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On best proximity points for set-valued contractions of Nadler type with respect to b-generalized pseudodistances in b-metric spaces

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Abstract

In this paper, in b-metric space, we introduce the concept of b-generalized pseudodistance which is an extension of the b-metric. Next, inspired by the ideas of Nadler (Pac. J. Math. 30:475-488, 1969) and Abkar and Gabeleh (Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 107(2):319-325, 2013), we define a new set-valued non-self-mapping contraction of Nadler type with respect to this b-generalized pseudodistance, which is a generalization of Nadler's contraction. Moreover, we provide the condition guaranteeing the existence of best proximity points for $T : A \to 2^B$. A best proximity point theorem furnishes sufficient conditions that ascertain the existence of an optimal solution to the problem of globally minimizing the error $\inf\{d(x, y) : y \in T(x)\}$, and hence the existence of a consummate approximate solution to the equation $T(x) = x$. In other words, the best proximity points theorem achieves a global optimal minimum of the map $x \to \inf\{d(x; y) : y \in T(x)\}$ by stipulating an approximate solution x of the point equation $T(x) = x$ to satisfy the condition that $\inf\{d(x; y) : y \in T(x)\} = \mathrm{dist}(A; B)$. The examples which illustrate the main result given. The paper includes also the comparison of our results with those existing in the literature.

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Keywords: b-metric spaces; b-generalized pseudodistances; global optimal minimum; best proximity points; Nadler contraction; set-valued maps

1 Introduction

A number of authors generalize Banach's [1] and Nadler's [2] result and introduce the new concepts of set-valued contractions (cyclic or non-cyclic) of Banach or Nadler type, and they study the problem concerning the existence of best proximity points for such contractions; see *e.g.* Abkar and Gabeleh [3–5], Al-Thagafi and Shahzad [6], Suzuki *et al.* [7], Di Bari *et al.* [8], Sankar Raj [9], Derafshpour *et al.* [10], Sadiq Basha [11], and Włodarczyk *et al.* [12].

In 2012, Abkar and Gabeleh [13] introduced and established the following interesting and important best proximity points theorem for a set-valued non-self-mapping. First, we recall some definitions and notations.

Let A, B be nonempty subsets of a metric space (X, d). Then denote: $\mathrm{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$; $A_0 = \{x \in A : d(x, y) = \mathrm{dist}(A, B)$ for some $y \in B\}$; $B_0 = \{y \in B :$

$d(x,y) = \operatorname{dist}(A,B)$ for some $x \in A\}$; $D(x,B) = \inf\{d(x,y) : y \in B\}$ for $x \in X$. We say that the pair (A,B) has the P-property if and only if

$$\{d(x_1,y_1) = \operatorname{dist}(A,B) \wedge d(x_2,y_2) = \operatorname{dist}(A,B)\} \quad \Rightarrow \quad d(x_1,x_2) = d(y_1,y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Theorem 1.1 (Abkar and Gabeleh [13]) *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$ and (A,B) has the P-property. Let $T : A \to 2^B$ be a multivalued non-self-mapping contraction, that is, $\exists_{0\leq\lambda<1}\forall_{x,y\in A}\{H(T(x),T(y)) \leq \lambda d(x,y)\}$. If $T(x)$ is bounded and closed in B for all $x \in A$, and $T(x_0) \subset B_0$ for each $x_0 \in A_0$, then T has a best proximity point in A.*

It is worth noticing that the map T in Theorem 1.1 is continuous, so it is u.s.c. on X, which by [14, Theorem 6, p.112], shows that T is closed on X. In 1998, Czerwik [15] introduced of the concept of a b-metric space. A number of authors study the problem concerning the existence of fixed points and best proximity points in b-metric space; see *e.g.* Berinde [16], Boriceanu *et al.* [17, 18], Bota *et al.* [19] and many others.

In this paper, in a b-metric space, we introduce the concept of a b-generalized pseudodistance which is an extension of the b-metric. The idea of replacing a metric by the more general mapping is not new (see *e.g.* distances of Tataru [20], w-distances of Kada *et al.* [21], τ-distances of Suzuki [22, Section 2] and τ-functions of Lin and Du [23] in metric spaces and distances of Vályi [24] in uniform spaces). Next, inspired by the ideas of Nadler [2] and Abkar and Gabeleh [13], we define a new set-valued non-self-mapping contraction of Nadler type with respect to this b-generalized pseudodistance, which is a generalization of Nadler's contraction. Moreover, we provide the condition guaranteeing the existence of best proximity points for $T : A \to 2^B$. A best proximity point theorem furnishes sufficient conditions that ascertain the existence of an optimal solution to the problem of globally minimizing the error $\inf\{d(x,y) : y \in T(x)\}$, and hence the existence of a consummate approximate solution to the equation $T(X) = x$. In other words, the best proximity points theorem achieves a global optimal minimum of the map $x \to \inf\{d(x;y) : y \in T(x)\}$ by stipulating an approximate solution x of the point equation $T(x) = x$ to satisfy the condition that $\inf\{d(x;y) : y \in T(x)\} = \operatorname{dist}(A;B)$. Examples which illustrate the main result are given. The paper includes also the comparison of our results with those existing in the literature. This paper is a continuation of research on b-generalized pseudodistances in the area of b-metric space, which was initiated in [25].

2 On generalized pseudodistance

To begin, we recall the concept of b-metric space, which was introduced by Czerwik [15] in 1998.

Definition 2.1 Let X be a nonempty subset and $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is b-metric if the following three conditions are satisfied: (d1) $\forall_{x,y\in X}\{d(x,y) = 0 \Leftrightarrow x = y\}$; (d2) $\forall_{x,y\in X}\{d(x,y) = d(y,x)\}$; and (d3) $\forall_{x,y,z\in X}\{d(x,z) \leq s[d(x,y) + d(y,z)]\}$.

The pair (X,d) is called a b-metric space (with constant $s \geq 1$). It is easy to see that each metric space is a b-metric space.

In the rest of the paper we assume that the b-metric $d : X \times X \to [0,\infty)$ is continuous on X^2. Now in b-metric space we introduce the concept of a b-generalized pseudodistance, which is an essential generalization of the b-metric.

Definition 2.2 Let X be a b-metric space (with constant $s \geq 1$). The map $J : X \times X \to [0,\infty)$, is said to be a *$b$-generalized pseudodistance on X* if the following two conditions hold:

(J1) $\forall_{x,y,z\in X}\{J(x,z) \leq s[J(x,y) + J(y,z)]\}$; and

(J2) for any sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X such that

$$\lim_{n\to\infty} \sup_{m>n} J(x_n, x_m) = 0 \tag{2.1}$$

and

$$\lim_{m\to\infty} J(x_m, y_m) = 0, \tag{2.2}$$

we have

$$\lim_{m\to\infty} d(x_m, y_m) = 0. \tag{2.3}$$

Remark 2.1 (A) If (X,d) is a b-metric space (with $s \geq 1$), then the b-metric $d : X \times X \to [0,\infty)$ is a b-generalized pseudodistance on X. However, there exists a b-generalized pseudodistance on X which is not a b-metric (for details see Example 4.1).

(B) From (J1) and (J2) it follows that if $x \neq y$, $x, y \in X$, then

$$J(x,y) > 0 \vee J(y,x) > 0.$$

Indeed, if $J(x,y) = 0$ and $J(y,x) = 0$, then $J(x,x) = 0$, since, by (J1), we get $J(x,x) \leq s[J(x,y) + J(y,x)] = s[0 + 0] = 0$. Now, defining $(x_m = x : m \in \mathbb{N})$ and $(y_m = y : m \in \mathbb{N})$, we conclude that (2.1) and (2.2) hold. Consequently, by (J2), we get (2.3), which implies $d(x,y) = 0$. However, since $x \neq y$, we have $d(x,y) \neq 0$, a contradiction.

Now, we apply the b-generalized pseudodistance to define the H^J-distance of Nadler type.

Definition 2.3 Let X be a b-metric space (with $s \geq 1$). Let the class of all nonempty closed subsets of X be denoted by $\mathrm{Cl}(X)$, and let the map $J : X \times X \to [0,\infty)$ be a b-generalized pseudodistance on X. Let $\forall_{u\in X}\forall_{V\in \mathrm{Cl}(X)}\{J(u,V) = \inf_{v\in V} J(u,v)\}$. Define $H^J : \mathrm{Cl}(X) \times \mathrm{Cl}(X) \to [0,\infty)$ by

$$\forall_{A,B\in \mathrm{Cl}(X)}\Big\{H^J(A,B) = \max\Big\{\sup_{u\in A} J(u,B), \sup_{v\in B} J(v,A)\Big\}\Big\}.$$

We will present now some indications that we will use later in the work.

Let (X,d) be a b-metric space (with $s \geq 1$) and let $A \neq \emptyset$ and $B \neq \emptyset$ be subsets of X and let the map $J : X \times X \to [0,\infty)$ be a b-generalized pseudodistance on X. We adopt the following denotations and definitions: $\forall_{A,B\in \mathrm{Cl}(X)}\{\mathrm{dist}(A,B) = \inf\{d(x,y) : x \in A, y \in B\}\}$ and

$$A_0 = \{x \in A : J(x,y) = \mathrm{dist}(A,B) \text{ for some } y \in B\};$$

$$B_0 = \{y \in B : J(x,y) = \mathrm{dist}(A,B) \text{ for some } x \in A\}.$$

Definition 2.4 Let X be a b-metric space (with $s \geq 1$) and let the map $J : X \times X \to [0, \infty)$ be a b-generalized pseudodistance on X. Let (A, B) be a pair of nonempty subset of X with $A_0 \neq \emptyset$.

(I) The pair (A, B) is said to have the P^J-property if and only if

$$\begin{aligned} &\bigl\{\bigl[J(x_1, y_1) = \operatorname{dist}(A, B)\bigr] \wedge \bigl[J(x_2, y_2) = \operatorname{dist}(A, B)\bigr]\bigr\} \\ &\quad \Rightarrow \quad \bigl\{J(x_1, x_2) = J(y_1, y_2)\bigr\}, \end{aligned}$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

(II) We say that the b-generalized pseudodistance J is associated with the pair (A, B) if for any sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X such that $\lim_{m\to\infty} x_m = x$; $\lim_{m\to\infty} y_m = y$, and

$$\forall_{m\in\mathbb{N}}\bigl\{J(x_m, y_{m-1}) = \operatorname{dist}(A, B)\bigr\},$$

then $d(x, y) = \operatorname{dist}(A, B)$.

Remark 2.2 If (X, d) is a b-metric space (with $s \geq 1$), and we put $J = d$, then:

(I) The map d is associated with each pair (A, B), where $A, B \subset X$. It is an easy consequence of the continuity of d.

(II) The P^d-property is identical with the P-property. In view of this, instead of writing the P^d-property we will write shortly the P-property.

3 The best proximity point theorem with respect to a *b*-generalized pseudodistance

We first recall the definition of closed maps in topological spaces given in Berge [14] and Klein and Thompson [26].

Definition 3.1 Let L be a topological vector space. The set-valued dynamic system (X, T), *i.e.* $T : X \to 2^X$ is called closed if whenever $(x_m : m \in \mathbb{N})$ is a sequence in X converging to $x \in X$ and $(y_m : m \in \mathbb{N})$ is a sequence in X satisfying the condition $\forall_{m\in\mathbb{N}}\{y_m \in T(x_m)\}$ and converging to $y \in X$, then $y \in T(x)$.

Next, we introduce the concepts of a set-valued non-self-closed map and a set-valued non-self-mapping contraction of Nadler type with respect to the b-generalized pseudodistance.

Definition 3.2 Let L be a topological vector space. Let X be certain space and A, B be a nonempty subsets of X. The set-valued non-self-mapping $T : A \to 2^B$ is called closed if whenever $(x_m : m \in \mathbb{N})$ is a sequence in A converging to $x \in A$ and $(y_m : m \in \mathbb{N})$ is a sequence in B satisfying the condition $\forall_{m\in\mathbb{N}}\{y_m \in T(x_m)\}$ and converging to $y \in B$, then $y \in T(x)$.

It is worth noticing that the map T in Theorem 1.1 is continuous, so it is u.s.c. on X, which by [14, Theorem 6, p.112], shows that T is closed on X.

Definition 3.3 Let X be a b-metric space (with $s \geq 1$) and let the map $J : X \times X \to [0, \infty)$ be a b-generalized pseudodistance on X. Let (A, B) be a pair of nonempty subsets of X.

The map $T : A \to 2^B$ such that $T(x) \in \mathrm{Cl}(X)$, for each $x \in X$, we call a set-valued non-self-mapping contraction of Nadler type, if the following condition holds:

$$\exists_{0\leq\lambda<1}\forall_{x,y\in A}\{sH^J(T(x),T(y)) \leq \lambda J(x,y)\}. \tag{3.1}$$

It is worth noticing that if (X,d) is a metric space (*i.e.* $s = 1$) and we put $J = d$, then we obtain the classical Nadler condition. Now we prove two auxiliary lemmas.

Lemma 3.1 *Let X be a complete b-metric space* (*with $s \geq 1$*). *Let (A,B) be a pair of nonempty closed subsets of X and let $T : A \to 2^B$. Then*

$$\forall_{x,y\in A}\forall_{\gamma>0}\forall_{w\in T(x)}\exists_{v\in T(y)}\{J(w,v) \leq H^J(T(x),T(y)) + \gamma\}. \tag{3.2}$$

Proof Let $x, y \in A$, $\gamma > 0$ and $w \in T(x)$ be arbitrary and fixed. Then, by the definition of infimum, there exists $v \in T(y)$ such that

$$J(w,v) < \inf\{J(w,u) : u \in T(y)\} + \gamma. \tag{3.3}$$

Next,

$$\begin{aligned}
&\inf\{J(w,u) : u \in T(y)\} + \gamma \\
&\quad\leq \sup\{\inf\{J(z,u) : u \in T(y)\} : z \in T(x)\} + \gamma \\
&\quad\leq \max\{\sup\{\inf\{J(z,u) : u \in T(y)\} : z \in T(x)\}, \\
&\qquad \sup\{\inf\{J(u,z) : z \in T(x)\} : u \in T(y)\}\} + \gamma \\
&\quad= H^J(T(x),T(y)) + \gamma.
\end{aligned}$$

Hence, by (3.3) we obtain $J(w,v) \leq H^J(T(x),T(y)) + \gamma$, thus (3.2) holds. □

Lemma 3.2 *Let X be a complete b-metric space* (*with $s \geq 1$*) *and let the sequence* $(x_m : m \in \{0\} \cup \mathbb{N})$ *satisfy*

$$\lim_{n\to\infty}\sup_{m>n} J(x_n,x_m) = 0. \tag{3.4}$$

Then $(x_m : m \in \{0\} \cup \mathbb{N})$ *is a Cauchy sequence on X.*

Proof From (3.4) we claim that

$$\forall_{\varepsilon>0}\exists_{n_1=n_1(\varepsilon)\in\mathbb{N}}\forall_{n>n_1}\{\sup\{J(x_n,x_m) : m > n\} < \varepsilon\}$$

and, in particular,

$$\forall_{\varepsilon>0}\exists_{n_1=n_1(\varepsilon)\in\mathbb{N}}\forall_{n>n_1}\forall_{t\in\mathbb{N}}\{J(x_n,x_{t+n}) < \varepsilon\}. \tag{3.5}$$

Let $i_0, j_0 \in \mathbb{N}$, $i_0 > j_0$, be arbitrary and fixed. If we define

$$z_n = x_{i_0+n} \quad \text{and} \quad u_n = x_{j_0+n} \quad \text{for } n \in \mathbb{N}, \tag{3.6}$$

then (3.5) gives

$$\lim_{n\to\infty} J(x_n, z_n) = \lim_{n\to\infty} J(x_n, u_n) = 0. \tag{3.7}$$

Therefore, by (3.4), (3.7), and (J2),

$$\lim_{n\to\infty} d(x_n, z_n) = \lim_{n\to\infty} d(x_n, u_n) = 0. \tag{3.8}$$

From (3.8) and (3.6) we then claim that

$$\forall_{\varepsilon>0} \exists_{n_2=n_2(\varepsilon)\in\mathbb{N}} \forall_{n>n_2} \left\{ d(x_n, x_{i_0+n}) < \frac{\varepsilon}{2s} \right\} \tag{3.9}$$

and

$$\exists_{n_3=n_3(\varepsilon)\in\mathbb{N}} \forall_{n>n_3} \left\{ d(x_n, x_{j_0+n}) < \frac{\varepsilon}{2s} \right\}. \tag{3.10}$$

Let now $\varepsilon_0 > 0$ be arbitrary and fixed, let $n_0(\varepsilon_0) = \max\{n_2(\varepsilon_0), n_3(\varepsilon_0)\} + 1$ and let $k, l \in \mathbb{N}$ be arbitrary and fixed such that $k > l > n_0$. Then $k = i_0 + n_0$ and $l = j_0 + n_0$ for some $i_0, j_0 \in \mathbb{N}$ such that $i_0 > j_0$ and, using (d3), (3.9), and (3.10), we get $d(x_k, x_l) = d(x_{i_0+n_0}, x_{j_0+n_0}) \leq sd(x_{n_0}, x_{i_0+n_0}) + sd(x_{n_0}, x_{j_0+n_0}) < s\varepsilon_0/2s + s\varepsilon_0/2s = \varepsilon_0$.

Hence, we conclude that $\forall_{\varepsilon>0} \exists_{n_0=n_0(\varepsilon)\in\mathbb{N}} \forall_{k,l\in\mathbb{N},k>l>n_0} \{d(x_k, x_l) < \varepsilon\}$. Thus the sequence $(x_m : m \in \{0\} \cup \mathbb{N})$ is Cauchy. □

Next we present the main result of the paper.

Theorem 3.1 *Let X be a complete b-metric space (with $s \geq 1$) and let the map $J : X \times X \to [0, \infty)$ be a b-generalized pseudodistance on X. Let (A, B) be a pair of nonempty closed subsets of X with $A_0 \neq \emptyset$ and such that (A, B) has the P^J-property and J is associated with (A, B). Let $T : A \to 2^B$ be a closed set-valued non-self-mapping contraction of Nadler type. If $T(x)$ is bounded and closed in B for all $x \in A$, and $T(x) \subset B_0$ for each $x \in A_0$, then T has a best proximity point in A.*

Proof To begin, we observe that by assumptions of Theorem 3.1 and by Lemma 3.1, the property (3.2) holds. The proof will be broken into four steps.

Step 1. *We can construct the sequences $(w^m : m \in \{0\} \cup \mathbb{N})$ and $(v^m : m \in \{0\} \cup \mathbb{N})$ such that*

$$\forall_{m\in\{0\}\cup\mathbb{N}} \{ w^m \in A_0 \wedge v^m \in B_0 \}, \tag{3.11}$$

$$\forall_{m\in\{0\}\cup\mathbb{N}} \{ v^m \in T(w^m) \}, \tag{3.12}$$

$$\forall_{m\in\mathbb{N}} \{ J(w^m, v^{m-1}) = \operatorname{dist}(A, B) \}, \tag{3.13}$$

$$\forall_{m\in\mathbb{N}} \left\{ J(v^{m-1}, v^m) \leq H^J(T(w^{m-1}), T(w^m)) + \left(\frac{\lambda}{s}\right)^m \right\} \tag{3.14}$$

and

$$\forall_{m\in\mathbb{N}} \{ J(w^m, w^{m+1}) = J(v^{m-1}, v^m) \}, \tag{3.15}$$

$$\lim_{n\to\infty}\sup_{m>n} J(w^n, w^m) = 0, \tag{3.16}$$

and

$$\lim_{n\to\infty}\sup_{m>n} J(v^n, v^m) = 0. \tag{3.17}$$

Indeed, since $A_0 \neq \emptyset$ and $T(x) \subseteq B_0$ for each $x \in A_0$, we may choose $w^0 \in A_0$ and next $v^0 \in T(w^0) \subseteq B_0$. By definition of B_0, there exists $w^1 \in A$ such that

$$J(w^1, v^0) = \mathrm{dist}(A, B). \tag{3.18}$$

Of course, since $v^0 \in B$, by (3.18), we have $w^1 \in A_0$. Next, since $T(x) \subseteq B_0$ for each $x \in A_0$, from (3.2) (for $x = w^0$, $y = w^1$, $\gamma = \lambda/s$, $w = v^0$) we conclude that there exists $v^1 \in T(w^1) \subseteq B_0$ (since $w^1 \in A_0$) such that

$$J(v^0, v^1) \leq H^J\bigl(T(w^0), T(w^1)\bigr) + \frac{\lambda}{s}. \tag{3.19}$$

Next, since $v^1 \in B_0$, by definition of B_0, there exists $w^2 \in A$ such that

$$J(w^2, v^1) = \mathrm{dist}(A, B). \tag{3.20}$$

Of course, since $v^1 \in B$, by (3.20), we have $w^2 \in A_0$. Since $T(x) \subseteq B_0$ for each $x \in A_0$, from (3.2) (for $x = w^1$, $y = w^2$, $\gamma = (\lambda/s)^2$, $w = v^1$) we conclude that there exists $v^2 \in T(w^2) \subseteq B_0$ (since $w^2 \in A_0$) such that

$$J(v^1, v^2) \leq H^J\bigl(T(w^1), T(w^2)\bigr) + \left(\frac{\lambda}{s}\right)^2. \tag{3.21}$$

By (3.18)-(3.21) and by the induction, we produce sequences $(w^m : m \in \{0\} \cup \mathbb{N})$ and $(v^m : m \in \{0\} \cup \mathbb{N})$ such that:

$$\begin{aligned}
&\forall_{m\in\{0\}\cup\mathbb{N}}\bigl\{w^m \in A_0 \wedge v^m \in B_0\bigr\},\\
&\forall_{m\in\{0\}\cup\mathbb{N}}\bigl\{v^m \in T(w^m)\bigr\},\\
&\forall_{m\in\mathbb{N}}\bigl\{J(w^m, v^{m-1}) = \mathrm{dist}(A, B)\bigr\}
\end{aligned}$$

and

$$\forall_{m\in\mathbb{N}}\left\{J(v^{m-1}, v^m) \leq H^J\bigl(T(w^{m-1}), T(w^m)\bigr) + \left(\frac{\lambda}{s}\right)^m\right\}.$$

Thus (3.11)-(3.14) hold. In particularly (3.13) gives $\forall_{m\in\mathbb{N}}\{J(w^m, v^{m-1}) = \mathrm{dist}(A, B) \wedge J(w^{m+1}, v^m) = \mathrm{dist}(A, B)\}$. Now, since the pair (A, B) has the P^J-property, from the above we conclude

$$\forall_{m\in\mathbb{N}}\bigl\{J(w^m, w^{m+1}) = J(v^{m-1}, v^m)\bigr\}.$$

Consequently, the property (3.15) holds.

We recall that the contractive condition (see (3.1)) is as follows:

$$\exists_{0\leq\lambda<1}\forall_{x,y\in A}\left\{sH^J\big(T(x),T(y)\big)\leq\lambda J(x,y)\right\}. \tag{3.22}$$

In particular, by (3.22) (for $x = w^m$, $y = w^{m+1}$, $m \in \{0\}\cup\mathbb{N}$) we obtain

$$\forall_{m\in\{0\}\cup\mathbb{N}}\left\{H^J\big(T(w^m),T(w^{m+1})\big)\leq\frac{\lambda}{s}J\big(w^m,w^{m+1}\big)\right\}. \tag{3.23}$$

Next, by (3.15), (3.14), and (3.23) we calculate:

$$\begin{aligned}\forall_{m\in\mathbb{N}}\Bigg\{J\big(w^m,w^{m+1}\big)&=J\big(v^{m-1},v^m\big)\leq H^J\big(T(w^{m-1}),T(w^m)\big)+\left(\frac{\lambda}{s}\right)^m\\&\leq\frac{\lambda}{s}J\big(w^{m-1},w^m\big)+\left(\frac{\lambda}{s}\right)^m=\frac{\lambda}{s}J\big(v^{m-2},v^{m-1}\big)+\left(\frac{\lambda}{s}\right)^m\\&\leq\frac{\lambda}{s}\left[H^J\big(T(w^{m-2}),T(w^{m-1})\big)+\left(\frac{\lambda}{s}\right)^{m-1}\right]+\left(\frac{\lambda}{s}\right)^m\\&=\frac{\lambda}{s}H^J\big(T(w^{m-2}),T(w^{m-1})\big)+2\left(\frac{\lambda}{s}\right)^m\\&\leq\left(\frac{\lambda}{s}\right)^2J\big(w^{m-2},w^{m-1}\big)+2\left(\frac{\lambda}{s}\right)^m=\left(\frac{\lambda}{s}\right)^2J\big(v^{m-3},v^{m-2}\big)+2\left(\frac{\lambda}{s}\right)^m\\&\leq\left(\frac{\lambda}{s}\right)^2\left[H^J\big(T(w^{m-3}),T(w^{m-2})\big)+\left(\frac{\lambda}{s}\right)^{m-2}\right]+2\left(\frac{\lambda}{s}\right)^m\\&=\left(\frac{\lambda}{s}\right)^2H^J\big(T(w^{m-3}),T(w^{m-2})\big)+3\left(\frac{\lambda}{s}\right)^m\\&\leq\left(\frac{\lambda}{s}\right)^3J\big(w^{m-3},w^{m-2}\big)+3\left(\frac{\lambda}{s}\right)^m\\&\leq\cdots\leq\left(\frac{\lambda}{s}\right)^mJ\big(w^0,w^1\big)+m\left(\frac{\lambda}{s}\right)^m\Bigg\}.\end{aligned}$$

Hence,

$$\forall_{m\in\mathbb{N}}\left\{J\big(w^m,w^{m+1}\big)\leq\left(\frac{\lambda}{s}\right)^mJ\big(w^0,w^1\big)+m\left(\frac{\lambda}{s}\right)^m\right\}. \tag{3.24}$$

Now, for arbitrary and fixed $n\in\mathbb{N}$ and all $m\in\mathbb{N}$, $m>n$, by (3.24) and (d3), we have

$$\begin{aligned}J\big(w^n,w^m\big)&\leq sJ\big(w^n,w^{n+1}\big)+sJ\big(w^{n+1},w^m\big)\\&\leq sJ\big(w^n,w^{n+1}\big)+s\big[sJ\big(w^{n+1},w^{n+2}\big)+sJ\big(w^{n+2},w^m\big)\big]\\&=sJ\big(w^n,w^{n+1}\big)+s^2J\big(w^{n+1},w^{n+2}\big)+s^2J\big(w^{n+2},w^m\big)\\&\leq\cdots\leq\sum_{k=0}^{m-(n+1)}s^{k+1}J\big(w^{n+k},w^{n+1+k}\big)\\&\leq\sum_{k=0}^{m-(n+1)}s^{k+1}\left[\left(\frac{\lambda}{s}\right)^{n+k}J\big(w^0,w^1\big)+(n+k)\left(\frac{\lambda}{s}\right)^{n+k}\right]\end{aligned}$$

$$= \sum_{k=0}^{m-(n+1)} \left[\left(\frac{\lambda^{n+k}}{s^{n-1}} \right) J(w^0, w^1) + (n+k) \left(\frac{\lambda^{n+k}}{s^{n-1}} \right) \right]$$
$$= \frac{1}{s^{n-1}} \sum_{k=0}^{m-(n+1)} \left[\lambda^{n+k} J(w^0, w^1) + (n+k)\lambda^{n+k} \right].$$

Hence

$$J(w^n, w^m) \leq \frac{1}{s^{n-1}} \sum_{k=0}^{m-(n+1)} \left[J(w^0, w^1) + (n+k) \right] \lambda^{n+k}. \tag{3.25}$$

Thus, as $n \to \infty$ in (3.25), we obtain

$$\lim_{n\to\infty} \sup_{m>n} J(w^n, w^m) = 0.$$

Next, by (3.15) we obtain $\lim_{n\to\infty} \sup_{m>n} J(v^n, v^m) = 0$. Then the properties (3.11)-(3.17) hold.

Step 2. *We can show that the sequence* $(w^m : m \in \{0\} \cup \mathbb{N})$ *is Cauchy.*

Indeed, it is an easy consequence of (3.16) and Lemma 3.2.

Step 3. *We can show that the sequence* $(v^m : m \in \{0\} \cup \mathbb{N})$ *is Cauchy.*

Indeed, it follows by Step 1 and by a similar argumentation as in Step 2.

Step 4. *There exists a best proximity point, i.e. there exists* $w_0 \in A$ *such that*

$$\inf\{d(w_0, z) : z \in T(w_0)\} = \mathrm{dist}(A, B).$$

Indeed, by Steps 2 and 3, the sequences $(w^m : m \in \{0\} \cup \mathbb{N})$ and $(v^m : m \in \{0\} \cup \mathbb{N})$ are Cauchy and in particularly satisfy (3.12). Next, since X is a complete space, there exist $w_0, v_0 \in X$ such that $\lim_{m\to\infty} w^m = w_0$ and $\lim_{m\to\infty} v^m = v_0$, respectively. Now, since A and B are closed (we recall that $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^m \in A \wedge v^m \in B\}$), thus $w_0 \in A$ and $v_0 \in B$. Finally, since by (3.12) we have $\forall_{m\in\{0\}\cup\mathbb{N}}\{v^m \in T(w^m)\}$, by closedness of T, we have

$$v_0 \in T(w_0). \tag{3.26}$$

Next, since $w_0 \in A$, $v_0 \in B$ and $T(A) \subset B$, by (3.26) we have $T(w_0) \subset B$ and

$$\begin{aligned} \mathrm{dist}(A, B) &= \inf\{d(a, b) : a \in A \wedge b \in B\} \leq D(w_0, B) \leq D(w_0, T(w_0)) \\ &= \inf\{d(w_0, z) : z \in T(w_0)\} \leq d(w_0, v_0). \end{aligned} \tag{3.27}$$

We know that $\lim_{m\to\infty} w^m = w_0$, $\lim_{m\to\infty} v^m = v_0$. Moreover by (3.13)

$$\forall_{m\in\mathbb{N}} \{J(w^m, v^{m-1}) = \mathrm{dist}(A, B)\}.$$

Thus, since J and (A, B) are associated, so by Definition 2.4(II), we conclude that

$$d(w_0, v_0) = \mathrm{dist}(A, B). \tag{3.28}$$

Finally, (3.27) and (3.28), give $\inf\{d(w_0, z) : z \in T(w_0)\} = \mathrm{dist}(A, B)$. □

4 Examples illustrating Theorem 3.1 and some comparisons

Now, we will present some examples illustrating the concepts having been introduced so far. We will show a fundamental difference between Theorem 1.1 and Theorem 3.1. The examples will show that Theorem 3.1 is an essential generalization of Theorem 1.1. First, we present an example of J, a generalized pseudodistance.

Example 4.1 Let X be a b-metric space (with constant $s = 2$) where b-metric $d : X \times X \to [0, \infty)$ is of the form $d(x, y) = |x - y|^2$, $x, y \in X$. Let the closed set $E \subset X$, containing at least two different points, be arbitrary and fixed. Let $c > 0$ such that $c > \delta(E)$, where $\delta(E) = \sup\{d(x, y) : x, y \in X\}$ be arbitrary and fixed. Define the map $J : X \times X \to [0, \infty)$ as follows:

$$J(x,y) = \begin{cases} d(x,y) & \text{if } \{x,y\} \cap E = \{x,y\}, \\ c & \text{if } \{x,y\} \cap E \neq \{x,y\}. \end{cases} \tag{4.1}$$

The map J is a b-generalized pseudodistance on X. Indeed, it is worth noticing that the condition (J1) does not hold only if some $x_0, y_0, z_0 \in X$ such that $J(x_0, z_0) > s[J(x_0, y_0) + J(y_0, z_0)]$ exists. This inequality is equivalent to $c > s[d(x_0, y_0) + d(y_0, z_0)]$ where $J(x_0, z_0) = c$, $J(x_0, y_0) = d(x_0, y_0)$ and $J(y_0, z_0) = d(y_0, z_0)$. However, by (4.1), $J(x_0, z_0) = c$ shows that there exists $v \in \{x_0, z_0\}$ such that $v \notin E$; $J(x_0, y_0) = d(x_0, y_0)$ gives $\{x_0, y_0\} \subset E$; $J(y_0, z_0) = d(y_0, z_0)$ gives $\{y_0, z_0\} \subset E$. This is impossible. Therefore, $\forall_{x,y,z\in X}\{J(x, y) \le s[J(x, z) + J(z, y)]\}$, *i.e.* the condition (J1) holds.

Proving that (J2) holds, we assume that the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X satisfy (2.1) and (2.2). Then, in particular, (2.2) yields

$$\forall_{0<\varepsilon<c}\exists_{m_0=m_0(\varepsilon)\in\mathbb{N}}\forall_{m\ge m_0}\{J(x_m, y_m) < \varepsilon\}. \tag{4.2}$$

By (4.2) and (4.1), since $\varepsilon < c$, we conclude that

$$\forall_{m\ge m_0}\{E \cap \{x_m, y_m\} = \{x_m, y_m\}\}. \tag{4.3}$$

From (4.3), (4.1), and (4.2), we get

$$\forall_{0<\varepsilon<c}\exists_{m_0\in\mathbb{N}}\forall_{m\ge m_0}\{d(x_m, y_m) < \varepsilon\}.$$

Therefore, the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ satisfy (2.3). Consequently, the property (J2) holds.

The next example illustrates Theorem 3.1.

Example 4.2 Let X be a b-metric space (with constant $s = 2$), where $X = [0, 3]$ and $d(x, y) = |x - y|^2$, $x, y \in X$. Let $A = [0, 1]$ and $B = [2, 3]$. Let $E = [0, \frac{1}{4}] \cup [1, 3]$ and let the map $J : X \times X \to [0, \infty)$ be defined as follows:

$$J(x,y) = \begin{cases} d(x,y) & \text{if } \{x,y\} \cap E = \{x,y\}, \\ 10 & \text{if } \{x,y\} \cap E \neq \{x,y\}. \end{cases} \tag{4.4}$$

Of course, since E is closed set and $\delta(E) = 9 < 10$, by Example 4.1 we see that the map J is the b-generalized pseudodistance on X. Moreover, it is easy to verify that $A_0 = \{1\}$ and

$B_0 = \{2\}$. Indeed, $\mathrm{dist}(A,B) = 1$, thus

$$A_0 = \bigl\{x \in A = [0,1] : J(x,y) = \mathrm{dist}(A,B) = 1 \text{ for some } y \in B = [2,3]\bigr\},$$

and by (4.4) $\{x,y\} \cap E = \{x,y\}$, so $J(x,y) = d(x,y)$, $x \in [0,1/4] \cup \{1\}$ and $y \in [2,3]$. Consequently $A_0 = \{1\}$. Similarly,

$$B_0 = \bigl\{y \in B = [2,3] : J(x,y) = \mathrm{dist}(A,B) = 1 \text{ for some } x \in A = [0,1]\bigr\},$$

and, by (4.4), $\{x,y\} \cap E = \{x,y\}$, so $J(x,y) = d(x,y)$, $y \in [2,3]$ and $x \in [0,1/4] \cup \{1\}$. Consequently $B_0 = \{2\}$.

Let $T : A \to 2^B$ be given by the formula

$$T(x) = \begin{cases} \{2\} \cup [\frac{11}{4},3] & \text{for } x \in [0,\frac{1}{4}], \\ [\frac{11}{4},3] & \text{for } x \in (\frac{1}{4},\frac{1}{2}), \\ [\frac{5}{2},3] & \text{for } x \in [\frac{1}{2},\frac{3}{4}), \\ [\frac{9}{4},3] & \text{for } x \in [\frac{3}{4},\frac{7}{8}), \\ \{2\} \cup [\frac{9}{4},3] & \text{for } x = \frac{7}{8}, \\ \{2\} & \text{for } x \in (\frac{7}{8},1], \end{cases} \quad x \in X. \tag{4.5}$$

We observe the following.

(I) *We can show that the pair* (A,B) *has the* P^J*-property.*

Indeed, as we have previously calculated $A_0 = \{1\}$ and $B_0 = \{2\}$. This gives the following result: for each $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, such that $J(x_1,y_1) = \mathrm{dist}(A,B) = 1$ and $J(x_2,y_2) = \mathrm{dist}(A,B) = 1$, since A_0 and B_0 are included in E, by (4.4) we have

$$J(x_1,x_2) = d(x_1,x_2) = d(1,1) = 0 = d(2,2) = d(y_1,y_2) = J(y_1,y_2).$$

(II) *We can show that the map* J *is associated with* (A,B).

Indeed, let the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X, such that $\lim_{m\to\infty} x_m = x$, $\lim_{m\to\infty} y_m = y$ and

$$\forall_{m\in\mathbb{N}} \bigl\{J(x_m,y_{m-1}) = \mathrm{dist}(A,B)\bigr\}, \tag{4.6}$$

be arbitrary and fixed. Then, since $\mathrm{dist}(A,B) = 1 < 10$, by (4.6) and (4.4), we have

$$\forall_{m\in\mathbb{N}} \bigl\{d(x_m,y_{m-1}) = J(x_m,y_{m-1}) = \mathrm{dist}(A,B)\bigr\}. \tag{4.7}$$

Now, from (4.7) and by continuity of d, we have $d(x,y) = \mathrm{dist}(A,B)$.

(III) *It is easy to see that* T *is a closed map on* X.

(IV) *We can show that* T *is a set-valued non-self-mapping contraction of Nadler type with respect* J *(for* $\lambda = 1/2$; *as a reminder*: *we have* $s = 2$).

Indeed, let $x, y \in A$ be arbitrary and fixed. First we observe that since $T(A) \subset B = [2,3] \subset E$, by (4.4) we have $H^J(T(x),T(y)) = H(T(x),T(y)) \leq 1$, for each $x,y \in A$. We consider the following two cases.

Case 1. If $\{x,y\} \cap E \neq \{x,y\}$, then by (4.4), $J(x,y) = 10$, and consequently $H^J(T(x),T(y)) \leq 1 < 10/4 = (1/4) \cdot 10 = (\lambda/s)J(x,y)$. In consequence, $sH^J(T(x),T(y)) \leq \lambda J(x,y)$.

Case 2. If $\{x,y\} \cap E = \{x,y\}$, then $x,y \in E \cap [0,1] = [0,/1/4] \cup \{1\}$. From the obvious property

$$\forall_{x,y\in[0,/1/4]}\bigl\{T(x) = T(y) \wedge T(1) \subset T(x)\bigr\}$$

can be deduced that $\forall_{x,y\in[0,/1/4]\cup\{1\}}\{H^J(T(x),T(y)) = 0\}$. Hence, $sH^J(T(x),T(y)) = 0 \leq \lambda J(x,y)$.

In consequence, T is the set-valued non-self-mapping contraction of Nadler type with respect to J.

(V) *We can show that $T(x)$ is bounded and closed in B for all $x \in A$.*

Indeed, it is an easy consequence of (4.5).

(VI) *We can show that $T(x) \subset B_0$ for each $x \in A_0$.*

Indeed, by (I), we have $A_0 = \{1\}$ and $B_0 = \{2\}$, from which, by (4.5), we get $T(1) = \{2\} \subseteq B_0$.

All assumptions of Theorem 3.1 hold. We see that $D(1,T(1)) = D(1,\{2\}) = 1 = \operatorname{dist}(A,B)$, *i.e.* 1 is the best proximity point of T.

Remark 4.1 (I) The introduction of the concept of b-generalized pseudodistances is essential. If X and T are like in Example 4.2, then we can show that T *is not a set-valued non-self-mapping contraction of Nadler type with respect to d*. Indeed, suppose that T *is a set-valued non-self-mapping contraction of Nadler type, i.e.* $\exists_{0\leq\lambda<1}\forall_{x,y\in X}\{sH(T(x),T(y)) \leq \lambda d(x,y)\}$. In particular, for $x_0 = \frac{1}{2}$ and $y_0 = 1$ we have $T(x_0) = [5/2,3]$, $T(y_0) = \{2\}$ and $2 = 2H(T(x_0),T(y_0)) = sH(T(x_0),T(y_0)) \leq \lambda d(x_0,y_0) = \lambda|1/2-1|^2 = \lambda \cdot 1/4 < 1/4$. This is absurd.

(II) If X is metric space ($s = 1$) with metric $d(x,y) = |x-y|$, $x,y \in X$, and T is like in Example 4.2, then we can show that T *is not a set-valued non-self-mapping contraction of Nadler type with respect to d*. Indeed, suppose that T *is a set-valued non-self-mapping contraction of Nadler type, i.e.* $\exists_{0\leq\lambda<1}\forall_{x,y\in X}\{H(T(x),T(y)) \leq \lambda d(x,y)\}$. In particular, for $x_0 = \frac{1}{2}$ and $y_0 = 1$ we have $2 = 2H(T(x_0),T(y_0)) = sH(T(x_0),T(y_0)) \leq \lambda d(x_0,y_0) = \lambda|1/2-1| = \lambda \cdot 1/2 < 1/2$. This is absurd. Hence, we find that our theorem is more general than Theorem 1.1 (Abkar and Gabeleh [13]).

Competing interests

The author declares that they have no competing interests.

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