)' \in H.$$

By Lagrange Lemma 3.12, for some $l \in L$,

$$(a, b') \leq (h(l), k(l)').$$

By the Comparison Lemma 3.11, this inequality implies

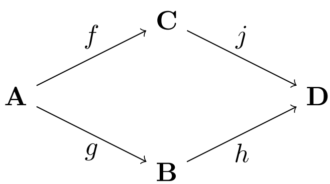
$$a \leq h(l) \quad \text{and} \quad b' \leq k(l)',$$

that is, $a \leq h(l)$ and $k(l) \leq b$.

Therefore, whenever $\phi(a) \leq \psi(b)$ in the amalgam, there exists $l \in L$ with $a \leq h(l)$ and $k(l) \leq b$. This is precisely the order-reflecting property required for super-amalgamation. □

Let us recall from Pitt [45] the notion of interpolation (we refer the reader also to the extensive discussion by Maksimova [35, 36, 37]).

Definition 6.3. Consider the commutative square

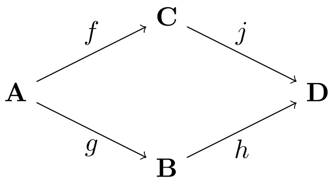


(6.2)

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are partially ordered algebras with order preserving maps.

We say that Diagram (6.3) enjoys the *interpolation property* if whenever for $b \in B, c \in C$ such that $h(b) \leq j(c)$, there is $a \in A$ so that $b \leq g(a)$ and $f(a) \leq c$.

THEOREM 6.4. *If a regular double Stone algebra* $\mathbf{A}$ *possesses a non-empty core, then for every* V*-formation* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *the commutative diagram:*



(6.3)

enjoys the interpolation property.

PROOF: Observe first that if $\mathbf{A}$ has a non-empty core, then every algebra appearing in the diagram must also have a non-empty core; otherwise, no homomorphisms between the algebras could exist, since the central element k can only be mapped to another central element.

Suppose that, for some $b \in B$ and $c \in C$, we have $h(b) \leq j(c)$ in $\mathbf{D}$, where $\mathbf{D}$ can be regarded as a strong amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, due to Theorem 5.6 and the fact that $\mathbf{A}$ possesses a non-empty core. By virtue of Theorem 2.4, b and c can be identified with the pairs $(\Box b, \Diamond b)$ and $(\Box c, \Diamond c)$, respectively. We may observe that in the V-formation

$$[S_K(\mathbf{A}), S_K(\mathbf{B}), S_K(\mathbf{C})]$$

the elements $\Box b, \Diamond b, \Box c, \Diamond c$ have Boolean interpolant elements a, d in the Boolean algebra $S_K(\mathbf{A})$. Specifically, we can assume that a, d satisfy

$$\Box b \leq g(a) \quad f(a) \leq \Box c$$

and

$$\Diamond b \leq g(d) \quad f(d) \leq \Diamond c.$$

Fix such elements $a, d \in S_K(\mathbf{A})$. In accordance with Theorem 2.4, since a, d are Boolean elements, they may be regarded as pairs of the form

$$a = (\Box a, \Diamond a) = (a, a), \quad d = (\Box d, \Diamond d) = (d, d).$$

This observation will play a pivotal role in the construction of the interpolant.

Since $\mathbf{A}$ has a non-empty core, by the construction in Display (A), $\mathbf{A}$ is of the form $[S_K(\mathbf{A}), S_K(A)]$.

Our goal is to identify an element $x = (\Box x, \Diamond x) \in A$ such that

$$b = (\Box b, \Diamond b) \leq g((\Box x, \Diamond x)) = g(x)$$

and

$$f(x) = f((\Box x, \Diamond x)) \leq (\Box c, \Diamond c) = c.$$

To this end, consider the element

$$x = (a \wedge d, a \vee d).$$

Clearly, $a \wedge d \leq a \vee d$, and since $\mathbf{A}$ has a non-empty core, this suffices to ensure that $(a \wedge d, a \vee d) \in A$, because both $a \wedge d, a \vee d \in S_K(A)$, and the universe of A is

$$\{(x, y) \in S_K(A)^2 : x \leq y\}.$$

We now verify that the element $x = (a \wedge d, a \vee d)$ is indeed the desired interpolant. To check that $b \leq g((a \wedge d, a \vee d))$, we focus on the maps g and f restricted to the sharp elements (see Theorem 2.4). From the fact that $\Box b \leq g(a)$ and $\Box b \leq \Diamond b \leq g(d)$, we obtain that

$$\Box b \leq g(a \wedge d).$$

Moreover, because $\Diamond b \leq g(d)$, then

$$\Diamond b \leq g(a \vee d).$$

Therefore, $b = (\Box b, \Diamond b) \leq g((a \wedge d, a \vee d))$. To verify that

$$f((a \wedge d, a \vee d)) \leq c,$$

the argument is dual. This establishes our claim. □

Therefore, we immediately obtain Theorem 6.5:

THEOREM 6.5. *The variety* $\mathbf{RDS}^k$ *enjoys the super-amalgamation property.*

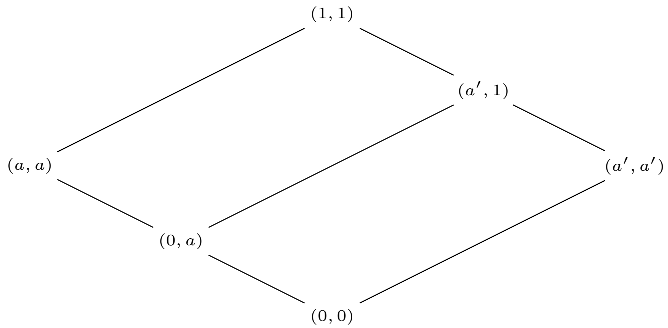
PROOF: By Theorem 4.1, every V-formation admits an amalgam that renders commutative the corresponding diagram. Moreover, by Theorem 6.4 any diagram satisfies the interpolation property. Therefore, the super-amalgamation property (Definition 6.1) follows. □

7. Examples and applications

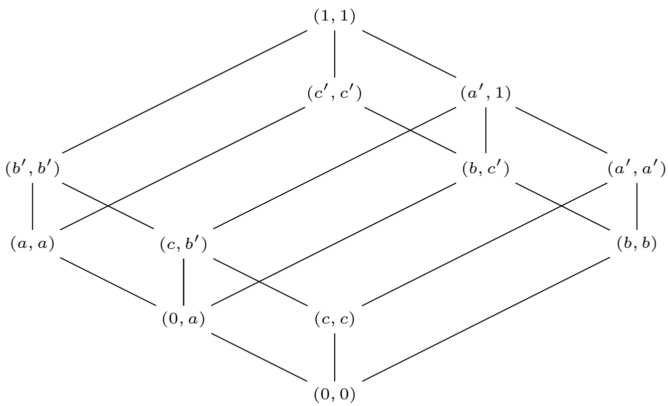
In this section we provide a collection of examples of regular double Stone algebras that will be expedient to describe concretely the construction of some interesting amalgams in the variety of regular double Stone algebras.

Furthermore, we elaborate on the significance of Theorem 4.4 by providing examples that show the existence of V-formations that fail to have a strong amalgam in the variety of regular double Stone algebras.

Example 7.1. Consider the 4-element Boolean algebra $\mathbf{4}$, whose carrier is $\{0, a, a', 1\}$ together with the filter $\{a', 1\}$. Then, the regular double Stone algebra $[\mathbf{4}, \{a', 1\}]$ will be the following:



Example 7.2. Consider the 8-element Boolean algebra **8**, whose carrier is $\{0, a, b, c, a', b', c', 1\}$ together with the filter $\{a', 1\}$. Then, the regular double Stone algebra $[\mathbf{8}, \{a', 1\}]$ will be the following:



As mentioned in Section 4, the variety of regular double Stone algebras does not fulfill the strong amalgamation property in general. However, in case a V-formation $(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2)$ is *homogeneous*, in the sense that all algebras in the formation possess a non-empty core, then the formation admits a strong amalgam. Actually, it is straightforward to observe that in case $\mathbf{A}_0$ possesses a non-empty core the whole V-formation must be homogeneous.

Example 7.1 will be relevant to this discourse.

In fact, it can be seen that $([\mathbf{4},\{a',1\}],[\mathbf{4},4],[\mathbf{8},\{a',1\}])$ is a V-formation, where the mappings f_1, f_2 are self-evident. To ease the notation, let us rename the V-formation as $(\mathbf{A},\mathbf{C}_1,\mathbf{C}_2)$. We can observe that

$$f_1((a',1)) = (f_1\restriction_{S_K(A)}(a'), f_1\restriction_{S_K(A)}(1)) = (a',1) \text{ in } C_1.$$

Clearly, $(\mathbf{4},\mathbf{4},\mathbf{8})$ is a Boolean V-formation of finite algebras. By Theorem 3.14, the algebra $\mathbf{4}\coprod\mathbf{8}/_H$ is an amalgam. As we have seen in Theorem 4.1 and Theorem 5.6 we construct

$$[\mathbf{4}\coprod\mathbf{8}/_H, \mathbf{4}\coprod\mathbf{8}/_H], \tag{7.1}$$

and we obtain a regular double Stone algebra with a non-empty core which amalgamates the formation

$$([\mathbf{4},\{a',1\}],[\mathbf{4},4],[\mathbf{8},\{a',1\}]).$$

Let us recall that the algebra in Display (7.1) is not a strong amalgam. Indeed, the same reasoning applies in Example 7.3.

Theorem 4.4 follows from general algebraic facts. However, it may be interesting to find a concrete case in which a V-formation of regular double Stone algebras does not have a strong amalgam within the same variety.

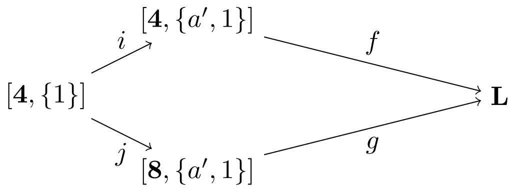
In Example 7.3, we propose a direct application of Theorem 4.4. In fact, making a straightforward use of the subdirectly irreducible members in the variety of regular double Stone algebras it can be shown that in general this variety does not fulfill the strong amalgamation property.

A more elaborated counterexample is Example 7.3. In fact, Example 7.3 presents the case of a V-formation homogeneous for not possessing a fixpoint which can not be strongly amalgamated in the variety of regular double Stone algebras.

Example 7.3. Consider the V-formation

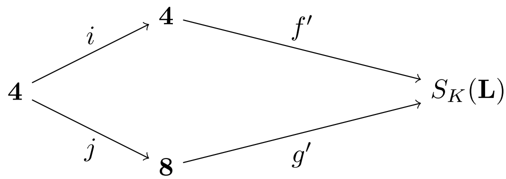
$$([\mathbf{4},\{1\}],[\mathbf{4},\{a',1\}],[\mathbf{8},\{a',1\}]),$$

where $[\mathbf{4}, \{a', 1\}]$ and $[\mathbf{8}, \{a', 1\}]$ are as in Example 7.1 and Example 7.2, respectively. Suppose that there exists a strong amalgam $\mathbf{L}$ as in Diagram (7.2):



(7.2)

Note that Diagram (7.2) is determined by Diagram (7.3):



(7.3)

Moreover, $S_K(\mathbf{L})$ is a strong Boolean amalgam. Indeed, for any element $x = (\Box x, \Diamond x)$, and any homomorphism in Diagram (7.2):

$$f((\Box x, \Diamond x)) = (f'(\Box x), f'(\Diamond x)).$$

By construction, the element $(a', 1)$ is in both $[\mathbf{4}, \{a', 1\}]$ and $[\mathbf{8}, \{a', 1\}]$. Moreover, by Condition (Regularity), $(a', 1)$ is the unique unsharp element in the interval $[(a', a'), (1, 1)]$.

Therefore, we have that

$$f((a', 1)) = (f'(a)', f'(1)) = (g'(a)', g'(1)) = g((a', 1)).$$

However, $(a', 1)$ is not the image of any element in $[\mathbf{4}, \{1\}]$, since $(a', 1)$ is unsharp, and $[\mathbf{4}, \{1\}]$ is a Boolean algebra.

Thus, the V-formation $([\mathbf{4}, \{1\}], [\mathbf{4}, \{a', 1\}], [\mathbf{8}, \{a', 1\}])$ does not admit any strong amalgam in the variety of regular double Stone algebras.

Acknowledgements. All authors thank Wesley Fussner for his email, mentioning Theorem 4.4 and its proof, as well as the references to Lemma 5.2 and Lemma 5.3.

References

[1] M. E. Adams, H. P. Sankappanavar, J. Vaz de Carvalho, *Regular double p-algebras*, **Mathematica Slovaca**, vol. 69(1) (2019), pp. 15–34, DOI: https://doi.org/10.1515/ms-2017-0200.

[2] R. Balbes, P. Dwinger, **Distributive Lattices**, University of Missouri Press, Columbia (1974).

[3] B. Banaschewski, *On the strong amalgamation of Boolean algebras*, **Algebra Universalis**, vol. 63(2–3) (2010), pp. 235–238, DOI: https://doi.org/10.1007/s00012-010-0072-5.

[4] R. Beazer, *The determination congruence on double p-algebras*, **Algebra Universalis**, vol. 6 (1976), pp. 121–129, DOI: https://doi.org/10.1007/BF02485882.

[5] G. Birkhoff, **Lattice Theory**, 3rd ed., vol. 25 of AMS Colloquium Publications, American Mathematical Society, Providence (1973).

[6] T. S. Blyth, **Lattices and Ordered Algebraic Structures**, Springer, Berlin (2005), DOI: https://doi.org/10.1007/b139095.

[7] L. Borowski (ed.), **Selected Works of J. Łukasiewicz**, North–Holland, Amsterdam (1970).

[8] S. Burris, H. P. Sankappanavar, **A Course in Universal Algebra**, vol. 78 of Graduate Texts in Mathematics, Springer, New York (1981).

[9] M. Campercholi, *Dominions and primitive positive functions*, **Journal of Symbolic Logic**, vol. 83(1) (2018), pp. 40–54, DOI: https://doi.org/10.1017/jsl.2017.53.

[10] I. Chajda, A. Ledda, G. Vergottini, *Representations of regular double Stone algebras. Theory and applications*, **Logic Journal of the IGPL**, in press.

[11] R. Cignoli, *Representation of Łukasiewicz and Post algebras by continuous functions*, **Colloquium Mathematicum**, vol. 24 (1972), pp. 127–138, DOI: https://doi.org/10.4064/cm-24-1-127-138.

[12] R. Cignoli, *Injective De Morgan and Kleene algebras*, **Proceedings of the American Mathematical Society**, vol. 47(2) (1975), pp. 269–278, DOI: https://doi.org/10.1090/S0002-9939-1975-0369188-9.

[13] S. Comer, *On connections between information systems, rough sets and algebraic logic*, [in:] **Algebraic Methods in Logic and in Computer Science**, vol. 28 of Banach Center Publications, Institute of Mathematics of the Polish Academy of Sciences (1993), pp. 117–124.

[14] J. M. Cornejo, M. Kinyon, H. P. Sankappanavar, *Regular Double p-Algebras: A Converse to a Katriňák Theorem and Applications*, **Mathematica Slovaca**, vol. 73(6) (2023), pp. 1373–1388, DOI: https://doi.org/10.1515/ms-2023-0085.

[15] J. M. Cornejo, H. P. Sankappanavar, *Amalgamation Property in the subvarieties of Gautama and Almost Gautama algebras*, **Journal of Mahani Mathematical Research**, (2025), online first; arXiv:2409.13124.

[16] I. Düntsch, *A logic for rough sets*, **Theoretical Computer Science**, vol. 179(1–2) (1997), pp. 427–436, DOI: https://doi.org/10.1016/S0304-3975(96)00173-1.

[17] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, **Residuated Lattices: An Algebraic Glimpse at Substructural Logics**, Elsevier, Amsterdam (2007), DOI: https://doi.org/10.1016/S0049-237X(06)80001-6.

[18] R. Giuntini, A. Ledda, G. Vergottini, *Generalizing orthomodularity to unsharp contexts: properties, blocks, residuation*, **Logic Journal of the IGPL**, (2024), DOI: https://doi.org/10.1093/jigpal/jzae076.

[19] G. Grätzer, **Lattice Theory: First Concepts and Distributive Lattices**, Courier Corporation, Chelmsford (2009).

[20] P. Hájek, **Metamathematics of Fuzzy Logic**, Kluwer, Dordrecht (1998), DOI: https://doi.org/10.1007/978-94-011-5300-3.

[21] W. Hodges, **Model Theory**, Cambridge University Press, Cambridge (1993), DOI: https://doi.org/10.1017/CBO9780511551574.

[22] E. Hoogland, **Definability and Interpolation: Model-Theoretic Investigations**, vol. DS-2001–05 of ILLC Dissertation Series, University of Amsterdam, Amsterdam (2001).

[23] P. T. Johnstone, **Stone Spaces**, vol. 3 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (1982), DOI: https://doi.org/10.1017/CBO9780511897719.

[24] P. T. Johnstone, *Vietoris locales and localic semilattices*, [in:] R. E. Hoffmann, K. H. Hofmann (eds.), **Continuous Lattices and Their Applications**, Marcel Dekker, New York (1985), pp. 155–180.

[25] T. Katriňák, *The structure of distributive double p-algebras. Regularity and congruences*, **Algebra Universalis**, vol. 3 (1973), pp. 238–246, DOI: https://doi.org/10.1007/BF02485442.

[26] T. Katriňák, *Construction of regular double p-algebras*, **Bulletin de la Société Royale des Sciences de Liège**, vol. 43 (1974), pp. 238–246.

[27] T. Katriňák, P. Mederly, *Constructions of p-algebras*, **Algebra Universalis**, vol. 17 (1983), pp. 288–316, DOI: https://doi.org/10.1007/BF01194937.

[28] S. Koppelberg (ed.), **Handbook of Boolean Algebras**, North–Holland, Amsterdam (1989).

[29] K. Kuratowski, T. Posament, *Sur l'isomorphie algébro-logique et les ensembles relativement boréliens*, **Fundamenta Mathematicae**, vol. 22 (1934), pp. 281–286.

[30] R. Lagrange, *Amalgamation and epimorphisms in m-complete Boolean algebras*, **Algebra Universalis**, vol. 4 (1974), pp. 277–279, DOI: https://doi.org/10.1007/BF02485823.

[31] A. Ledda, G. Vergottini, *A survey on unsharp orthomodular lattices: A unifying framework*, **Journal of Algebraic Hyperstructures and Logical Algebras**, vol. 5(1) (2024), pp. 19–33, DOI: https://doi.org/10.61838/kman.jahla.5.1.4.

[32] A. Ledda, G. Vergottini, *Orthomodular and Unsharp Orthomodular Lattices: A Categorical Equivalence*, **Studia Logica**, (2025), DOI: https://doi.org/10.1007/s11225-025-10191-z, published online first.

[33] S. Mac Lane, **Categories for the Working Mathematician**, 2nd ed., vol. 5 of Graduate Texts in Mathematics, Springer, New York (1998), DOI: https://doi.org/10.1007/978-1-4757-4721-8.

[34] J. T. Madarász, *Interpolation and amalgamation; pushing the limits. Part I*, **Studia Logica**, vol. 61 (1998), pp. 311–345, DOI: https://doi.org/10.1023/A:1005151808900.

[35] L. L. Maksimova, *Craig interpolation in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras*, **Algebra i Logika**, vol. 16 (1977), pp. 643–681.

[36] L. L. Maksimova, *Interpolation theorems in modal logics and amalgamable varieties of topological Boolean algebras*, **Algebra i Logika**, vol. 18 (1980), pp. 556–586.

[37] L. L. Maksimova, *Interpolation theorems in modal logics: sufficient conditions*, **Algebra i Logika**, vol. 19 (1980), pp. 194–213.

[38] G. C. Moisil, *Recherches sur les logiques non chrysippiennes*, **Annales Scientifiques de l'Université de Jassy**, vol. 26 (1940), pp. 431–466.

[39] A. Monteiro, L. Monteiro, **Three-valued Łukasiewicz Algebras**, Universidad Nacional del Sur, Bahía Blanca (1963), lecture notes.

[40] J. Pavelka, *On fuzzy logic I: many-valued rules of inference*, **Zeitschrift für mathematische Logik und Grundlagen der Mathematik**, vol. 25 (1979), pp. 45–52, DOI: https://doi.org/10.1002/malq.19790250105.

[41] J. Pavelka, *On fuzzy logic II: enriched residuated lattices and semantics of propositional calculi*, **Zeitschrift für mathematische Logik und Grundlagen der Mathematik**, vol. 25 (1979), pp. 119–134, DOI: https://doi.org/10.1002/malq.19790250205.

[42] J. Pavelka, *On fuzzy logic III: semantical completeness of some many-valued propositional calculi*, **Zeitschrift für mathematische Logik und Grundlagen der Mathematik**, vol. 25 (1979), pp. 447–464, DOI: https://doi.org/10.1002/malq.19790251502.

[43] Z. Pawlak, *Information systems — theoretical foundations*, **Information Systems**, vol. 6(3) (1981), pp. 205–218, DOI: https://doi.org/10.1016/0306-4379(81)90023-5.

[44] Z. Pawlak, *Rough sets*, **International Journal of Computer and Information Sciences**, vol. 11(5) (1982), pp. 341–356, DOI: https://doi.org/10.1007/BF01001956.

[45] A. M. Pitts, *Amalgamation and interpolation in the category of Heyting algebras*, **Journal of Pure and Applied Algebra**, vol. 29 (1983), pp. 155–165, DOI: https://doi.org/10.1016/0022-4049(83)90052-4.

[46] B. Poizat, **A Course in Model Theory**, Springer (2000).

[47] J. Pomykala, J. A. Pomykala, *The Stone algebra of rough sets*, **Bulletin of the Polish Academy of Sciences: Mathematics**, vol. 36 (1988), pp. 495–508.

[48] P. Ribenboim, *Characterization of the sup-complement in a distributive lattice with last element*, **Summa Brasiliensis Mathematicae**, vol. 2 (1949), pp. 43–49.

[49] H. P. Sankappanavar, *Expansions of semi-Heyting algebras I: Discriminator varieties*, **Studia Logica**, vol. 98(1–2) (2011), pp. 27–81, DOI: https://doi.org/10.1007/s11225-010-9289-0.

[50] R. Sikorski, *On an analogy between measures and homomorphisms*, **Rocznik Polskiego Towarzystwa Matematycznego**, vol. 23 (1950), pp. 1–20.

[51] A. R. J. Srikanth, R. V. G. Kumar, *Centre of Core Regular Double Stone Algebra*, **European Journal of Pure and Applied Mathematics**, vol. 10 (2017), pp. 717–729.

[52] J. Varlet, *A regular variety of type* $\langle 2,2,1,1,0,0\rangle$, **Algebra Universalis**, vol. 2 (1972), pp. 218–223, DOI: https://doi.org/10.1007/BF02485712.

Antonio Ledda
Università degli Studi di Cagliari
Facoltà di Studi Umanistici
Cagliari, Italia
e-mail: antonio.ledda@unica.it

Hanamantagouda P. Sankappanavar
State University of New York, New Paltz
Department of Mathematics
NY, USA
e-mail: sankapph@hawkmail.newpaltz.edu

Gandolfo Vergottini
Universitá di Urbino Carlo Bo
Dipartimento di Scienze Pure e Applicate
Urbino, Italia
e-mail: gandolfo.vergottini@uniurb.it

Funding information: A. Ledda and G. Vergottini were supported by PLEXUS, (Grant Agreement no. 101086295) a Marie Sklodowska-Curie action funded by the EU under the Horizon Europe Research and Innovation Programme.
The same authors express their gratitude for the support of the Italian Ministry of Education, University and Research through the PRIN 2022 project "Developing Kleene Logics and their Applications" (DeKLA), project code: 2022SM4XC8, and Fondazione di Sardegna, through the project "An algebraic approach to hyperintensionality", project code: F23C25000360007.

Conflict of interests: None.

Ethical considerations: The Authors assure of no violations of publication ethics and take full responsibility for the content of the publication.

The percentage share of the author in the preparation of the work: Antonio Ledda $33\frac{1}{3}\%$, Hanamantagouda P. Sankappanavar $33\frac{1}{3}\%$, Gandolfo Vergottini $33\frac{1}{3}\%$

Declaration regarding the use of GAI tools: Not used.