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LIMIT LAWS FOR MULTIVALUED RANDOM VARIABLES

Abstract. In the probability theory, the strong law of large numbers and the central limit theorem are the most important convergence theorems.

Given a probability measure space (Ω, *A*, *P*), random variable in classical definition is a mapping from Ω to **R**. Multivalued random variable is a mapping from Ω to all subset of *X*. For a real separable Banach space *X* with dual space X^*, let $L^p(\Omega, A)$, for $1 \leqslant p \leqslant \infty$, denote the *X*-valued L^p-space. In this paper we introduce the multivalued $\mathbf{L}^p$ space, next the integral for multifunction and some property of the sequences in *X* with respect to the Hausdorff distance convergence.

Probabilistic law for multifunctions are available, when multifunctions are viewed as point-valued mapping into appriopriate space in which the sets are embedded. In this paper, we will discuss limit laws for multivalued random variables whose values are compact or weakly compact in Banach space.

Key words: multivalued random variable, multivalued function, multivalued L^p space, Banach space.

1. INTRODUCTION

In the probability theory, the strong law of large numbers and the central limit theorem are the most important convergence theorems.

Probabilistic law for multifunctions are available, when multifunctions are viewed as point-valued mapping into appriopriate space in which the sets are embedded. In this paper, we will discuss a limit law for random variables whose values are weakly compact in Banach space.

The paper is organized as follows. In section 2 we display the multivalued random variable; some properties of which are discuss in section 3. The limits law in Banach space is presented in section 4.

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2. MULTIVALUED RANDOM VARIABLE

Given a probability measure space (Ω, A, P), a random variable in classical definition is a mapping from Ω to $\mathbf{R}$. Multivalued random variable is a mapping from Ω to all closed subset of X.

We have a real separable Banach space X with dual space X^*. For any nonempty and closed sets A, $B \subset X$ we define the excess $e(A, B)$ of A over B, the Hausdorff distance $h(A, B)$ of A and B, the norm $\|A\|$ of A, and the support function $s(A, \cdot)$ of A.

Definition 1. The excess for two nonempty and closed sets is defined by

$$e(A, B) = \sup_{x \in A} d(x, B), \text{ where } d(x, B) = \inf_{y \in B} \| x - y \|$$

the Hausdorff distance of A and B is given by

$$h(A, B) = \max\{(eA, B), e(B, A)\},$$

the norm $\| \mathrm{A} \|$ of set A we get as

$$\| A \| = h(A, \{0\}) = \sup_{x \in A} \| x \|$$

and the support function:

$$s(A, x^*) = \sup_{x \in A} < x, x^* >, \quad x^* \in X^*.$$

The set of all nonempty and closed subsets of X is a metric space with the Hausdorff distance. The set of all nonempty and compact subsets of X is a complete, separable metric space with the metric h.

For sequence of nonempty and closed subsets of X besides the Hausdorff distance, we use some notion for convergence sequence of sets (Hausdorf 1957; Salinetti, Wets 1979).

Given a sequence $\{A_n\}$ of nonempty subsets of X let:

– s–lim inf A_n be a set of all $x \in X$ such that $\| x_n - x \| \longrightarrow 0$ for some $x_n \in A_n$, $n \geq 1$

– w–lim sup A_n be a set of all $x \in X$ such that $x_k \longrightarrow x$ (weakly) for some $x_k \in A_{n_k}$ $(k \geq 1)$ and some subsequence $\{A_{n_k}\}$ of $\{A_n\}$.

Clearly

$$s\text{–lim inf } A_n \subset w\text{–lim sup } A_n.$$

For a sequence of nonempty and closed sets $\{A_n\}$, s–lim inf A_n is also nonempty and closed.

Definition 2. The sequence $\{A_n\}$ converges to A, denoted by $\lim_{n\to\infty} A_n = A$, if s–lim inf $A_n = A = w$–lim sup A_n.

Definition 3. A multivalued function $\varphi\colon \Omega \longrightarrow 2^X$ with nonempty and closed values, is said to be (weakly) measurable if φ satisfies the following equivalent conditions:

a) $\varphi^{-1}(C) = \{\omega\in\Omega\colon \varphi(\omega)\cap C \neq \varnothing\} \in A$ for every open subset C of X,
b) $d(x, \varphi(\omega))$ is measurable in ω for every $x \in X$,
c) there exists a sequence $\{f_n\}$ of measurable functions $f_n\colon \Omega \longrightarrow X$ such that $\varphi(\omega) = \mathrm{cl}\{f_n(\omega)\}$ for all $\omega\in\Omega$.

Definition 4. A measurable multivalued function $\varphi\colon \Omega \longrightarrow 2^X$ with nonempty and closed values is called a multivalued random variable.

A multivalued function is called strongly measurable if there exists a sequence $\{\varphi_n\}$ of simple functions (measurable functions having a finite number of values in 2^X), such that $h(\varphi_n(\omega), \varphi(\omega)) \longrightarrow 0$ a.s.

Since the set of all nonempty and compact (or convex and compact) subsets of X is a complete separable metric space with the metric h, so multifunction $\varphi\colon \Omega \longrightarrow 2^X$ is measurable if and only if it is strongly measurable. This is equivalent to the Borel measurability of φ.

Let $K(X)$ denote all nonempty and closed subsets of X. As the σ-field on $K(X)$, we get the σ-field G generated by $\varphi^{-1}(C) = \{\omega\in\Omega\colon \varphi(\omega)\cap C \neq \varnothing\}$, for every open subset C of X.

Definition 5. We say that a sequence of multivalued random variables $\varphi_n\colon \Omega \longrightarrow 2^{K(X)}$ is independent if so is $\{\varphi_n\}$ considered as measurable functions from (Ω, A, P) to $(K(X), G)$.

Definition 6. Two multivalued random variables φ, $\psi\colon \Omega \longrightarrow 2^{K(X)}$ are identically distributed if $\varphi(\omega) = \psi(\omega)$ a.s.

Particularly for φ_n with compact values independence (identical distributedness) of $\{\varphi_n\}$ coincides with that considered as Borel measurable functions to all nonempty, compact subsets of X.

In this case, $\{\varphi_n\}$ is independent if and only if

$$P\left(\bigcap_{i=1}^{n}\{\omega\colon \varphi_i(\omega) \subset G_i\}\right) = \prod_{i=1}^{n} P(\{\omega\colon \varphi_i(\omega) \subset G_i\})$$

for each $n \geqslant 1$ and each open subsets $G_1, \ldots, G_n \subset X$.

3. MEAN OF MULTIVALUED RANDOM VARIABLE

Let $L^p(\Omega, A)$, for $1 \leqslant p \leqslant \infty$, denote the X – valued L^p – space. We introduce the multivalued $\mathbf{L}^p$ space.

Definition 7. The multivalued space $\mathbf{L}^p[\Omega, K(X)]$, for $1 \leqslant p \leqslant \infty$ denotes the space of all measurable multivalued functions $\varphi: \Omega \longrightarrow 2^{K(X)}$, such that $\|\varphi\| = \|\varphi(\cdot)\|$ is in $\mathbf{L}^p$.

Then $\mathbf{L}^p[\Omega, K(X)]$ becomes a complete metric space with the metric H_p given by

$$H_p(\varphi, \psi) = \{ \int_\Omega h(\varphi(\omega), \psi(\omega)^p dP\}^{1/p}, \text{ for } 1 \leqslant p \leqslant \infty$$

$$H_p(\varphi, \psi) = \operatorname*{ess\,sup}_{\omega \in \Omega} h(\varphi(\omega), \psi(\omega),$$

where φ and ψ are considered to be identical if $\varphi(\omega) = \psi(\omega)$ a.s.

We can define similarly other $\mathbf{L}^p$ space for the set of different subsets of X (convex and closed, weakly compact or compact). We denote by $\mathbf{L}^p[\Omega, K(X)]$ the space of all strongly measurable functions in $\mathbf{L}^p[\Omega, K(X)]$. Then all this space is complete metric space with the metric H_p.

Definition 8. The mean $E(\varphi)$, for a multivalued random variables $\varphi: \Omega \longrightarrow 2^{K(X)}$ is given as the integral $\int_\Omega \varphi dP$ of φ defined by

$$\int_\Omega \varphi dP = \{ \int_\Omega f dP: f \in S(\varphi)\}$$

where

$$S(\varphi) = \{f \in \mathbf{L}^1[\Omega, X]: f(\omega) \in \varphi(\omega) \text{ a.s.}\}$$

The mean $E(\varphi)$ exists if $S(\varphi)$ is nonempty. If φ have an integral the $E(\varphi)$ is compact.

This multivalued integral was introduced by Aumann (1965). For detailed arguments concerning the measurability and integratin of multifunction we refer to (Castaing, Valadier 1977; Debren 1967; Hiai, Umegaki 1978). Now we present some properties of mean of multivalued random variables.

Let $\varphi, \psi: \Omega \longrightarrow 2^{K(X)}$ be two multivalued random variables with nonempty $S(\varphi)$ and $S(\psi)$, then:

1. cl $E(\varphi \cup \psi) = \text{cl}(E(\varphi) + E(\psi))$, where $(\varphi \cup \psi)(\omega) = \text{cl}(\varphi(\omega) + \psi(\omega))$.
2. cl $E(\overline{\text{co}}\varphi) = \overline{\text{co}}E(\varphi)$, where $(\overline{\text{co}}\varphi)(\omega) = \overline{\text{co}}\varphi(\omega)$, the closed convex hull.
3. $s(\text{cl } E(\varphi), x^*) = E(s(\varphi(\cdot), x^*))$ for every $x^* \in X^*$.
4. Let $Wc(X)$ denote all nonempty and weakly compact subsets of X. If $\varphi = \mathbf{L}^1[\Omega, Wc(X)]$, then $E(\varphi) \in Wc(X)$.

4. MULTIVALUED STRONG LAW OF LARGE NUMBERS

A multivalued strong law of large numbers was proved by Arstein and Vitale (1975) for a sequence of independent identically distributed random variables having values in compact subset of R^n.

Given a probability measure space (Ω, A, P) and a Banach space X, we have a theorem (Serfling 1991, p. 134):

If $\{f_n\}$ is a sequence of independent identically distributed random variables in $L^1[\Omega, X]$, then

$$\lim_{n\to\infty} \| n^{-1} \sum_{i=1}^{n} f_i(\omega) - m \| = 0 \text{ a.s.} \tag{*}$$

where $m = E(f_n)$.

We now establish a strong law for multivalued random variables, which are generalization of this theorem, for the case of independent identically distributed random variables with values are weakly compact subset of Banach space. We start by presenting two lemmas.

Lemma 1. If $\varphi \in L^1[\Omega, K(X)]$ and $E(\varphi) = \{x\}$, then there exists an $f \in L^1[\Omega, X]$ such that $\varphi(\omega) = \{f(\omega)\}$ a.s.

Proof. We can choose the sequence $\{x_j^*\}$ in X^* which separates points of X

By point 3) in section 3 we get

$$E(s(\varphi(\cdot), x_j^*)) = <x, x_j^*> = E(\inf_{x\in\varphi(\cdot)} <x, x_j^*>),$$

and hence $s(\varphi(\omega), x_j^*) = \inf_{x\in\varphi(\cdot)} <x, x_j^*>$ a.s. for $j \geqslant i$. This implies $\varphi(\omega)$ is a single point for a.s. $\omega \in \Omega$. Thus the lemma is proved.

For each $A \in Wc(X)$ and $x^* \in X^*$, we define $\Phi(X, x^*). \in Wc(X)$ by

$$\Phi(A, x^*) = \{x \in X: <x, x^*> = s(A, x^*)\}$$

Lemma 2. For each $x^* \in X^*$, the mapping $\Phi(\cdot, x^*)$: $Wc(X) \longrightarrow Wc(X)$ is measurable with respect to $(G|Wc(X), G|Wc(X))$.

Proof. Let $\mathbf{G} = G|Wc(X)$. Since each open subset of X is a countable union of closed balls, it suffices to show that $\{A \in Wc(X): \Phi(A, x^*). \cap V \neq \varnothing\} \in \mathbf{G}$ for any closed ball V.

Let $V = \{x: \|x-y\| \leqslant r\}$, since

$\{A \in Kc(X): s(A, x^*) > \alpha\} = \{A \in Kc(X): A \cap \{x: \langle x, x^* \rangle > \alpha\} \neq \emptyset\} \in \mathbf{G}$ for every $\alpha \in R$, so the mapping $A \mapsto s(A, x^*)$ from $Kc(X)$ to $\mathbf{R}$ is G measurable.

Let $V_n = \{x: \|x-y\| \leqslant r+n^{-1}\}$ for $n \geqslant 1$. If $A \in Wc(X)$ and $A \cap V_n \neq \emptyset$ for $n \geqslant 1$, then $\{A \cap \mathrm{cl}\, V_n\}$ is a decreasing sequence of nonempty weakly compact subsets of X, so that $A \cap V = \bigcap_{n=1}^{n} \{A \cap V_n\} \neq \emptyset$.

Hence we get

$$\{A \in Wc(X): A \cap V \neq \emptyset\} = \bigcap_{n=1}^{n} \{A \in Wc(X): A \cap V_n \neq \emptyset\} \in \mathbf{G}.$$

Moreover, for any closed ball $V' = \{x: \|x-y\| \leqslant r'\}$, we similarly get

$$\{A \in Wc(X): A \cap V + V' \neq \emptyset\} = \bigcap_{n=1}^{n} \{A \in Wc(X): A \cap V_n \cap V'_n \neq \emptyset\} \in \mathbf{G},$$

where $V'_n = \{x: \|x-y'\| \leqslant r'+n^{-1}\}$.

Hence the mapping $A \mapsto A \cap V$ from $Wc(X)$ to $Wc(X)$ is measurable with respect to $(\mathbf{G}, \mathbf{G})$. Thus we have

$$\{A \in Wc(X): \Phi(A, x^*). \cap V \neq \emptyset\} = \{A \in Wc(X): A \cap V \neq \emptyset, s(A \cap V, x^*) = s(A, x^*)\} \in \mathbf{G}.$$

The lemma is proved.

Theorem 1. If $\{\varphi_n\}$ is a sequence of independent identically distributed random variables in $L^1[Wc(X)]$, then

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \varphi(\omega) = M \text{ a.s.}$$

where $M = E(\varphi_n)$.

Proof. For $\{\varphi_n\}$, let $\Psi_n(\omega) = n^{-1} \sum_{i=1}^{n} \varphi_i(\omega)$ for $n \geqslant 1$. We get $E(\varphi_n) \in Wc(X)$ by point 4) in section 3. As noted in the proof of Lemma 2 the mapping $A \mapsto s(A, x^*)$ from $Kc(X)$ to $\mathbf{R}$ is G measurable, it follows that $\{s(\varphi_n(\cdot), x^*)\}$ is identically distributed for every $x^* \in X^*$. Hence, by point 3) in section 2, we get that $M = E(\varphi_n)$ is independent of n.

We first show that $M \subset s\text{-}\lim\inf \Psi_n(\omega)$ a.s.

Since M is a closed convex hull of its strongly exposed points and $s\text{-}\lim\inf \Psi_n(\omega)$ is closed and convex, it suffices to show that any exposed point of M is contained in $s\text{-}\lim\inf \Psi_n(\omega)$ for a.s. $\omega \in \Omega$.

Let x be any exposed point of M, then there is an $x^* \in X^*$ with $\Phi(M, x^*) = \{x\}$. Lemma 2 implies that $\{\Phi(\varphi_n(\cdot), x^*)\}$ is a sequence of i.i.d.

random variables in $L^1[\Omega, Wc(X)]$. If $f \in S(\Phi(\varphi_n(\cdot), x^*))$, then we have $E(f) \in M$ and $<E(f), x^*> = s(M, x^*)$, so that $E(f) = \Phi(M, x^*) = \{x\}$. Hence $E(\Phi(\varphi_n(\cdot), x^*)) = \{x\}$. By Lemma 1, there exists an $f_n \in L^1[\Omega, X]$ such that $\Phi(\varphi_n(\cdot), x^*) = \{f_n(\omega)\}$ a.s. We thus obtain $\{f_n\}$ of independent identically distributed random variables in $L^1[\Omega, X]$ with $E(f_n) = x$. It follows from (*) that $\lim_{n\to\infty} \|n^{-1}\sum_{i=1}^{n} f_i(\omega) - x\| = 0$ a.s. Since $n^{-1}\sum_{i=1}^{n} f_i(\omega) \in \Psi_n(\omega)$ a.s., we get $x \in s$-lim inf $\Psi_n(\omega)$ a.s

Now, we show that w–lim inf $\Psi_n(\omega) \in M$ a.s.

Since X is separable we can choose a sequence $\{x_j^*\}$ in X^* such that if $<x, x_j^*> \leqslant s(M, x_j^*)$ for all $j \geqslant i$ then $x \in M$. The sequence $\{s(\varphi(\cdot), x_j^*))$ is a sequence of i.i.d. random variables in L^1 with the mean $s(M, x_j^*)$. Hence there exist $B \in A$ with $P(B) = 0$ such that

$$\lim_{n\to\infty} s(\Psi_n(\omega), x_j^*) = \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} s(\varphi_i(\omega), x_j^*) = s(M, x_j^*)$$

for every $j \geqslant i$ and $\omega \in \Omega/B$. If $x \in w$–lim inf $\Psi_n(\omega)$ with $\omega \in \Omega/B$ then $x_k \longrightarrow x$ (weakly) for some $x_k \in s \cdot \Psi_{n_k}(\omega)$ for $k \geqslant 1$. Since

$$<x, x_j^*> = \lim_{n\to\infty} <x_k, x_j^*>) = \lim_{n\to\infty} s(\Psi_{n_k}(\omega), x_j^*) = s(M, x_j^*), \text{ for } j \geqslant i,$$

we get $x \in M$. Thus w–lim inf $\Psi_n(\omega) \in M$ a.s.

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TWIERDZENIA GRANICZNE DLA WIELOWARTOŚCIOWYCH ZMIENNYCH LOSOWYCH

W teorii prawdopodobieństwa mocne prawo wielkich liczb oraz twierdzenie graniczne stanowią podstawowe teorie konwergencji.

Przy danej mierze przestrzeni probabilistycznej (Ω, A, P) zmienna losowa definiowana jest klasycznie jako odwzorowanie Ω na $\mathbf{R}$. Wielowartościowa zmienna losowa jest odwzorowaniem Ω na wszystkie podzbiory X. W pracy wprowadzono pojęcie przestrzeni L^p, całki dla funkcji wielowartościowej oraz niektóre własności ciągu w przestrzeni Banacha ze względu na zbieżność według metryki Hausdorffa.

Rozważane są również moce prawa wielkich liczb dla wielowartościowych zmiennych losowych i twierdzenia graniczne w przestrzeni Banacha.