ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 141, 1997

*Maria Jadamus-Hacura**

BAYESIAN ESTIMATION OF A SHIFT POINT IN A TWO-PHASE REGRESSION MODEL

Abstract. The purpose of this paper is to carry out the Bayesian analysis of a two-phase regression model with an unknown break point. Essentially, there are two problems associated with a changing linear model. Firstly, one will want to be able to detect a break point, and secondly, assuming that a change has occurred, to be able to estimate it as well as other parameters of the model. Much of the classical testing procedure for the parameter constancy (as the Chow test, CUSUM, CUSUMSQ, tests and their modifications, predictions tests for structural stability) indicate only that the regression coefficients shifted, without specifying a break point.

In this study we adopt the Bayesian methodology of investigating structural changes in regression models. The break point is identified as the largest posterior mass density, the peak of the posterior discrete distribution of a break point. It seems to work well with artificially generated data.

The Bayesian framework also seems to be promising for extending the analysis of a single break to that of multiple breaks.

Key words: two-phase-regression model, changing linear model, detection a break point, Bayesian estimation, test for structural stability.

1. INTRODUCTION

The definition of structural changes has not been given clearly in studies, but mainly it has been considered as the problem of non-constancy of regression parameters over the sample period. In this paper we adopt this concept. It is obvious that the use of sample where the regression coefficients are not constant, leads to biased testing results, because testing procedures are based on the full sample estimates. Under the possibility of structural changes during the sample period, it is important to segment a full sample into sub samples, where the regression parameters are constant. This problem has been attacked in various ways.

* Department of Econometrics, Academy of Economics, Katowice.

Our analysis is based on the Bayesian estimation of a shift point presented by Bromeling and Tsurumi (1987). The marginal posterior density function of the shift point is used for testing the stability of parameters in regression model and some numerical studies are performed.

2. MODEL AND THE POSTERIOR DISTRIBUTION OF THE SHIFT POINT

The model we shall consider is the two-phase regression model:

$$y_i = \begin{cases} x_i\beta_1 + \varepsilon_i & i = 1, 2, ..., m \\ x_i\beta_2 + \varepsilon_i & i = m+1, m+2, ..., n \end{cases} \tag{1}$$

where:

- y_i $(i = 1, 2, ..., n)$ is the ith observation of the dependent variable,
- x_i is the $1 \times p$ vector of given observations of independent variables,
- β_1, β_2 are the $p \times 1$ vectors of the regression coefficients considered to shift from the first regime to the second regime,
- m is the unknown shift point,
- ε_i is the error term; it is assumed that the ε_i are normally and independently distributed with zero mean and common unknown precision τ $(E(\varepsilon) = 0,\ E(\varepsilon'\varepsilon) = \tau^{-1}I_n)$.

If $m = n$, no change has occurred, while if $m \in \{1, 2, ..., n-1\}$ exactly one change has occurred. Note, that for $m \neq n$ we can rewrite the model (1) as follows:

$$\begin{bmatrix} Y_1(m) \\ Y_2(m) \end{bmatrix} = \begin{bmatrix} X_1(m) & 0 \\ 0 & X_2(m) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \varepsilon \tag{2}$$

where:

$$X_1(m) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix},\ X_2(m) = \begin{bmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{bmatrix},\ Y_1(m) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},\ Y_2(m) = \begin{bmatrix} y_{m+1} \\ y_{m+2} \\ \vdots \\ y_n \end{bmatrix}.$$

On the other hand, for $m = n$ the model is given by:

$$Y = X\beta_1 + \varepsilon \tag{3}$$

where:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

As it is well known, in the Bayesian approach all inferences about unknown parameter vector θ are based on the posterior probability density function (*pdf*) for parameter vector θ, given the sample information y ($p(\theta/y)$). We would obtain this posterior *pdf* $p(\theta/y)$ by combining prior *pdf* ($p(\theta)$), which represents our initial beliefs about θ and the likelihood function $L(\theta/y)$ representing the sample information.

In the analysis of linear regression models usually two types of the prior probability density functions have been used: a noninformative prior, when an informative or a subjective prior is not available (see, for example Keying, 1993) and an informative prior named "natural conjugate" prior, which is often useful in representing prior information, relatively simple, and mathematically tractable (see De Groot, 1981).

In our model (1) parameter vector $\theta = (m, \beta_1, \beta_2, \tau)$. For our analysis we consider the following natural conjugate, the normal-gamma *pdf* for θ:

$$p(m, \beta_1, \beta_2, \tau) = \begin{cases} q(2\pi)^{-p/2}|\Lambda|^{1/2}\tau^{p/2}\exp(-\tau/2(\beta_1-\beta_{10})'\Lambda_{11}(\beta_1-\beta_{10}))\cdot \\ \cdot b^a/\Gamma(a)\tau^{a-1}\exp(-\tau b) \qquad \text{dla } m = n \\ \cdot(1-q)/(n-1)(2\pi)^{-p}|\Lambda|^{1/2}\tau^{p}\exp(-\tau/2(\beta-\beta_0)'\Lambda(\beta-\beta_0))\cdot \\ {}^{*}b^a/\Gamma(a)\tau^{a-1}\exp(-\tau b) \qquad \text{dla } 1 \leqslant m \leqslant n-1 \end{cases} \tag{4}$$

For $m = n$ parameters (β_1, τ) are assigned the normal-gamma density i.e. the conditional distribution for β_1 is the p-dimensional normal *pdf* with mean vector β_{10} and precision matrix $\tau\Lambda_{11}$ (Λ_{11} is a positive-definite matrix of order p), while the marginal distribution for τ is the gamma *pdf* with parameters a and b. From (4) it is visible that when $1 < m < n-1$, the conditional distribution for $\beta_0 = (\beta_1, \beta_2)$ is $2p$-dimensional normal *pdf* with mean vector $\beta_0 = (\beta_{10}, \beta_{20})$ and positive-definite precision matrix Λ of order $2p$, where:

$$\Lambda = \begin{bmatrix} \Lambda_{11}\Lambda_{12} \\ \Lambda_{21}\Lambda_{22} \end{bmatrix}$$

Furthermore, the marginal prior density function of m is:

$$p(m) = \begin{cases} q & \text{dla } m = n \\ (1-q)/(n-1) & \text{dla } 1 \leqslant m \leqslant n-1 \end{cases} \tag{5}$$

Note, that for $m = n$, the model is stable and q represents the probability of this event. When we are not sure that the model is stable, $q = 0.5$ seems to be appropriate prior choice.

Our main interest is to find the marginal posterior density function of m and to compare it with the marginal prior density function of m given by (5).

Under the above assumptions the likelihood function is:

$$L(q/Y) = \begin{cases} (2\pi)^{-n/2}\tau^{n/2}\exp(-\tau/2(Y-X\beta_1)'(Y-X\beta_1)) & \text{dla } m=n \\ (2\pi)^{-n/2}\tau^{n/2}\exp(-\tau/2(Y-X(m)\beta)'(Y-X(m)\beta)) & \text{dla } 1 \leqslant m \leqslant n \end{cases} \tag{6}$$

where:

$$X(m) = \begin{bmatrix} X_1(m) & 0 \\ 0 & X_2(m) \end{bmatrix}$$

Then, combining (4) and (6) we obtain the joint posterior *pdf* for θ. This function is a basis for making inferences about β_1, β_2, τ and m. Integrating the joint posterior *pdf* with respect to β and τ, the posterior distribution of the shift point is obtained as:

$$p(m/Y) \propto \begin{cases} q|\Lambda_{11}|^{0,5}D(n)^{-(n+2a)/2}|X'X+\Lambda_{11}|^{-0,5} & \text{dla } m=n \\ (1-q)/(n-1)|\Lambda|^{0,5}D(m)^{-(n+2a)/2}|X'(m)X(m)+\Lambda|^{-0,5} & \\ & \text{dla } 1 \leqslant m \leqslant n-1 \end{cases} \tag{7}$$

where:

$$D(n) = b + 0,5[Y'Y + \beta_{10}'\Lambda_{11}\beta_{10} - \beta_1^{*\prime}(X'X+\Lambda_{11})\beta_1^*]$$

$$D(m) = b + 0,5\{[Y - X(m)\beta^*(m)]'Y + [\beta_0 - \beta^*(m)]'\Lambda\beta_0\} \quad \text{for } 1 \leqslant m \leqslant n-1$$

and

$$\beta_1^* = [X'X + \Lambda_{11}]^{-1}[\Lambda_{11}\beta_{10} + X'Y];$$

$$\beta^*(m) = [X'(m)X(m) + \Lambda]^{-1}[\Lambda\beta_0 + X'(m)Y].$$

The comparison of the posterior probability density function of m with the prior probability density function (5) let us deduce about the model parameters stability as well as specify the shift point m^*. When the posterior probability of no change $p(m=n)/Y$ is less than the prior probability of no change equal q, the model is unstable; in the opposite case the hypothesis about the model stability should be accepted.

3. A NUMERICAL STUDY

For illustrative purposes, we perform some numerical studies in this section. The posterior distribution of the shift point m was obtained with the data generated by the following model:

$$y = \begin{cases} 3x_i + \varepsilon_i, & i = 1, 2, ..., m \\ (3+\Delta)x_i + \varepsilon_i, & i = m+1,\ m+2, ..., 20 \end{cases} \tag{8}$$

where:

– x_i ($i = 1, 2, ..., 20$) are generated from the uniform distribution over $\langle 1, 20 \rangle$,

– $\varepsilon_i \sim N(0, 1)$,
– for $\Delta \neq 0$ a structural change occurs at point $m = m^*$.

In the numerical studies we put $m^* = 5$, 10, 15 and $\Delta = 0.1$, 0.2, 0.4 and 0.5 i.e. the angle between regression lines is respectively equal 0.01 rad, 0.02 rad, 0.03 rad and 0.04 rad.

The prior distribution for this example is such that $q = 0.5$, m is independent of β and τ, β given τ is four dimensional normal *pdf* with mean vector $\beta_0 = (3, 0, 3, 0)$ i.e. the model is stable, and precision matrix is $\tau\ I_4$ ($\Lambda = I_4$ and $\Lambda_{11} = \Lambda_{22} = I_2$, $\Lambda_{12} = \Lambda_{21} = 0$) and the marginal distribution of τ is the gamma *pdf* with parameters $a = 2$, $b = 1$.

Table 1

Marginal posterior distribution of m when $m^* = 5$

$m \backslash \Delta$	0.1	0.2	0.4	0.5
1	0.120	0.001	0.000	0.000
2	0.038	0.002	0.000	0.000
3	0.022	0.003	0.000	0.000
4	0.100	**0.391**	0.062	0.010
5	0.044	0.162	0.318	0.431
6	0.043	0.201	**0.400**	**0.435**
7	0.053	0.238	0.220	0.124
8	0.009	0.000	0.000	0.000
9	0.019	0.000	0.000	0.000
10	0.011	0.000	0.000	0.000
11	0.006	0.000	0.000	0.000
12	0.006	0.000	0.000	0.000
13	0.006	0.000	0.000	0.000
14	0.002	0.000	0.000	0.000
15	0.001	0.000	0.000	0.000
16	0.002	0.000	0.000	0.000
17	0.005	0.000	0.000	0.000
18	0.005	0.000	0.000	0.000
19	0.006	0.000	0.000	0.000
20	**0.502**	**0.001**	**0.000**	**0.000**

The objective is to test the null hypothesis of no change by using the marginal posterior density function (7).

The posterior distributions of m are shown in Tab. 1–3 and in Fig. 1–3. From these tables and figures we can see the following facts:

– only in two cases: for $\Delta = 0.1$, and $m^* = 5$ and $m^* = 15$ the posterior probability of no change $(p(m = 20)/Y)$ is larger than $q = 0.5$. In other cases this posterior probability is close to zero, which indicates the model instability.

– we can estimate the shift point m^* as the mode of the marginal posterior density function of m;

Table 2

Marginal posterior distribution of m when $m^* = 10$

$m\backslash\Delta$	0.1	0.2	0.4	0.5
1	0.002	0.000	0.000	0.000
2	0.001	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000
4	0.001	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.000
7	0.001	0.000	0.000	0.000
8	0.001	0.000	0.000	0.000
9	0.166	0.046	0.000	0.000
10	**0.330**	**0.716**	**0.979**	**0.995**
11	0.161	0.118	0.011	0.003
12	0.134	0.103	0.010	0.002
13	0.093	0.017	0.000	0.000
14	0.001	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000
16	0.000	0.000	0.000	0.000
17	0.001	0.000	0.000	0.000
18	0.001	0.000	0.000	0.000
19	0.001	0.000	0.000	0.000
20	**0.106**	**0.000**	**0.000**	**0.000**

if Δ increases, the probability $P(m = m^*)/Y$, is larger.

– in general a shift in the centre of the data is easier to detect (see Tab. 2), the problem arises only if the break point is at the beginning or the end and the value of Δ is relatively small.

Table 3

Marginal posterior distribution of m when $m^* = 15$

$m\backslash\Delta$	0.1	0.2	0.4	0.5
1	0.013	0.000	0.000	0.000
2	0.004	0.000	0.000	0.000
3	0.002	0.000	0.000	0.000
4	0.002	0.000	0.000	0.000
5	0.002	0.000	0.000	0.000
6	0.002	0.000	0.000	0.000
7	0.002	0.000	0.000	0.000
8	0.002	0.000	0.000	0.000
9	0.025	0.010	0.000	0.000
10	0.028	0.016	0.000	0.000
11	0.021	0.011	0.000	0.000
12	0.019	0.012	0.000	0.000
13	0.037	0.047	0.000	0.000
14	0.028	**0.785**	0.168	0.007
15	0.005	0.064	**0.830**	**0.993**
16	0.003	0.007	0.001	0.000
17	0.007	0.000	0.000	0.000
18	0.007	0.000	0.000	0.000
19	0.009	0.001	0.000	0.000
20	**0.782**	0.045	0.000	0.000

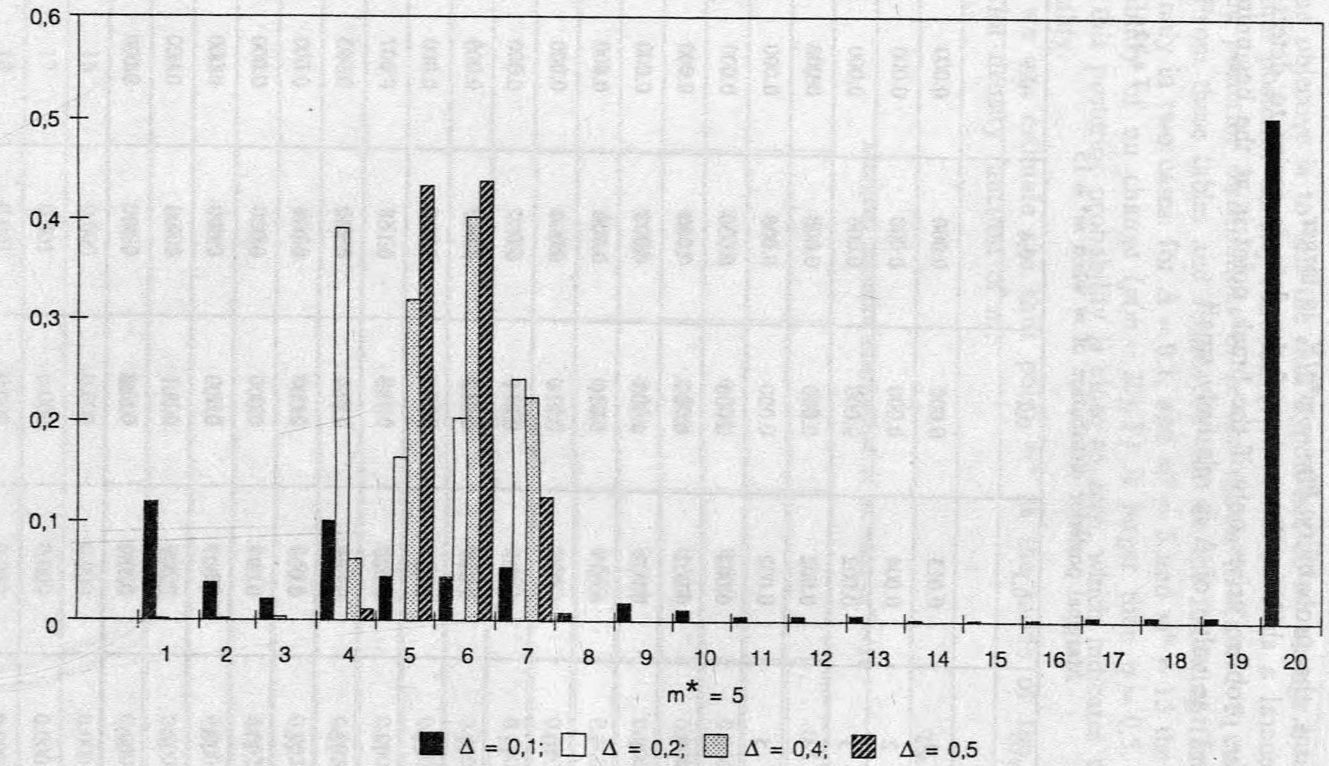

Fig. 1. Marginal posterior distribution of *m*

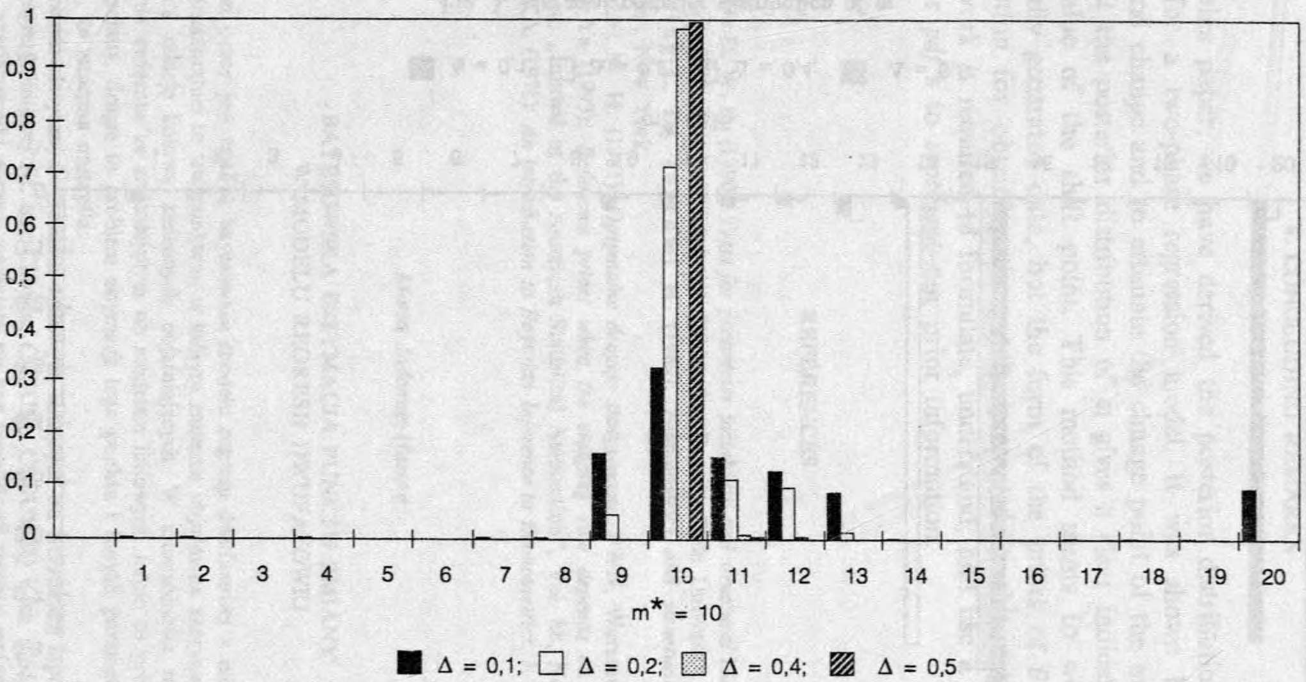

Fig. 2. Marginal posterior distribution of *m*

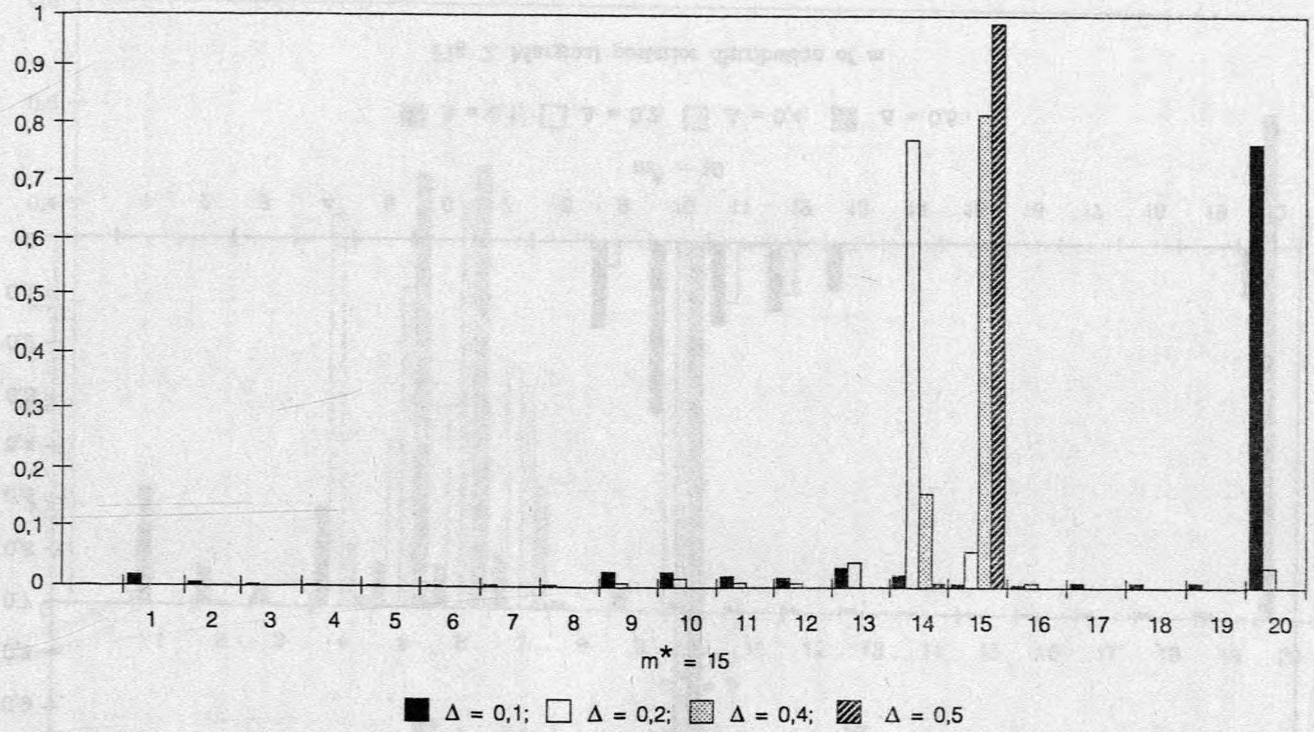

Fig. 3. Marginal posterior distribution of *m*

4. CONCLUDING REMARKS

In this paper, we have derived the posterior distribution of the shift point for a two-phase regression model. It was shown how to detect structural change and to estimate the change point of the model. We may see that the posterior distribution of m gives a clear indication about the true value of the shift point. This method seems to work well with artificially generated data, but the form of the priors of θ is the crucial assumption for obtaining the results presented in this paper. Therefore, more work is required to formulate, understand, and use a broader range of prior *pdf*'s to represent our prior information.

REFERENCES

Andrews D. W. K. (1990): *Tests for parameter instability and structural change with unknown change point*, „Cowles Foundation Discussion Paper", Yale University, No 943.

Broemeing L. D., Tsurumi H. (1987): *Econometrics and structural change*, Marcel Dekker, New York.

De Groot M. H. (1981): *Optymalne decyzje statystyczne*, PWN, Warszawa.

Keying Ye (1993): *Reference priors when the stopping rule depends on the parameter of interest*, „Journal of the American Statistical Association", Vol. 88, No 421.

Zellner A. (1971): *An introduction to Bayesian Inference in Econometrics*, J. Wiley, New York.

Maria Jadamus-Hacura

BAYESOWSKA ESTYMACJA PUNKTU ZMIANY W MODELU REGRESJI DWUFAZOWEJ

Celem pracy jest analiza bayesowska modelu regresji dwufazowej z nieznanym punktem zmiany strukturalnej to jest punktem, w którym zmienna objaśniana zaczyna być kształtowana przez inną relację liniową zmiennych objaśniających. W szczególności rozważane są dwa zagadnienia związane ze zmieniającym się modelem liniowym. Jedno to problem wykrywania punktu zmiany, drugie to problem estymacji tego punktu i innych parametrów modelu przy założeniu, że zmienna nastąpiła.

Większość klasycznych procedur testowych służących do weryfikacji hipotezy o stabilności modelu regresji liniowej (tj. test Chow'a, CUSUM, CUSUMSQ i ich modyfikacje) wskazuje tylko, że współczynniki regresji zmieniają się bez specyfikacji punktu zmiany.

W pracy tej zastosowano metodę bayesowską badania strukturalnych zmian w modelach regresji. Przy estymacji bayesowskiej tychże modeli przyjęto klasyczne założenia o gamma-normalnym rozkładzie a priori szacowanych parametrów.

W ostatnim punkcie pracy przedstawiony jest przykład ilustrujący opisaną metodę.